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First Passage Time Model Based on Lévy Process for Contingent Convertible Bond Pricing

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Abstract: This paper develops a general Lévy framework to reduce the pricing problem of contingent convertible (CoCos) bonds to the problem of the first pass time of the triggering process. We consider two Lévy models driven by the derived Brownian motion and the spectrally negative Lévy process. These two Lévy models keep the form of the Lévy process unchanged under the measure transform, which avoids the difficulty that only rare forms of Lévy processes solved the first passage time problem. We use single and double Laplace transform in combination with numerical Fourier inversion to find closed form expressions for the price of CoCos bonds. The results show that the model driven by the spectrally negative Lévy process would provide a more accurate CoCos bonds price when taking into account the phenomenon of jumps in the financial market. Indeed, negative jumps play a much critical role in the pricing of CoCos bonds. This paper underlines the importance of the evaluation of the CoCos bonds by the Lévy process.

Keywords: CoCos bonds, Lévy process, Financial market, Laplace transform, Fourier inversion.

Introduction

In this paper, we review all the work carried out so far by the authors, and we introduce new questions concerning jumps. Most models are Brownian driven, we will focus on jump driven models like Madan and Schoutens (2007). In addition, we illustrate the structural hop diffusion model with a study of the capital structure of a bank that issues CoCos bonds. CoCos bonds can be converted into equity when the issuing bank encounters financial difficulties. This loss absorbing debt is invented to protect the taxpayer from bank bailouts in the event of a financial crisis. Basel III encourages banks to issue this financial instrument, and many Asian and European banks have issued it for regulatory purposes. This complex product has a variety of designs, focusing on trigger events and conversion mechanisms. The price of CoCos bonds follows the Brownian motion process and the Lévy process. We consider the jump risks in order to compare these two models, including the Black-Scholes model (derived Brownian motion) and the exponential jump diffusion model (spectrally negative Lévy process).

In this sense, a Lévy model was examined to intuitively show the hybrid nature of CoCos bonds and to reduce their pricing problem to the problem of the first time of the triggering process. We return to Merton (1974), Black and Scholes (1973) and Black and Cox (1976).The main pricing models offered by CoCos bonds include the intensity model and the first pass model. The intensity model is also called a credit derivatives model in some papers and the first transit time model can be considered to be a structural model and an equity derivatives model because they all use a barrier approach based on the first passing time distribution. The intensity model has been studied by De Spiegeleer and Schoutens (2012) and Cheridito and Xu (2015). However, most studies focus on the first pass time model with a triggering process, which can be either an accounting ratio or a market price.

Several papers use structural models to derive the price of CoCos bonds, but their focus is more on the role and behaviors of CoCos bonds. Pennacchi (2010) investigates the influence of contractual terms and different sources of risk of the issuing firm on the value of CoCos bonds by simulation in a structural jump diffusion model. Koziol and Lawrenz (2012) show that CoCos bonds reduce incentives for risk taking and possibly create negative externalities under certain conditions. Berg and Kaserer (2015) show that CoCos bonds would agitate shareholders' risk-taking incentives and the problem of over-indebtedness. Hilscher and Raviv (2014) point out that banks issuing CoCos bonds would have a lower probability of default and that proper design of CoCos bonds could completely eliminate shareholder risk-taking incentives. Himmelberg et al. (2014) show that CoCos bonds can, if well designed, induce banks to seek conservative capital structures to avoid the risk of dilution resulting from forced conversion and reduce the problem of over-indebtedness. Albul et al. (2010) use a structural model to determine the role of CoCos bonds in the optimal capital structure of the issuing company in a context of infinite maturity and analyze the behaviors of CoCos bonds in different scenes. Metzler and Reesor (2015) indicate that the terms of the conversion fundamentally change the nature of CoCos bonds thanks to the Merton-type structural model.

Some more theoretical work studies equity derivatives models and Wilkens and Bethke (2014) shows that this model is the most practical for pricing and risk management of CoCos bonds by comparing the credit derivatives model, the structural model and the equity derivatives model through empirical studies based on a broadly adapted market price.De Spiegeleer and Schoutens (2012) approximate the event of non-compliance with the accounting trigger level by an event where the price of observable securities is below an implicit obstacle and model the stock market process of the issuing company as a geometric Brownian movement and derive a closed-form expression for the price of CoCos bonds. Cheridito and Xu (2015) develop a general continuous model and obtain the price of CoCos bonds by solving a parabolic partial differential equation with Dirichlet boundary conditions. These equity derivative models assume that the triggering process evolves continuously, thereby neglecting sudden movements in which most of the risk is concentrated.

Discontinuous models describing the risk of jumping due to exogenous shock are worth studying, and De Spiegeleer and Schoutens (2012) also suggest that their Black-Scholes model be extended to the Lévy model to incorporate jumps and heavy tails for CoCos bond prices. Corcuera et al. (2013) propose the family-driven model of the Lévy process and exploit the Weiner-Hopf Monte Carlo method to price CoCos bonds. Corcuera and Valdivia (2016) propose a one-sided treatment of CGMY Lévy-like stock price dynamics and give the CoCos bond price by combining the closed-form expression of the CoCos bond price at the Laplace transform with a numerical inversion of Fourier.

This paper references this work and proposes two first passage time models guided by the derivative Brownian motion and the spectrally negative Lévy process for the pricing of CoCos bonds. Here, the equity derivative models are called the first-passage-time models because their results are easily extended to structural models to study the role and behaviors of CoCos bonds.Since the pricing of CoCos bonds is our primary concern and the equity derivatives model has proven to be more practical, only equity derivative models focusing on Brownian motion, the spectrally negative Lévy process will be discussed. The derived Brownian motion provides closed form expressions while the spectrally negative scattering process possesses them up to the Laplace transform whose results are given by combining with numerical Fourier inversion. Although the model based on the derived Brownian motion was given by De Spiegeleer and Schoutens (2012), the result given in the general framework of Lévy can provide different indications, and it would be more practical to compare the model of Black and Scholes with the spectrally negative Lévy process in the general Lévy framework. We assume in our paper that the defect occurs when this process reaches a given level. The calculation of the probability of default in finite time is thus done from the first passage time, in an equivalent way, from the minimum of the Lévy process.

To achieve this, this paper presents an extended version of the framework proposed by Rogers (2000), which allowed the calculation of the probability of the first passage time of the spectrally negative Lévy process. Indeed, we develop the formulas of the calculation of the probability of default in order to show in particular how these formulas are obtained via the simple or double Laplace transform and the numerical inversion of Fourier to determine the pricing of CoCos bonds.

This paper is organized as follows. The characteristics of the contract are specified in section 2. In addition, a general Lévy framework of the first transit time model for CoCo bond pricing is presented to establish the general pricing problem. Section 3 presents two advanced models. On the one hand, a Brownian motion is built into the stock price dynamics, and the corresponding prices are subsequently obtained by a barrier option approach. On the other hand, a spectrally negative Lévy model, and the obtaining of the corresponding prices are processed by a Fourier method exploiting the Wiener-Hopf factorization for the Lévy processes. Finally, section 4 will discuss the numerical results.

General Lévy Framework for CoCos Bonds Pricing

A CoCo bond is a bond issued by a financial institution where an automatic conversion into a predetermined number of shares takes place upon the occurrence of a trigger event, linked to a distress of the institution. By considering only the structure of CoCos bonds, a general expression of the price of CoCos bonds can be derived in this chapter. Instead of thinking of CoCos bonds as fixed income securities, we can think of them from the equity side.

This chapter will describe and analyze the approach to equity derivatives, introduced by De Spiegeleer and Schoutens (2012). We will follow closely the notations of De Spiegeleer and Schoutens (2012) to present closed-form solutions to equity derivatives for the pricing of CoCos bonds. The intuition behind the equity derivatives approach is that current and existing equity derivatives are added together to replicate the payout structure of a CoCo bond. To value the CoCo bond using equity derivatives, the CoCo bond is divided into different parts, which will add up to the total value of the CoCo bond.

Defining a CoCo bond requires specifying its face value (C) and maturity (T), as well as the random time (τ) at which the conversion will take place. Assuming that (m) coupons are attached to the CoCo bond. The coupon structure of the CoCo bond$(c_i,\ t_i)_{i=1}^{m}$ is defined so that the amount c_i is paid at time $t_i$provided that $\tau > t_i$. In our case, we will assume that the issuer of the CoCo bond pays dividends according to a deterministic function (κ) and that, on the side of the investors, no dividend is paid after the conversion time τ. We therefore assume below that there is no dividend after the conversion time τ and receiving βS_τ at τ is equivalent to receiving βS_T at T. Let r be the risk-free interest rate. Recall that in the approach to equity derivatives, the price of CoCos bonds is broken down into three parts, which all boil down to the total value of the instrument. As a result, the price of CoCos bonds under arbitrage can be broken down into three parts:

The value of the principal payment at maturity

$$V_p = \mathbb{E}^Q\left[Ce^{-rT}\mathbb{1}_{\{\tau>T\}}\right] \quad (1)$$

The value of coupon payments

$$V_c = \sum_{i=1}^{m} \mathbb{E}^Q[c_i e^{-rt_i}\mathbb{1}_{\{\tau>t_i\}}] \quad (2)$$

The value of converting

$$V_e = \mathbb{E}^Q\left[\beta e^{-rT} S_T \mathbb{1}_{\{\tau\leq T\}}\right] \quad (3)$$

Hence the value of a CoCo bond is expressed as follows:

$$\begin{aligned} V = V_p + V_c + V_e &= \mathbb{E}^Q\left[Ce^{-rT}\mathbb{1}_{\{\tau>T\}}\right] + \sum_{i=1}^{m} \mathbb{E}^Q[c_i e^{-rt_i}\mathbb{1}_{\{\tau>t_i\}}] + \mathbb{E}^Q\left[\beta e^{-rT} S_T \mathbb{1}_{\{\tau\leq T\}}\right] \\ &= Ce^{-rT}Q(\tau > T) + \sum_{i=1}^{m} c_i e^{-rt_i}\, Q(\tau > t_i) + \beta e^{-rT}\mathbb{E}^Q\left[S_T\mathbb{1}_{\{\tau\leq T\}}\right] \quad (4) \end{aligned}$$

Let $\Psi(x,t,\mu,\sigma,\upsilon) = Q(min_{0\leq s\leq t} X(s) \leq x)$, then the value of a CoCo bond can be written in the following form

$$\begin{aligned} V = Ce^{-rT}\left(1 - \Psi\left(log\, {}^{k}/_{S_0}, T, \mu, \sigma, \upsilon\right)\right) + \sum_{i=1}^{m} c_i e^{-rt_i}\left(1 - \Psi\left(log\, {}^{k}/_{S_0}, t_i, \mu, \sigma, \upsilon\right)\right) \\ + \beta e^{-rT}\mathbb{E}^Q\left[S_T\mathbb{1}_{\{\tau\leq T\}}\right] \quad (5) \end{aligned}$$

It suffices to calculate the probability of conversion for the valuation of the CoCo bond. In fact, the key step in calculating the conversion value V_e is the determination of the joint modeling of the conversion time τ and the stock price S_τ. In the CoCos bond pricing framework, we show how to derive the probability of conversion. In effect, we are applying the measurement change that simplifies the pricing formula for conversion value by eliminating the stock price dependence. First, we fix a probability space $(\Omega, (F_t)_{t\geq 0}, Q)$ in which Q is a neutral risk measure. All underlying processes of our model are assumed to be observable and suitable for $(F_t)_{t\geq 0}$ filtration. We consider a financial institution issuing a contingent convertible bond to avoid financial deterioration or bankruptcy when a trigger event occurs. In this section, we will consider an exponential Lévy model for stock price. We assume that this triggering process is a stock price process that follows an exponential Lévy process under the risk-neutral probability measure Q;

$$S_t = S_0 e^{X_t}, \qquad t \geq 0 \quad (6)$$

where S_0 is the initial value of the stock price and X_t is a general Lévy process that starts at zero and has independent and stationary increments. For more information on Lévy processes, see the book by Bertoin (1996). By defining a price action barrier, k, such that $k < S_0$. When the trigger processS_t crosses the barrier k, the fault occurs, and the conversion will take place. Therefore, the trigger event occurs at the first passage time if$X_t$exceeds the boundary condition $log\left({}^{k}/_{S_0}\right)$ and is given by

$$\tau = inf\{t \geq 0, S_t \leq k\} = inf\left\{t \geq 0, X_t \leq log\left({}^{k}/_{S_0}\right)\right\} \quad (7)$$

From the Lévy-Khintchine formula, we see that in general a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part and a pure jump part. According to the Lévy-Itô decomposition, X_t can be written in the following form:

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{|x|\geq 1} x J_x(ds, dx) + \int_0^t \int_{|x|<1} x(J_x(ds, dx) - \upsilon(dx)ds) \quad (8)$$

with

$$\mu = r - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}} (e^x - 1 - xI_{\{|x|\leq 1\}})\upsilon(dx)$$

The first part designates the diffusive part with W_t being a standard Brownian motion ($W_0 = 0$) and the constants μ and $\sigma > 0$ constitute the drift and the volatility of the diffusive part of the price dynamics. The second part identifies the small jumps which describe the daily jitters caused by minor stock price fluctuations, while the third represents the large jumps which describe the large stock price fluctuations caused by major market disturbances. $J_x(ds, dx)$ is a random Poisson measure on $[0, \infty[$ and $(J_x(ds, dx) - \upsilon(dx)ds)$ represents its compensated Poisson measure. The measure υ, called Lévy measure, is a positive measure on $\mathbb{R}\backslash\{0\}$ which determines the progress of the jumps and verifies the following condition

$$\int_{-\infty}^{+\infty} inf\{1, x^2\}\upsilon(dx) = \int_{-\infty}^{+\infty} (1 \wedge x^2)\upsilon(dx) < \infty$$

Processes W_t and $J_x$are assumed to be independent.
The Lévy process satisfies $\int_{|x|<1} |x|\upsilon(dx) < \infty$ and $\int_0^t \int_{|x|<1} xJ_x(ds, dx) < \infty$.

If we set $d = \mu - \int_{|x|<1} x\upsilon(dx)$

Then the Lévy process can be written as follows:

$$X_t = dt + \sigma W_t + \int_0^t \int_{\mathbb{R}} xJ_x(ds, dx) \quad (9)$$

We define a stock price adjustment process $\widetilde{S_T} = S_T e^{-rT}$. Girsanov's theorem allows us to show that there is a probability Q^* equivalent to Q under which the discounted price $\widetilde{S_T}$ is a martingale. Indeed, if the market is free of arbitrage, there is a probability measure Q^* equivalent to Q under which the updated asset $\widetilde{S_T}$ is a martingale. AOA (Absence of Arbitrage Opportunity) implies that the present value of any asset is a martingale under the measure, which is called a martingale measure. Thus, any AOA pricing rule is given by an equivalent martingale measure. Therefore, since $S_T e^{-rT}$ is a martingale under the neutral risk probability measure Q, we can take $S_T e^{-rT}$ to construct a change of measure. We assume a new probability measure Q^* such that:

$$Z_t = \frac{dQ^*}{dQ}\bigg|_{F_t} = \frac{\widetilde{S_T}}{\widetilde{S_0}} = \frac{S_T e^{-rT}}{S_0} = \frac{S_0 e^{X_t} e^{-rT}}{S_0} = e^{X_t - rT} \quad (10)$$

Under the stock price measure Q^* and assuming that the dynamics of the process of X_t is specified by (2), X_t is a new Lévy process under the probability measure Q^*. Indeed, we refer to proposition 9.8 in Tankov (2003), the elements of the new Lévy triplet are given by:

$$\mu^* = \mu + \sigma^2 + \int_{-1}^{1} x(e^x - 1)\upsilon(dx)$$

$$\sigma^* = \sigma$$

$$\upsilon^*(dx) = e^x \upsilon(dx)$$

If $\int_{|x|<1} |x|\upsilon^*(dx) < \infty$, we note that $d^* = \mu^* - \int_{|x|<1} x\upsilon^*(dx)$ and the relation between μ^* and μ can be simplified by

$$d^* = d + \sigma^2 = \mu + \sigma^2 + \int_{-1}^{1} x(e^x - 1)\upsilon(dx) - \int_{-1}^{1} xe^x\,\upsilon(dx)$$

So, $X_t$under Q^*can be expressed in the following form:

$$X_t = \mu^* t + \sigma^* W_t^* + \int_0^t \int_{|x|\geq 1} xJ_x{}^*(ds, dx) + \int_0^t \int_{|x|<1} x(J_x{}^*(ds, dx) - \upsilon^*(dx)ds) \quad (11)$$

Therefore, the value of the conversion is given by:

$$\beta e^{-rT}\mathbb{E}^Q\left[S_T \mathbb{1}_{\{\tau\leq T\}}\right] = \beta e^{-rT}\mathbb{E}^{Q^*}\left[S_T \mathbb{1}_{\{\tau\leq T\}}/Z_t\right] = \beta e^{-rT}\mathbb{E}^{Q^*}\left[S_0 e^{rT} \mathbb{1}_{\{\tau\leq T\}}\right] = \beta S_0 Q^*(\tau \leq T)$$

Finally, we get the following formula for the value of a CoCo bond

$$V = Ce^{-rT}\left(1 - \Psi\left(log\, {}^{k}/_{S_0}, T, \mu, \sigma, \upsilon\right)\right) + \sum_{i=1}^{m} c_i e^{-rt_i}\left(1 - \Psi\left(log\, {}^{k}/_{S_0}, t_i, \mu, \sigma, \upsilon\right)\right) + \beta S_0 \Psi\left(log\, {}^{k}/_{S_0}, T, \mu^*, \sigma^*, \upsilon^*\right) \quad (12)$$

This equation shows that the CoCos bond pricing problem is a problem of the first time of the passage of two Lévy processes. It reflects the hybrid nature of the CoCo bond. Indeed, the closed-form expression of the price of CoCos bonds consists that the Lévy processes (8) and (11) have closed-form expressions for their distributions of the first time of the passage.

Valuation of a Contingent Convertible Bond according to the First Passage Time and the Probability of Default

Brownian Motion

In our context, we are interested in the value of the probability of default where default occurs at maturity for the first time. Merton's original model (1974) does not allow for premature default, in the sense that default can only occur at the maturity of the claim. We will therefore examine the version of the first passage time models following Black and Cox (1976). They represent an important extension of Merton (1974) in many respects.

First, they form security covenants that allow creditors to take over the borrowing company when its value is below a certain threshold. The stock is no longer a European call option on the borrower's assets. Rather, the stock is a "down-and-out" call option on the company's assets, implying that the presence of security covenants transfers the value of the stock to creditors and allows the issuance of debt with higher circulation. Also, they consider senior and subordinate debts. Then, they develop an approach to value risky bonds paying with the limit of default, with and without asset sale restrictions, and demonstrate that security covenants and asset sale restrictions can improve creditor rights and increase debt values. Indeed, instead of only admitting the possibility of default at maturity (T), Black and Cox (1976) postulated that default occurs at the first time the asset value of the business drops below a certain barrier. In most of these models, the time to default is given as the first time the process passes from the value of assets S_t to a deterministic or random barrier. Default can therefore occur at any time before or on the maturity date of the CoCo bond (T). However, the fault occurs the first time $(S_t)_{t\geq 0}$ touches barrier k such that $\tau = inf\{t \geq 0, S_t \leq k\}$. Following Black and Cox (1976), let $m_t = min_{0<t\leq T} S_t$ be the first time the asset value process crosses the bankruptcy barrier and let $\upsilon = 0$ lead at $\upsilon^* = 0$ then $X_t = \mu t + \sigma W_t$is a Brownian motion. Let $f(y)$ be the probability density of S_t such that:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-(y-\mu t)^2/2\sigma^2 t} \qquad (13)$$

and $g(y, x)$ is the joint probability density with $x = log\left({}^{k}/{}_{S_0}\right)$ such that:

$$g(y, x) = \frac{1}{\sigma\sqrt{2\pi t}} e^{2\mu x/\sigma^2} e^{-(y-2x-\mu t)^2/2\sigma^2 t} \qquad (14)$$

The probability of default is given by:

$$\mathbb{P}(\tau_x \leq t) = \mathbb{P}\left(\min_{0<t\leq T} S_t \leq x\right) = \mathbb{P}(S_t \leq x) + \mathbb{P}\left(\min_{0<t\leq T} S_t \leq x, S_t > x\right) = \int_{-\infty}^{x} f(y)dy + \int_{x}^{+\infty} g(y, x)dy$$

$$= N\left(\frac{log\left({}^{k}/{}_{S_0}\right) - \mu t}{\sigma\sqrt{t}}\right) + \left(\frac{k}{S_0}\right)^{2\mu/\sigma^2} N\left(\frac{log\left({}^{k}/{}_{S_0}\right) + \mu t}{\sigma\sqrt{t}}\right) (15)$$

Where $N()$ is the cumulative function of the normal distribution. Then, the first passage time model driven by the derived Brownian motion yields the following CoCos bond price:

$$V = Ce^{-rT}\left(1 - N\left(\frac{log\left({}^{k}/{}_{S_0}\right) - \mu T}{\sigma\sqrt{T}}\right) + \left(\frac{k}{S_0}\right)^{2\mu/\sigma^2} N\left(\frac{log\left({}^{k}/{}_{S_0}\right) + \mu T}{\sigma\sqrt{T}}\right)\right)$$

$$+ \sum_{i=1}^{m} c_i e^{-rt_i}\left(1 - N\left(\frac{log\left({}^{k}/{}_{S_0}\right) - \mu t_i}{\sigma\sqrt{t_i}}\right) + \left(\frac{k}{S_0}\right)^{2\mu/\sigma^2} N\left(\frac{log\left({}^{k}/{}_{S_0}\right) + \mu t_i}{\sigma\sqrt{t_i}}\right)\right)$$

$$+ \beta S_0\left(N\left(\frac{log\left({}^{k}/{}_{S_0}\right) - \mu^* T}{\sigma^*\sqrt{T}}\right) + \left(\frac{k}{S_0}\right)^{2\mu^*/\sigma^{*2}} N\left(\frac{log\left({}^{k}/{}_{S_0}\right) + \mu^* T}{\sigma^*\sqrt{T}}\right)\right) (16)$$

This equation represents the closed-form expression of the first passage time of the price of CoCos bonds with the model of Black and Scholes (1973).

Spectrally Negative Lévy Process

General Framework

The mathematical notion on which the modeling of asset prices is based is the notion of stochastic process, modeling price trajectories amounts to specifying a family of stochastic processes. The common point shared by a large part of the stochastic processes used in finance is to belong to the sub-family of diffusion processes, which is based on Brownian motion. The best known of these models is the Black and Scholes (1973) model, which models price trajectories as an exponential of Brownian motion. However, the observation of prices reveals the presence of visible jumps, which the Black and Scholes model (1973) does not allow to reproduce. These phenomena can be of great importance for risk management because they correspond to periods of crisis.

The inadequacy of the Black and Scholes (1973) model to the reality of the markets can also be seen graphically when, instead of considering price trajectories, we compare "returns", i.e. differentials to time course of the logarithm of prices. The returns of most financial assets take on much more dispersed values than those of the Black and Scholes (1973) model, with frequent peaks corresponding to "jumps" in prices. We are interested here in the models of the first transit time based on the Lévy process. Indeed, several models incorporating jumps in the dynamics of firm value are described in the literature. Hilberink and Rogers (2002) opt for an extension of

Leland (1994), using Lévy processes which only allow downward jumps in the value of the firm. Kou and Wang (2003) showed how to use fluctuation identities from Lévy process theory to path-dependent options on assets driven by jump diffusions with exponentially distributed Poisson jumps. Madan and Schoutens (2007) work with downward jumping strategies, and thus allow situations where the default barrier is not only hit, but crossed by a jump. They detail the theory of spectrally negative Lévy processes in general and detail some popular examples. We return to the dynamics of the stock price process $(S_t)_{t\geq 0}$which is described by the exponential of a (non-Brownian) Lévy process. Indeed, we will consider a numerical approach to the pricing of CoCos bonds based on the explanation of the de Wiener-Hopf factorization of the Lévy process $(X_t)_{t\geq 0}$. This approach has been applied recently to set the contract price as in Corcuera et al (2013) and Madan and Schoutens (2007).

Let $\sigma, \upsilon \neq 0$ and $\upsilon([0,\infty[) = 0$ then $(X_t)_{t\geq 0}$ becomes a spectrally negative Lévy process. Since these conditions lead to $\sigma^*, \upsilon^* \neq 0$ and $\upsilon^*([0,\infty[) = 0$, (11) is always a spectrally negative Lévy process.

The closed-form expression of the Laplace transform of the distribution of the first passage time of the negative Lévy process on the spectral plane was given by Rogers (2000) thanks to the Wiener-Hopf factorization. Rogers (2000) determines the time distribution of the first pass by approximating the standard inverse Fourier transform, which exploits the change in the appropriate integration limit to avoid the difficulty of directly solving the Laplace exponent equation.

Let $= \inf\{t \geq 0, X_t \leq x\}$, with $x = log\left({}^{k}/_{S_0}\right)$.

Since the default is triggered by the crossing of a low barrier, or by the point where the minimum current will cross this level. The distribution of the current maximum and minimum of the Lévy process $(X_t)_{t\geq 0}$ will be as follows:

$$\overline{X_t} = sup_{s\leq t}\, X_s$$
$$\underline{X_t} = inf_{s\leq t}\, X_s$$

We assume that e_λ is an exponential distribution with parameter λ, independent of $(X_t)_{t\geq 0}$. Then the default probability of a Lévy process $(X_t)_{t\geq 0}$ is factored into a Laplace transform of the current minimum and maximum taken at exponential time.

However, we can write $\mathbb{E}(e^{zX_t}) = e^{t\varphi_x(z)}$.
Where $\varphi_x(z)$ is the Laplace exponent of the spectrally negative Lévy process (Bertoin, 1996) and is represented by:

$$\varphi_x(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{0} [e^{zx} - 1 - zx\mathbb{1}_{\{|x|\leq 1\}}]\upsilon(dx) \qquad (17)$$

Therefore, the Laplace transform of the process $(X_t)_{t\geq 0}$ taken at exponential time is given by:

$$\mathbb{E}\left(e^{zX_{e_\lambda}}\right) = \frac{\lambda}{\lambda - \varphi_x(z)} \qquad (18)$$

According to this equation and that the Wiener-Hopf factorization (Rogozin, 1966) is valid for general Lévy processes, we then have:

$$\frac{\lambda}{\lambda - \varphi_x(z)} = \mathbb{E}\left(e^{z\overline{X_{e_\lambda}}}\right)\mathbb{E}\left(e^{z\underline{X_{e_\lambda}}}\right) = \varphi_\lambda^+(z)\varphi_\lambda^-(z) \qquad (19)$$

Based on the knowledge of one of the two factors of this equation, we can obviously establish the other. Moreover, the classical theory of the Lévy process (Bertoin 1996, Sato 1999 or Kyprianou 2006) shows that, for a spectrally negative process, the right-hand Wiener-Hopf factor is expressed by:

$$\varphi_\lambda^+(z) = \frac{\chi^*}{\chi^* - z} \qquad (20)$$

with χ^* is a constant independent of λ and χ^* is the solution of $\varphi_x(\chi) = \lambda$.

So, once the current maximum taken at exponential time is exponentially distributed with the parameter $\chi^* = \varphi_x^{-1}(\lambda)$, we have calculated χ^* explicitly and we get the following expression for the left Wiener-Hopf factor:

$$\varphi_{\lambda}^{-}(z) = \mathbb{E}\left(e^{z\underline{X_{e_\lambda}}}\right) = \frac{\lambda}{\lambda - \varphi_x(z)}\frac{\chi^* - z}{\chi^*} (21)$$

This equation can be related to the distribution function of ($\underline{X_t}$) (current minimum) following a partial integration. Indeed, we have:

$$\varphi_{\lambda}^{-}(z) = \int_{0}^{+\infty} \lambda e^{-\lambda t} dt \int_{-\infty}^{0} e^{zx} f_{\underline{X_t}}(x)dx = \int_{0}^{\infty} \lambda e^{-\lambda t} dt \int_{-\infty}^{0} z e^{zx} \mathbb{P}(\underline{X_t} > x)dx$$
$$= \lambda z \int_{t=0}^{t=+\infty} \int_{x=0}^{x=+\infty} e^{-\lambda t - zx} f(t,x)dtdx = \lambda z \tilde{f}(\lambda, z) \qquad (22)$$

However, we have defined the default time by $\tau_x = inf\{t \geq 0, X_t \leq x\}$ and according to Rogers (2000),

$$f(t,x) = \mathbb{P}(\tau_{-x} > t) = \mathbb{P}(\underline{X_t} > -x)$$

This equation represents the probability that the current minimum remains above $(-x)$ in t time units. Note that $\tilde{f}(\lambda, z)$ is the Laplace transform of $f(t,x)$. Indeed, the double Laplace transform of $f(t,x)$ is determined as follows:

$$\tilde{f}(\lambda, z) = \int_{t=0}^{t=+\infty} \int_{x=0}^{x=+\infty} e^{-\lambda t - zx} f(t,x)dtdx = \int_{t=0}^{t=+\infty} \int_{x=-\infty}^{x=0} e^{-\lambda t + zx}\, \mathbb{P}\left(\underline{X_t} > x\right) dtdx$$
$$= \frac{\chi^*(\lambda) - z}{(\lambda - \varphi_x(z))\chi^*(\lambda) z} \qquad (23)$$

As Madan and Schoutens (2007) show, it is possible from this equation to show that $f(t,x)$, the probability that the minimum stays above $(-x)$ in t time units, can be obtained by the double inverse Fourier transform. The Fourier transform pricing method is a widely used method for valuing options in financial models when the risk-neutral density of the underlying asset is not given in an analytically tractable form, however the characteristic function, which describes the probabilistic behavior of the underlying, can be easily assessed. So far, there is a wide variety of Fourier-based pricing algorithms, but we limit the discussion to one of the most common versions.

A method based on Monte Carlo simulation is inefficient, due to slow convergence due to the large amplitude of the jumps, and the inherent difficulties in identifying the optimal exercise policy. Knowing the characteristic function of the Lévy process paves the way for a Fourier approach to evaluating options on the spot. The algorithm for evaluating the first passage time distribution of the spectrally negative Lévy process can be summarized as follows. We set $\lambda_1, \lambda_2, z_1$and z_2 which are real numbers with $\lambda_1, z_1 > 0$such that $\lambda = \lambda_1 - i\lambda_2$and $z = z_1 - iz_2$.

Then, we can write $\tilde{f}(\lambda, z)$ in the following form:

$$\tilde{f}(\lambda_1 - i\lambda_2, z_1 - iz_2) = \int_{t=0}^{x=+\infty} \int_{x=0}^{x=+\infty} e^{-(\lambda_1 - i\lambda_2)t - (z_1 - iz_2)x} f(t,x)dtdx$$
$$= \int_{t=0}^{t=+\infty} \int_{x=0}^{x=+\infty} e^{i\lambda_2 t + iz_2 x} e^{-\lambda_1 t - z_1 x} f(t,x)dtdx (24)$$

This new function represents the double Fourier transform of $e^{-\lambda_1 t - z_1 x} f(t,x)$. As a result, following the inverse Fourier transform, we then have:

$$e^{-\lambda_1 t - z_1 x} f(t,x) = \frac{1}{(2\pi)^2} \int_{\lambda_2=-\infty}^{\lambda_2=+\infty} \int_{z_2=-\infty}^{z_2=+\infty} e^{-(i\lambda_2 t + iz_2 x)}\, \tilde{f}(\lambda_1 - i\lambda_2, z_1 - iz_2)\, dz_2 d\lambda_2 (25)$$

or,

$$f(t,x) = \frac{1}{(2\pi)^2} \int_{\lambda_2=-\infty}^{\lambda_2=+\infty} \int_{z_2=-\infty}^{z_2=+\infty} e^{(\lambda_1 - i\lambda_2)t + (z_1 - iz_2)x}\, \tilde{f}(\lambda_1 - i\lambda_2, z_1 - iz_2)\, dz_2 d\lambda_2$$

Therefore,

$$f(t,x) = -\frac{1}{(2\pi)^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\lambda t+\tau x}\frac{\chi^*(\lambda) - z}{(\lambda - \varphi_x(z))\chi^*(\lambda)z}d\lambda dz (26)$$

Where the limits Γ_1 and Γ_2 are defined as follows:

$$\Gamma_1 = \{\lambda_1 + i\lambda_2 \backslash \lambda_2 = -\infty \dots \infty\}$$
$$\Gamma_2 = \{z_1 + iz_2 \backslash z_2 = -\infty \dots \infty\}$$

This problem is solved by performing a boundary change following Rogers (2000) and using the Abate and Whit (1992) approximation.

Lemma 1.
For fixed t and x;

$$S_N(t,x) = \frac{h_1 h_2}{(2\pi)^2}\sum_{n=-N}^{N}\sum_{m=-N}^{N} h'(a_1 + inh_1)\tilde{f}(h(a_1 + inh_1), a_2 + imh_2)e^{h(a_1+inh_1)t+(a_2+imh_2)x}$$

with i being the imaginary unit, $h = \varphi \circ \varphi_0^{-1}$, h' is its derivative and

$$\tilde{f}(h(\varsigma), z) = \frac{\varphi_0^{-1}(\varsigma) - z}{(h(\varsigma) - \varphi_x(z))\varphi_0^{-1}(\varsigma)z},$$

Where $\varphi_0(z) = \mu z + (1/2)\sigma^2 z^2$ and $\varphi_0^{-1}(z) = (\sqrt{\mu^2 + 2\sigma^2 z} - \mu)/\sigma^2$.

Given the set of parameters (A_1, A_2, l_1, l_2, N) and following Madan and Schoutens (2007), it is suggested to take

$a_1 = {}^{A_1}/_{2tl_1},\ a_2 = {}^{A_2}/_{2xl_2}$
$h_1 = {}^{\pi}/_{tl_1},\ h_2 = {}^{\pi}/_{xl_2}$
$A_1 = A_2 = 22$
$l_1 = l_2 = 1$

and Rogers (2000) suggests that choosing $N = 6$ gives satisfactory results. The double sum in the previous equation is used as an initial approximation of $f(t,x)$ and the parameters (A_1, A_2, l_1, l_2) are positive real numbers chosen large enough to control the error of spectrum aliasing. Finally, it is incited to take an Euler sum

$$f(t,x) \doteq \sum_{k=0}^{M} 2^{-M}\binom{M}{k} S_{N+k}(t,x)$$

with $M = 9$ and the symbol $\doteq$ indicates an Euler sum. This final Euler sum is used to improve the accuracy of the raw approximation $S_N(t,x)$. Given the expression for the first passage time distribution of the spectrally negative Lévy process, the expression for the CoCos bond pricing valuation can be expressed as:

$$V = Ce^{-rT}\left(1 - \Psi\left(log\,{}^{k}/_{S_0}, T, \mu, \sigma, \upsilon\right)\right) + \sum_{i=1}^{m} c_i e^{-rt_i}\left(1 - \Psi\left(log\,{}^{k}/_{S_0}, t_i, \mu, \sigma, \upsilon\right)\right) + \beta S_0 \Psi\left(log\,{}^{k}/_{S_0}, T, \mu^*, \sigma^*, \upsilon^*\right) \ (27)$$

with $\Psi(x,t,\mu,\sigma,\upsilon) = \mathbb{P}(\tau_x \le t) = 1 - f(t,-x)$.

The spectrally negative Lévy process is a large family of Lévy processes including various forms of jumping. To price of CoCos bonds in this case, the jump form of the spectrally negative Lévy process need not remain unchanged under the measure transform. The one-sided CGMY Lévy model proposed by Corcuera and Valdivia (2016) is a practical and specific case of this work. Since downside risk should always be considered in the financial market, the spectrally negative Lévy process has been widely applied in finance. From there, the expression for the CoCos bond pricing valuation enriches the financial applications of the spectrally negative Lévy process. A simple example is given below.

Exponential Jump Diffusion Process

In this example, the jumps of the spectrally negative Lévy process have an exponential distribution with the parameter η, arriving at the rate λ. The jump shape does not change under the measuretransform. The density function of the exponential distribution is given by:

$$f_X(x) = \eta e^{\eta x} I_{\{x<0\}} (28)$$

The Lévy measure of an exponential jump diffusion process can be expressed as

$$v(dx) = \lambda f_X(x) dx (29)$$

Returning to the general equation for the Laplace exponent above and following integration by parts, the Laplace exponent can be expressed as

$$\varphi_x(\tau) = \left(\mu + \lambda\left(\frac{1 - e^{-\eta}}{\eta} - e^{-\eta}\right)\right) z + \frac{1}{2}\sigma^2 z^2 - \frac{\lambda z}{\eta + z} \quad (30)$$

By noting $\omega = \mathbb{E}^Q(e^X) = \eta/(\eta + 1)$

After the measure transform, the jumps have an exponential distribution with a new rate $\lambda^* = \lambda \mathbb{E}^Q(e^X) = \lambda\eta/(\eta + 1)$ and the new density function under the probability measure Q^* is in the following form:

$$f_{X^*}(x) = (\eta+1) e^{(\eta+1)x} I_{\{x<0\}} \quad (31)$$

The jump distribution under the probability measure Q^* is always an exponential distribution with parameters $\eta^* = \eta + 1$, which proves that the exponential jump diffusion process under the measure transform is always an exponential jump. Then the Laplace exponent can be expressed as

$$\varphi_x(\tau) = \left(\mu + \sigma^2 + \lambda\left(\frac{1 - e^{-\eta}}{\eta} - e^{-\eta}\right)\right) z + \frac{1}{2}\sigma^2 z^2 - \frac{\lambda\eta z}{(\eta + 1)(\eta + z + 1)} \quad (32)$$

Then the price of CoCos bonds can be evaluated through the previous expression.

Numerical Example

In this section, we try to compare the two Lévy models. Indeed, we study the difference in the price of CoCos bonds between the two Lévy models and how the price of CoCos bonds changes with the parameters of the models. For a simple statement, the models driven by the derived Brownian motion and the spectrally negative Lévy process are abbreviated as BS and SN, respectively. To make the comparison results more explainable, the jumps of the spectrally negative Lévy process are chosen to arrive at a limited speed and have an exponential distribution. The parameters mainly refer to Rogers (2000) and Kou and Wang (2003). Since these Lévy models all satisfy $\int_{|x|<1} |x| v(dx) < \infty$ under the assumptions, X_t can be expressed as (9) in the Lévy models. μ is chosen to satisfy the martingale condition as

$$\mu = r - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}} (e^x - 1 - x I_{\{|x|\leq 1\}}) v(dx).$$

Like Rogers (2000) and Kou and Wang (2003), σ is equal to 0.2 for both models. The risk-free rate is assumed to be 0.05. The numerical inverse Fourier parameters for SN are chosen as follows:

$$l1 = l2 = 1, A1 = A2 = 22, N = 6 \text{ and } M = 9.$$

CoCos bonds are supported with a 10-year maturity and a fixed coupon of 0.16 paid annually. Let the principal C of the CoCos bonds be 100, the initial price $S_0 = 10$ and the implied market price barrier $k = 8.5$. Once the trigger event occurs, the CoCos bonds per share are converted into 20 shares. In this section, we analyze the effect of λ on the price of CoCos, the effect of volatility on the price of CoCos and the effect of the conversion ratio on the price of CoCos.

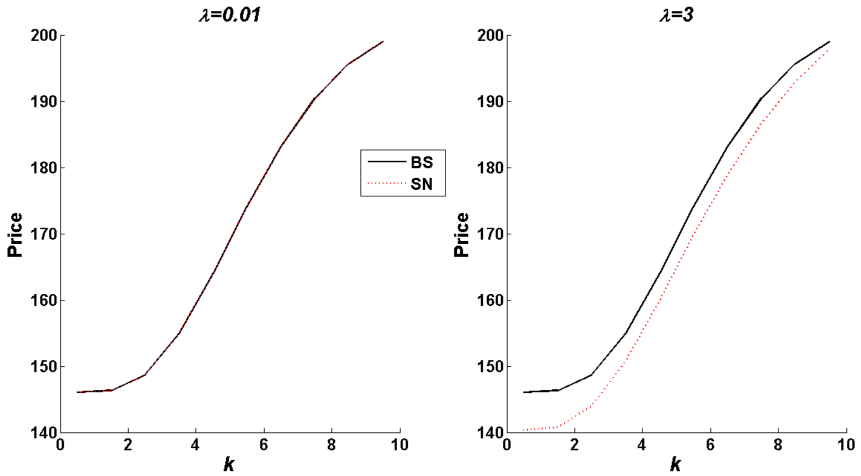

Figure 1. The price sensitivity of CoCos when (a) $\lambda = 0.01$ and (b) $\lambda = 3$.

Figure 1. Examines the price of CoCos bonds at different k values when $\lambda = 0.01$ and $\lambda = 3$. If λ is small enough, the jumps in SN are so few that the price evaluated from this term is almost identical to that of BS. The charts are not monotonous, reflecting the hybrid nature of CoCos bonds. If $\lambda = 3$, the SN curve almost overlaps. This result shows that jumps have a limited influence on the pricing of CoCos bonds and the difference in CoCos bond prices between BS and the other spectrally negative Lévy model. SN is a natural improvement over BS as they embed a jump structure in BS for the characterization of the jump phenomenon in the financial market. The difference between these two models shows the value and importance of introducing the SN model which has a lower price of CoCos bonds than BS, which shows that BS would overestimate the price of CoCos bonds to compress the information on the jumps in volatility. The result of BS overestimation goes against the intuitive understanding that BS would underestimate the price of CoCos bonds without considering jump risk. Using the Lévy SN model, we clearly observe a significant improvement over the BS model. We can conclude that the more flexible Lévy processes are more suitable than the normal distribution.

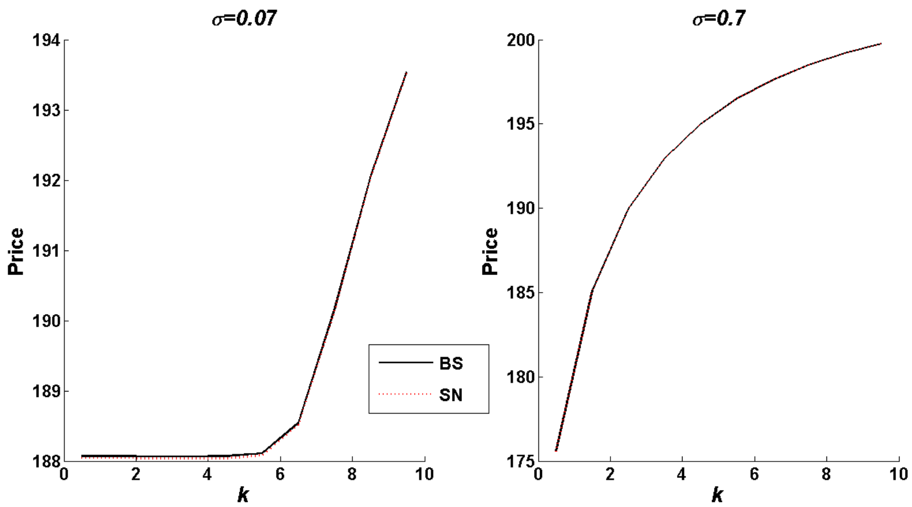

Figure 2. The price sensitivity of CoCos when (a) $\sigma = 0.07$ and (b) $\sigma = 0.7$.

Figure 2 shows that when σ is larger, we note an increase in the price of CoCos bonds in both models. Indeed, it indicates that an increase in volatility will increase the price of CoCos bonds. In addition, the increasing speed decreases for the upper bound of the trigger probability. The increase in σ plays an increasingly critical role in the pricing of CoCos, smooths the influence of jumps, and narrows the spread between CoCos bond prices among these patterns.

Figure 3 proves that the conversion ratio has a significant effect on the price of CoCos bonds for each of the two models, BS and SN. We note that if the conversion ratio is higher, the price of the CoCos bonds rises, while if the conversion ratio is lower, no progress is noted for the price of the CoCos bonds. Therefore, the conversion ratio plays an important role in the pricing of CoCos regardless of the process used.

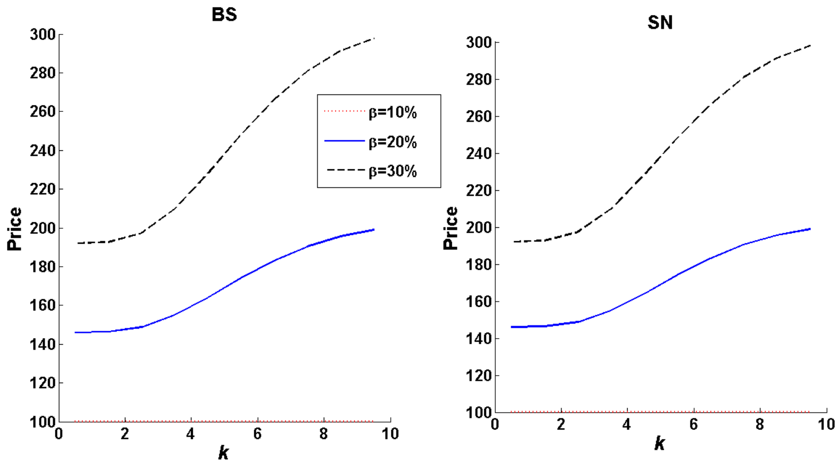

Figure 3. CoCos bond prices of the black-scholes model and the spectrally negative lévy process, respectively, for a range of the conversion ratio β.

Conclusion

Despite its success, the formula of Black and Scholes (1973) is often criticized for its inadequacy to the realities of financial markets. Several families of models have thus been proposed to remedy its deficiencies, leading in particular to relaxing the assumption of continuity of price trajectories. This chapter develops a general Lévy framework for pricing CoCos bonds. Lévy's framework intuitively shows the hybrid nature of CoCos bonds and reduces the problem of pricing CoCos bonds to the problem of the first passage time of the triggering process. According to the characteristics of the new Lévy measure after the measure transform, two Lévy models driven by the derived Brownian motion and the spectrally negative Lévy process are proposed. These two Lévy models keep the form of the Lévy process unchanged under the measure transform, which avoids the difficulty that only rare forms of Lévy processes solved the first passage time problem. These Lévy models provide closed form expressions for the price of CoCos while one owns it up to the double Laplace transform, the pricing results of which are given by combining with the numerical Fourier inversion.

The numerical results show that negative jumps play a much critical role in the pricing of CoCos bonds. The Black-Scholes model compresses all the information about jumps in volatility, which makes a big difference in the price of CoCos bonds between the Black-Scholes model and the SN model. The model driven by the spectrally negative Lévy process only compresses the information of positive jumps in volatility. Without the characterization of jumps in the triggering process, the Black-Scholes model would overestimate the price of CoCos bonds. The model driven by the spectrally negative Lévy process would provide a more accurate CoCos bond price taking into account the phenomenon of jumps in the financial market. The proposed Lévy models can capture the short-term behavior of the triggering process. However, the long-term phenomenon such as volatility clustering is not characterized. In addition, stochastic volatility Lévy models and regime-switching Lévy models, which can capture long-term behavior, deserve further study. Some special regime-switching Lévy models solved the first passage time problem, and the next step can extend Lévy models to regime-switching Lévy models. Since different designs of CoCos bonds lean towards different pricing models, Levy models for more complex designs of CoCos with features such as multi-variate trigger are also for further study.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPESS journal belongs to the authors.

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