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Dynamical Analysis of a Local Lengley-Epstein System Coupled with Fractional Delayed Differential Equations

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Abstract

We consider a system of fractional delayed differential equations. The ordinary differential version of the system without delay is introduced in the Lengyel-Epstein reaction-diffusion system. We evaluate the system with and without delay and explore the stability of the unique positive equilibrium. We also prove the existence of Hopf bifurcation for both cases. Furthermore, the impacts of Caputo fractional order parameter and time delay parameter on the dynamics of the system are investigated with numerical simulations. It is also concluded that for different values of time delay parameter, the decreament of the Caputo fractional order parameter has opposite effects on the system in terms of stability.

1. Introduction

Fractional calculus is considered as a generalization of ordinary calculus to non-integer orders. Fractional derivative operator is a non-local operator in nature. As a result, fractional differential equations are associated with memory and hereditary attributes, which are present in many real processes. Thus, there are many applications of fractional differential equations in various research fields such as chemistry [1,2], physics [3,4], biology [5,6], epidemic modelling [7–9] mechanical engineering [10] and network theory [11]. The best-known definitions of fractional order derivative are Riemann-Liouville and Caputo definitions. These definitions are more reliable in terms of non-locality and uncovering memory effects despite the fact that there are relatively new approches like conformable fractional derivative, Caputo-Fabrizio derivative etc. [12] On the other hand, Riemann-Liouville fractional derivative requires fractional initial conditions due to fact that Rimann-Liouville derivative of a constant is not zero. This is not the case for Caputo sense fractional differential equations which requires standard initial conditions same as in ordinary differential equations (ODEs). This property makes Caputo definition more appealing while modelling physical or biological facts. The Caputo fractional derivative of order $\alpha > 0$ of a real valued fuction h is defined as

$$D^{\alpha}h(s) = \frac{1}{\Gamma(k-\alpha)}\int_0^s (s-\zeta)^{k-\alpha-1}h^{(k)}(\zeta)d\zeta,$$

where k is an integer and $k-1<\alpha<k$.

Time delay is another useful tool to describe processes that also depends on the past data [13], which exist in many real systems such as chemical processes, technical processes, biosciences, economics and other branches [14,15]. Since both time delays and fractional derivatives allow past data to affect the current state, fractional delayed differential equations (FDDEs) are very effective for constructing strongly realistic models of systems with memory and hereditary properties. There are some works on stability conditions of FDDEs. But, the existing stability conditions for FDDEs do not comprise effective algebraic criteria or algorithms for testing of stability of FDDEs [14,16,17]. Some studies on dynamical analysis of FDDEs can be found in [11,18–20]. In [11], authors worked on fractional complex-valued neural network with delays and provided a detailed numerical analysis. Li et al. [18] investigated the dynamical behaviours of a prey-predator model with double delays and proved the existence of Hopf bifurcation depending on the time delay parameter. In [19], authors also considered a prey-predator model with incorporating the dispersal of prey and analyzed numerically the relation between the fractional order and the time delay parameters.

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In this work, we study Caputo fractional order version of the following system of delay differential equations

$$\begin{cases} u' = a - u - 4\dfrac{uv(t-\tau)}{1+u^2}, \\ v' = \sigma b\left(u - \dfrac{uv(t-\tau)}{1+u^2}\right) \end{cases} \tag{1.1}$$

where $\tau \geq 0$ is the delay parameter. When $\tau = 0$, the system (1.1) reduces to local Lengyel-Epstein system which is reaction-diffusion system that is used to describe chlorite-iodide malonic-acid (CIMA) chemical reaction [21, 22]. Here, $u(t)$ and $v(t)$ represents concentrations of the activator iodine (I^-) and the inhibitor chlorite (CIO_2^-). The positive parameters a and b are correlated to the feed concentrations; $\sigma > 0$ is a rescaling coefficient depending on the concentration of the starch.

For $\tau = 0$, both of this ODE and the associated PDE model are studied in [23]. Yi et al. derived the conditions about Turing instability and they proved the existence of Hopf bifurcation together with direction of bifurcation [23]. In [24], authors make Hopf bifurcation analysis of the system (1.1) by applying normal form theory given in [25]. In [26], authors consider the fractional version of Lengley-Epstein system by replacing left hand side ordinary derivatives by Caputo sense fractional derivatives. They established the conditions necessary for local and global asymptotic stability of the steady state [26].

Time delays can have a major influence on the dynamic behavior of systems and may cause instability and chaos [15]. On the other hand, using time delays in fractional differential equations is a relatively recent topic which is the main interest of this work. The aim is to study fractional order version of the system (1.1) with $\tau = 0$ and $\tau > 0$. Hopf bifurcation analysis is performed for both cases. Then, we give numerical simulations to illustrate and verify our theoretical results. We also focus on the relation between time delay parameter τ and fractional order parameter α.

2. Fractional Order System without Delay

Firstly, we take fractional order version of the system (1.1) with $\tau = 0$:

$$\begin{cases} D^{\alpha}u(t) = a - u - 4\dfrac{uv}{1+u^2}, \\ D^{\alpha}v(t) = \sigma b\left(u - \dfrac{uv}{1+u^2}\right) \end{cases} \tag{2.1}$$

where $\alpha \in (0,1)$ is the order of the Caputo sense fractional derivative. In order to find equilibrium points of the system (2.1), we solve the following system:

$$\begin{cases} D^{\alpha}u(t) = 0 \\ D^{\alpha}v(t) = 0 \end{cases}.$$

The system (2.1) has a unique positive equilibrium $(u^*, v^*) = (\delta, 1+\delta^2)$ with $\delta = \frac{a}{5}$.

Theorem 2.1. *[27] Consider the n-dimensional system*

$$D_a^{\alpha}h(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $\alpha \in (0,1)$, D_a^{α} represents the Caputo fractional derivative of order α. Let x^* be the equilibrium point of the system and $J(x^*)$ be the Jacobian matrix about the equilibrium point x^*. Then, the equilibrium point x^* is locally asymptotically stable if and only if all the eigenvalues λ_i, $i = 1, 2, \ldots, n$ of $J(x^*)$ satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

Theorem 2.2. *The equilibrium point $(u^*, v^*) = (\delta, 1+\delta^2)$ of the system (2.1) is locally asymptotically stable if one of the following coonditions holds.*

i) $\delta \leq \sqrt{\frac{5}{3}}$,

ii) $\delta > \sqrt{\frac{5}{3}}$ *and* $b > \frac{-5+3\delta^2}{\delta\sigma}$,

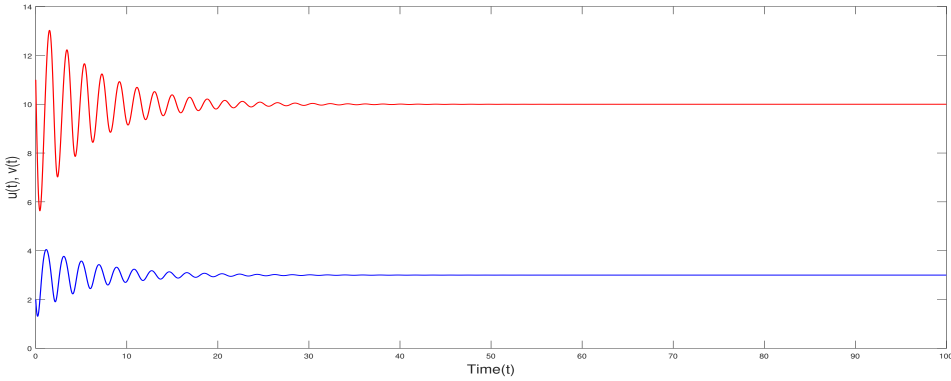
iii) $\delta > \sqrt{\frac{5}{3}}$, $\frac{5+13\delta^2}{\delta\sigma} - 4\sqrt{\frac{10(1+\delta^2)}{\sigma^2}} < b < \frac{-5+3\delta^2}{\delta\sigma}$ *and* $|tan^{-1}\left(\dfrac{\sqrt{-\left(\dfrac{3\delta^2-5-\sigma b\delta}{1+\delta^2}\right)^2 + \dfrac{20\sigma b\delta}{1+\delta^2}}}{\dfrac{3\delta^2-5-\sigma b\delta}{1+\delta^2}}\right)| > \frac{\alpha\pi}{2}$.

Proof. The jacobian matrix of the system (2.1) evaluated at (u^*, v^*) is

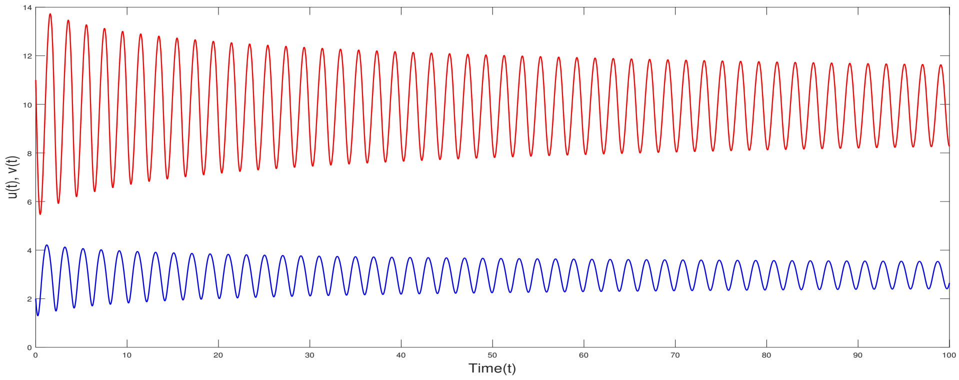
$$J(u^*, v^*) = \begin{pmatrix} \frac{3\delta^2-5}{1+\delta^2} & \frac{-4\delta}{1+\delta^2} \\ \frac{2\sigma b\delta^2}{1+\delta^2} & \frac{-\sigma b\delta}{1+\delta^2} \end{pmatrix}$$

and the corresponding characteristic polynomial is given by

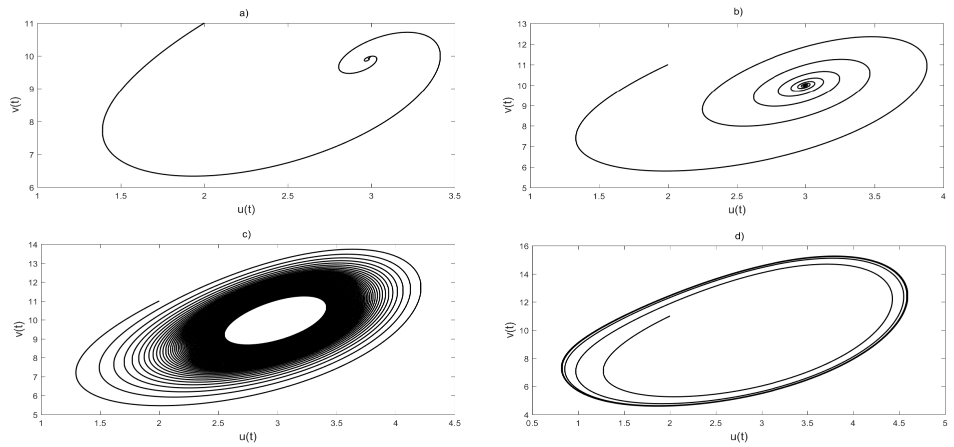
$$\lambda^2 + \rho_1\lambda + \rho_0 = 0$$



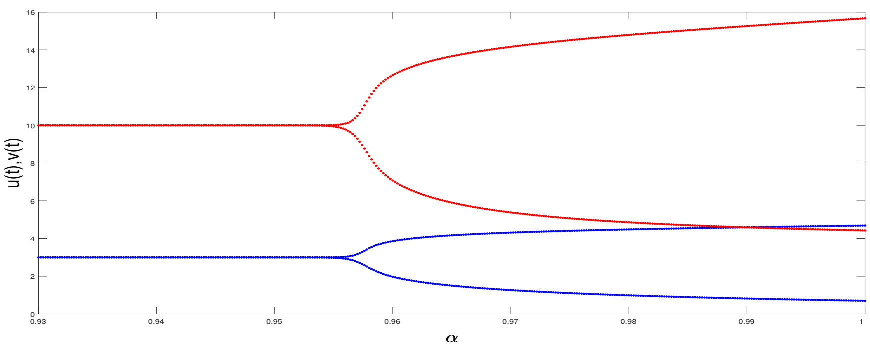
(a) Solution of system (2.1) where $\alpha = 0.90$; $u(t)$ and $v(t)$ displayed by blue and red lines respectively with the initial condition $(2,11)$.



(b) Solution of system (2.1) where $\alpha = \alpha^{*} = 0.9575$; $u(t)$ and $v(t)$ displayed as blue and red lines respectively with the initial condition $(2,11)$.



(c) Phase portraits of system (2.1) with varying the fractional order α where $\alpha = 0.80$ a), $\alpha = 0.90$ b), $\alpha = 0.9575$ c), $\alpha = 0.99$ d) and initial condition $(2,11)$.



(d) Bifurcation diagram of (2.1) depending on α with the initial condition $(2,11)$.

Figure 2.1: Numerical simulations of system (2.1)

where

$$\rho_1 = \frac{5+\sigma b\delta - 3\delta^2}{1+\delta^2}, \quad \rho_0 = \frac{5\sigma b\delta}{1+\delta^2}.$$

Since all the parameters in the system (2.1) are positive, $\rho_0 > 0$. If the conditions i) or ii) holds we have that $\rho_1 > 0$. This implies that the eigenvalues $\lambda_{1,2}$ of $J(u^*,v^*)$ are negative real numbers or complex numbers with negative real part. So, they satify $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$. Moreover, under the conditions $\delta > \sqrt{\frac{5}{3}}$, $b < \frac{-5+3\delta^2}{\delta\sigma}$, $b > \frac{5+13\delta^2}{\delta\sigma} - 4\sqrt{\frac{10(1+\delta^2)}{\sigma^2}}$, we have that $\rho_1 < 0$ and $\rho_1^2 - 4\rho_0 < 0$. So, the eigenvalues $\lambda_{1,2}$ of $J(u^*,v^*)$ are complex numbers with positive real part. If $|arg(\lambda_{1,2})| = |tan^{-1}(\frac{4\rho_0-\rho_1^2}{\rho_1})| > \frac{\alpha\pi}{2}$, the equilibrium point (u^*,v^*) is locally asymptotically stable and thus proving the theorem. □

Theorem 2.3. *[17, 28] When the fractional order parameter α passes through the critical value $\alpha^* \in (0,1)$, Hopf bifurcation occurs for the system (2.1) around the equilibrium point if the followings satisfied:*

(a) The jacobian matrix of (2.1) at the equilibrium point has a pair of complex conjugate eigenvalues $\lambda_{1,2} = \theta + i\gamma$, where $\theta > 0$;
(b) $m(\alpha^*) = 0$, where $m(\alpha) = \frac{\alpha\pi}{2} - min_{1\le i\le 2}|arg(\lambda_i)|$;
(c) $\frac{dm(\alpha)}{d\alpha}|_{\alpha=\alpha^*} \neq 0$. (transversality condition)

Theorem 2.4. *Assume $\delta > \sqrt{\frac{5}{3}}$, $b < \frac{-5+3\delta^2}{\delta\sigma}$, $b > \frac{5+13\delta^2}{\delta\sigma} - 4\sqrt{\frac{10(1+\delta^2)}{\sigma^2}}$. Then the system (2.1) undergoes a Hopf bifurcation about the equilibrium point (u^*,v^*) when $\alpha = \alpha^* = \frac{2}{\pi}tan^{-1}(\frac{\sqrt{4\rho_0-\rho_1^2}}{-\rho_1})$.*

Proof. We again consider the jacobian matrix $J(u^*,v^*)$ (2.2) and the characteristic equation (2.2). The conditions $\delta > \sqrt{\frac{5}{3}}$ and $b < \frac{-5+3\delta^2}{\delta\sigma}$ ensures that $\rho_1 < 0$. Moreover the condition $b > \frac{5+13\delta^2}{\delta\sigma} - 4\sqrt{\frac{10(1+\delta^2)}{\sigma^2}}$ guarantees that the eigenvalues $\lambda_{1,2} = \frac{-\rho_1 \pm \sqrt{\rho_1^2 - 4\rho_0}}{2}$ of the $J(u^*,v^*)$ are complex conjugates with positive real part. So, $min_{1\le i\le 2}|arg(\lambda_i)| = tan^{-1}(\frac{\sqrt{4\rho_0-\rho_1^2}}{-\rho_1})$. For $\alpha = \alpha^*$, $m(\alpha^*) = \frac{\alpha^*\pi}{2} - min_{1\le i\le 2}|arg(\lambda_i)| = 0$. Finally, the transverality condition $\frac{dm(\alpha)}{d\alpha}|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0$ is also satisfied, which proves the theorem.

□

3. Fractional Order System with Delay

In this section, the fractional order version of the system (1.1) is considered with $\tau > 0$:

$$\begin{cases} D^\alpha u(t) = a - u - 4\dfrac{uv(t-\tau)}{1+u^2}, \\ D^\alpha v(t) = \sigma b\left(u - \dfrac{uv(t-\tau)}{1+u^2}\right). \end{cases} \tag{3.1}$$

Here $\alpha \in (0,1)$ is the order of the Caputo fractional derivative. We shall investigate the stability and Hopf bifurcation of the system (3.1) by setting the parameter τ as a bifurcation parameter. We firstly note that, the system (3.1)) with delay has also the same equilibirum point with the system (2.1) without delay, which is $(u^*,v^*) = (\delta, 1+\delta^2)$ where $\delta = \frac{a}{5}$.

Theorem 3.1. *[29, 30] Consider the delayed, Caputo fractional fractional order system as*

$$D^\alpha y(t) = Ay(t) + By(t-\tau), \quad y(t) = \Phi(t), t \in [-\tau, 0], \tag{3.2}$$

where $\alpha \in (0,1]$, $y \in R^n$, $A, B \in R^{n\times n}$, and $\Phi(t) \in R_+^{n\times n}$. The characteristic equation of the system (3.2) is given as

$$det|s^\alpha I - A - Be^{-s\tau}| = 0. \tag{3.3}$$

If all the roots of (3.3) have negative real parts, then the zero solution of system (3.2) is locally asymptotically stable.

By linearizing (3.1) about the positive equilibrium (u^*,v^*), we obtain

$$\begin{cases} D^\alpha u(t) = \dfrac{3\delta^2-5}{1+\delta^2}u(t) + -\dfrac{4\delta}{1+\delta^2}v(t-\tau), \\ D^\alpha v(t) = \dfrac{2\sigma b\delta^2}{1+\delta^2}u(t) - \dfrac{\sigma b\delta}{1+\delta^2}v(t-\tau). \end{cases} \tag{3.4}$$

The characteristic matrix of the system (3.4) is

$$J(u^*,v^*) = \begin{pmatrix} s - \frac{3\delta^2-5}{1+\delta^2} & \frac{4\delta}{1+\delta^2}e^{-s\tau} \\ \frac{-2\sigma b\delta^2}{1+\delta^2} & s + \frac{\sigma b\delta}{1+\delta^2}e^{-s\tau} \end{pmatrix}$$

and the corresponding characteristic equation is

$$s^{2\alpha} - ms^\alpha + (ns^\alpha + 5n)e^{-s\tau} = 0 \tag{3.5}$$

where

$$m = \frac{3\delta^2-5}{1+\delta^2} \quad \text{and} \quad n = \frac{\sigma b\delta}{1+\delta^2}.$$

Theorem 3.2. *Assuming that the inequality (3.12) and the conditions of Theorem 2.2 are fullfilled, the following results hold for the fractional delayed system (3.1):*

(i) The equilibrium point (u^*,v^*) is locally asymptotically stable for $\tau<\tau_0$ where $\tau_0=min\{\tau_k^j\}$ and

$$\tau_k^j=\frac{1}{\omega_k}\left[cos^{-1}\left(\frac{m^2\omega_k^{2\alpha}-n^2\omega_k^{2\alpha}-\omega_k^{4\alpha}+25n^2}{10mn\omega_k^{\alpha}+2n\omega_k^{3\alpha}}\right)-\frac{\alpha\pi}{2}+2j\pi\right].$$

(ii) The system undergoes a Hopf bifurcation about the equilibrium point (u^*,v^*) for $\tau=\tau_0$.

Proof. Assume that the characteristic equation (3.5) has a pair of pure imaginary roots $s_{1,2}=\pm i\zeta$, $\zeta>0$. By putting $s_1=i\zeta$ into the equation (3.5), we obtain

$$(i\zeta)^{2\alpha}-m(i\zeta)^{\alpha}+(n(i\zeta)^{\alpha}+5n)e^{-i\zeta\tau}=0. \tag{3.6}$$

By seperating real and imaginary parts of (3.6), one has

$$\begin{aligned}\zeta^{2\alpha}cos\alpha\pi-m\zeta^{\alpha}cos\frac{\alpha\pi}{2}&=-5ncos\tau\zeta-n\zeta^{\alpha}cos\left(\frac{\alpha\pi}{2}-\tau\zeta\right),\\ \zeta^{2\alpha}sin\alpha\pi-m\zeta^{\alpha}sin\frac{\alpha\pi}{2}&=5nsin\tau\zeta-n\zeta^{\alpha}sin\left(\frac{\alpha\pi}{2}-\tau\zeta\right).\end{aligned} \tag{3.7}$$

Squaring and adding two equations in (3.7) yields to the equality

$$\zeta^{4\alpha}-2m\zeta^{3\alpha}cos\frac{\alpha\pi}{2}+\zeta^{2\alpha}(m^2-n^2)-10n^2\zeta^{\alpha}cos\frac{\alpha\pi}{2}-25n^2=0. \tag{3.8}$$

Since $-25n^2<0$, the Eq. (3.8) has at least one positive root. Denote this positive root as ζ_k. Substituting ζ_k in (3.7) gives

$$\begin{aligned}\zeta^{2\alpha}cos\alpha\pi+n\zeta^{\alpha}cos\left(\frac{\alpha\pi}{2}-\tau\zeta\right)&=-5ncos\tau\zeta+m\zeta^{\alpha}cos\frac{\alpha\pi}{2},\\ \zeta^{2\alpha}sin\alpha\pi+n\zeta^{\alpha}sin\left(\frac{\alpha\pi}{2}-\tau\zeta\right)&=5nsin\tau\zeta+m\zeta^{\alpha}sin\frac{\alpha\pi}{2}.\end{aligned} \tag{3.9}$$

Squarring and adding two equations in (3.9), we obtain

$$\zeta_k^{4\alpha}+\zeta_k^{3\alpha}2ncos\left(\frac{\alpha\pi}{2}+\tau\zeta_k\right)+\zeta_k^{2\alpha}(n^2-m^2)+\zeta_k^{\alpha}10mncos\left(\frac{\alpha\pi}{2}+\tau\zeta_k\right)-25n^2=0. \tag{3.10}$$

From (3.10), τ_k can be obtained as

$$\tau_k^j=\frac{1}{\omega_k}\left[cos^{-1}\left(\frac{m^2\omega_k^{2\alpha}-n^2\omega_k^{2\alpha}-\omega_k^{4\alpha}+25n^2}{10mn\omega_k^{\alpha}+2n\omega_k^{3\alpha}}\right)-\frac{\alpha\pi}{2}+2j\pi\right],$$

where j=0,1,2, ... We define $\tau_0=min\{\tau_k^j\}$. For $\tau<\tau_0$ all the roots of the characteristic equation (3.5) have negative real parts and the equilibrium point (u^*,v^*) is locally asymptotically stable.
Now, we check the transversality condition. Let us rewrite the characteristic equation (3.5) as

$$Q_1(s)+Q_2(s)e^{-s\tau}=0, \tag{3.11}$$

where $Q_1(s)=s^{2\alpha}-ms^{\alpha}$ and $Q_2(s)=ns^{\alpha}+5n$. We differentiate (3.11) with respect to τ to get

$$\frac{ds}{d\tau}(Q_1'(s)+Q_2'(s)e^{-s\tau}-Q_2(s)e^{-s\tau}\tau)-Q_2(s)e^{-s\tau}s=0$$

and

$$\frac{ds}{d\tau}=\frac{Q_2(s)e^{-s\tau}s}{Q_1'(s)+Q_2'(s)e^{-s\tau}-Q_2(s)e^{-s\tau}\tau}=\frac{A(s)}{B(s)},$$

where

$$\begin{aligned}A(s)=&(ns^{\alpha}+5n)e^{-s\tau}s=A_1+iA_2,\\ B(s)=&2\alpha s^{2\alpha-1}-m\alpha s^{\alpha-1}+n\alpha s^{\alpha-1}e^{-s\tau}-(ns^{\alpha}+5n)e^{-s\tau}\tau=B_1+iB_2.\end{aligned}$$

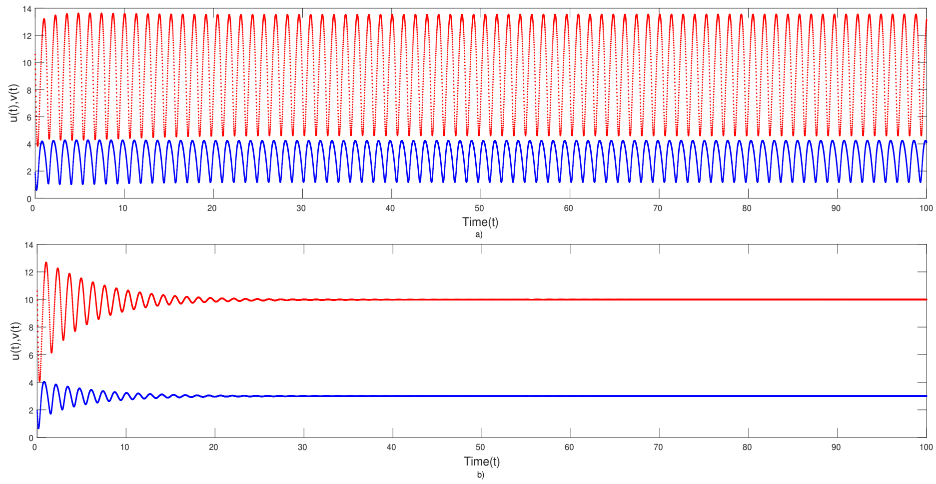
Let $s(\tau)=\nu(\tau)+i\zeta(\tau)$ be the root of equation (3.11) with $\nu(\tau_j)=0$, $\zeta(\tau_j)=\zeta_0$. Then, we can obtain

$$Re\left[\frac{ds}{d\tau}\right]\Bigg|_{(\tau=\tau_0,\zeta=\zeta_0)}=\frac{A_1B_1+A_2B_2}{B_1^2+B_2^2}.$$

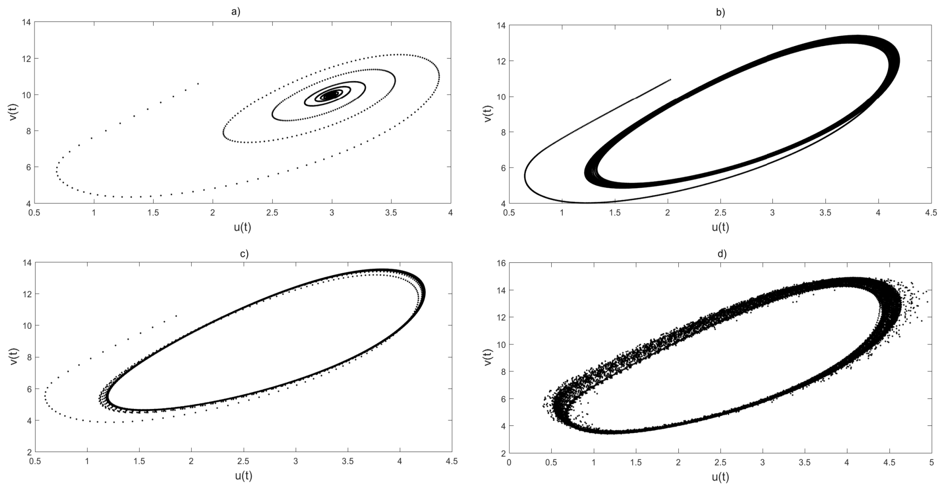
So, under the condition

$$\frac{A_1B_1+A_2B_2}{B_1^2+B_2^2}\neq0,$$

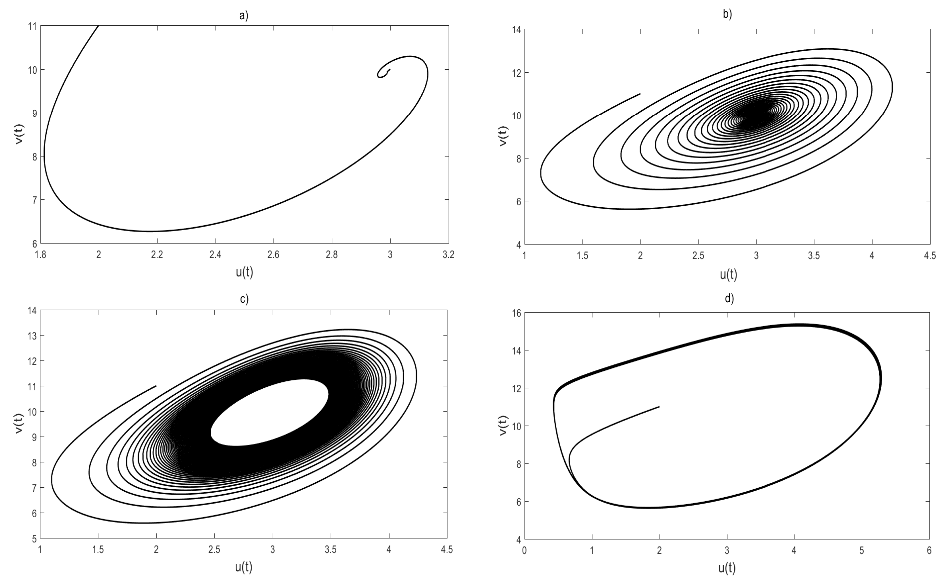
the transversality condition holds. □



(a) Solution of system (3.1) where $\alpha = 0.80$, $\tau = 0.11$ in a), $\tau = 0.10$ in b); $u(t)$ and $v(t)$ displayed by blue and red lines resp. with initial condition $(2, 11)$.



(b) Phase portraits of system (3.1) with varying the time delay τ as $\tau = 0.09$ a), $\tau = 0.103$ b), $\tau = 0.11$ c), $\tau = 0.13$ d); where $\alpha = 0.80$ and initial condition $(2, 11)$.

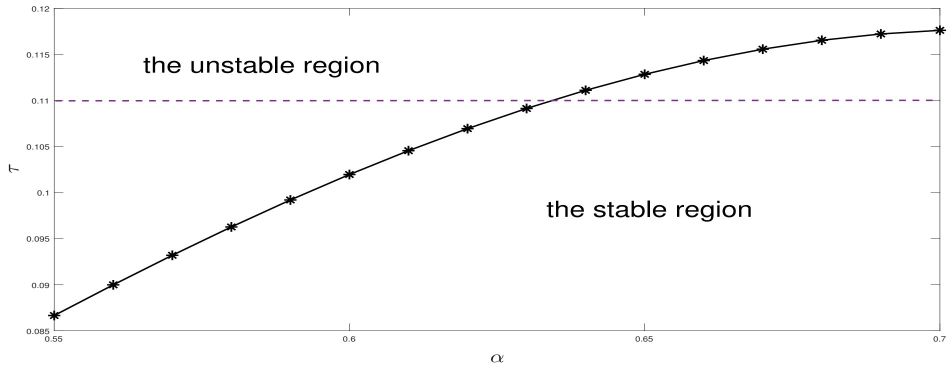


(c) Phase portraits of system (2.1) with varying the parameter b where $b = 3$ a), $b = 0.8$ b), $b = 0.765879$ c), $b = 0.4$ d); $\alpha = 0.90$ and initial condition $(2, 11)$.

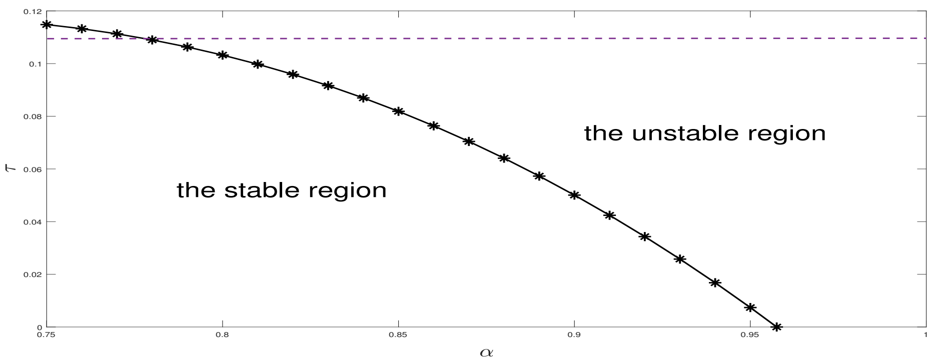
Figure 3.1: Numerical simulations of system (3.1)

4. Numerical Results

In this section, numerical illustrations are displayed to support theoretical results. Firstly, we give numerical simulations about the system (2.1) using predictor corrector (PECE) method. The PECE method, referred to as fractional Adams-Bashforth-Moulton methods [31], has proven to be an accurate and powerful method to find approximate solutions of FDEs.
For numerical simulations, we pick parameter values as $a = 15$, $b = 1$, $\sigma = 6$. In Figure 2.1a, we observe that the equilibrium point $(u^*, v^*) = (3, 10)$ is locally asymptotically stable for $\alpha = 0.90$. The critical bifurcation value of fractional order α in Theorem 2.4 is calculated as $\alpha^* = 0.9575$. By setting $\alpha = \alpha^*$, the system (2.1) undergoes a Hopf bifurcation. We observe the oscillatory behavior of the solutions (Figure 2.1b).



(a) The curve represent the correlation between the fractional order parameter α and the critical value of time delay τ_0 for the system (3.1) with $0.75 \leq \alpha \leq 1$



(b) The curve represent the correlation between the fractional order parameter α and the critical value of time delay τ_0 for the system (3.1) with $0.50 \leq \alpha \leq 0.70$

Figure 4.1: Fractional order parameter α versus delay parameter τ in terms of stability

In Figure 2.1c, we give phase diagrams of the system (2.1) for several values of fractional order α. There exist a limit cycle for $\alpha = \alpha^* = 0.9975$ (Figure 2.1c,c). Moreover the system (2.1) is locally asymptotically stable for smaller values of α (Figure 2.1c,a,b) and unstable otherwise (Figure 2.1c,d). In Figure 2.1d, we display the corresponding bifurcation diagram. In Figure 3.1c, we give phase diagrams of the system (2.1) for different values of b. We obtain the critical value of b as 0.765879 for which we obtain a limit cycle (Figure 3.1c,c). The unique positive equilibrium point of the system is locally asymptotically stable for bigger values of b (Figure 3.1c,a,b) and unstable otherwise (Figure 3.1c,d).

Now, we give numerical simulations of the fractional delayed differential system (3.1). We again pick parameter values as $a = 15$, $b = 1$, $\sigma = 6$. For $\alpha = 0.80$, the critical bifurcation value is calculated as $\tau_0 = 0.1032287$. For $\tau < \tau_0$, the system (3.1) shows stable behaviour (Figure 3.1a,b, Figure 3.1b,a). For $\tau \approx \tau_0$, we observe a periodic solution caused by Hopf bifurcation(Figure 3.1a,a, Figure 3.1b,b). For $\tau > \tau_0$, the system continues to exhibit oscillatory behavior and the equilibrium $(3, 10)$ of the system (3.1) is unstable (Figure 3.1b,c,d).

Figure 4.1a and Figure 4.1b represent correlation between the fractional order parameter α and the critical value of time delay τ_0 for the system (3.1). In general, we can say that the smaller fractional order enlarges the regions of stability for a system without delay. This is due to the stability condition $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$ in Theorem 2.1. But, we cannot extend this statement for fractional delayed differential systems. In [11], the authors work on a fractional delayed network system and conclude that the Hopf bifurcation appearance is delayed as the order increases. So, for some values of τ, mentioned system exhibit stable behaviour for bigger fractional order and exhibit unstable behaviour for smaller fractional order. On the other hand, in [18], authors conclude that the occurrence of bifurcation can be delayed with the decrease of the fractional order for the introduced fractional delayed predator-prey system. The system (3.1) we are examining in this article comprises both of these situations. For $0.75 \leq \alpha \leq 0.9575$, the critical value of time delay parameter τ_0 decreases while α increases (Figure 4.1b). On the contrary, for $0.50 \leq \alpha \leq 0.70$, τ_0 increases while fractional order parameter α increases (Figure 4.1a). This behavioral change occurs approximately at $\alpha = 0.707$ due to the characteristic equation (3.5) and the equation (3.8). We call this situation as "conflict of memory effects" to emphasize that both fractional derivatives and time delays are using to reflect memory effects in modeling dynamical systems that also depend on past data. To better understand the destabilizing (Figure 4.1b) and stabilizing (Figure 4.1a) effect of increment of fractional order, we draw the line $\tau = 0.11$ in both figures. For exemple, for $\tau = 0.11$, we observe that the system (3.1) is unstable for $\alpha = 0.55$ and stable if $\alpha = 0.65$ (Figure 4.2).

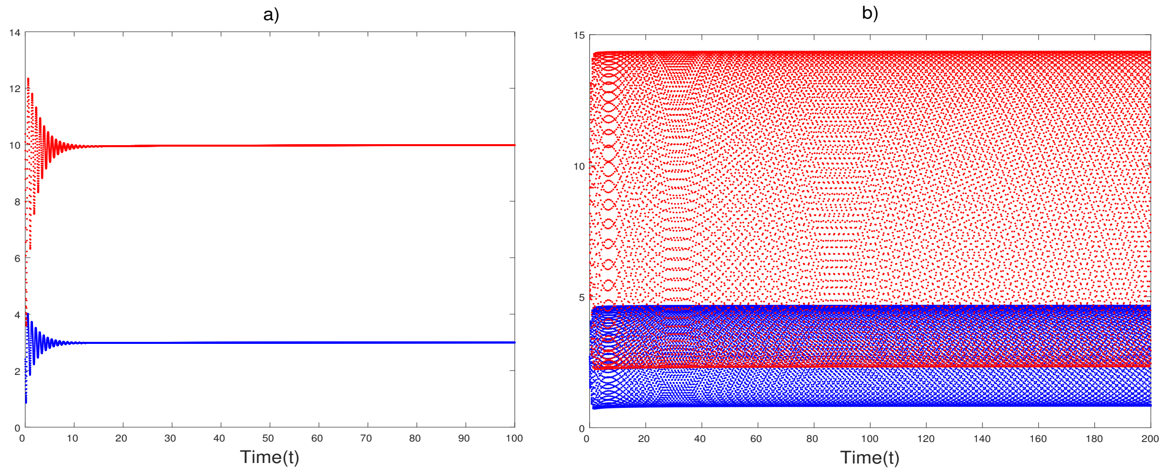


Figure 4.2: The equilibrium $(u^*, v^*) = (3, 10)$ of the system (3.1) is locally asymptotically stable for $\alpha = 0.65$ in a) and untable for $\alpha = 0.55$ in b) where $\tau = 0.11$.

5. Conclusions

This paper has analyzed local Lengley-Epstein system with fractional delayed differential equations. For the case $\tau = 0$, system (2.1) undergoes a Hopf bifurcation depending on α. The stabilizing influence of the decreament in fractional order for the system (2.1) is exhibited with the help of numerical exemples. Then, the impact of time delay parameter on the system (3.1) is investigated. The ciritical τ_0 value is determined such that the equilibrium point is locally asymptotically stable for $\tau < \tau_0$, and undergoes a Hopf bifurcation for $\tau = \tau_0$. Time delays and fractional derivatives are both used to include memory effects to the model if the current state of the system depends on past data. We conclude that for different values of time delay τ, the decreament of the fractional order α has opposite effects on the system (3.1) in terms of stability. We named this situation as "conflict of memory effects". Time delays are present in many chemical processes. By incorporating time delays into the local Lengley-Epstein system with the fractional order derivative, we observed the presence of oscillatory behavior, which is also encountered in chemical models, depending on the critical parameters.

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