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Research Article

Reflections from the generalization strategies used by gifted students in the growing geometric pattern task[1]

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Abstract

One of the cognitive characters emphasized by different researchers in mathematically gifted students is generalization of mathematical structures and patterns. In particular, experience with growing geometric patterns is important for initiating and developing algebraic thinking. In this context, this study aimed to explore the generalization strategies used by gifted students in the growing geometric pattern task. The study was designed in a case study. The participants of the study are five eighth grade students who were diagnosed as gifted through diagnostic tests. The data of the study were collected with the "Geometric Pattern Task Form" consisting of open-ended problems. The geometric pattern task consists of linear and quadratic patterns. Data were collected by task-based interview method and analyzed with thematic analysis. The results of the study show that gifted students exhibit figural and numerical approaches while solving pattern problems. In particular, for quadratic (non-linear) pattern, gifted students used functional strategy in all problems of finding near, far terms, and general rule of pattern. However, in the problems of finding the number of white balls (linear pattern), different strategies (e.g., recursive, chunking, contextual) than the functional strategy were also used. Based on the results of the study, it is suggested that geometric pattern tasks involving linear and non-linear relationships may be centralized in the development of functional thinking and generalization skills of gifted students in classroom practices.

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Introduction

In recent years, the concept of giftedness and the educational needs of gifted students have attracted attention in the world (Paz-Baruch et al., 2022). Traditionally, researchers initially defined giftedness as high general intelligence as measured by high intelligence scores (Terman 1924; cited in Pitta-Pantazi, 2017). However, later on, taking into account social or educational needs, giftedness began to be defined according to social needs. For example, according to Sternberg and Davidson (1986), giftedness is "not something we discover, but something we invent. It is what a society wants it to be, and so its conceptualization can change over time and space". Contemporary conceptualizations of giftedness, on the other hand, suggest that this phenomenon is multidimensional beyond the concepts of intelligence level (Sternberg & Grigorenko, 2004). These multidimensional definitions combine several factors: above-average ability, commitment to task, wisdom, intelligence, and creativity (Renzulli, 1978; Stenberg et al., 2021).

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Along with the heterogeneous nature of the abilities that gifted students show, there is no clear definition of mathematical giftedness, which is a domain-specific concept (Paz-Baruch et al., 2022). Mathematical giftedness is defined by a certain directionality of the whole mind as well as an increasingly specific mathematical abilities (Assmus & Fritzlar, 2022). Leikin (2018) suggested that mathematical giftedness is a combination of high mathematical performance and mathematical creativity. Researchers have revealed some cognitive characteristics of mathematically gifted students. One of the cognitive characters emphasized by different researchers in mathematically gifted students is generalization of mathematical structures and patterns (Assmus & Fritzlar, 2022; Krutetskii, 1976; Leikin, 2021; Leikin et al., 2017; Paz-Baruch et al., 2022; Singer et al., 2016; Sriraman, 2003).

Generalization is the process of drawing conclusions and induction from certain situations (Sriraman, 2003). Generalization is a potentially creative process as it leads to the discovery of new situations (Assmus & Fritzlar, 2022). In addition to being the "heartbeat of mathematics" (Mason, 1996), generalization is also one of the basic elements of algebraic thinking (Radford, 2018). Determining the relationship of change in quantities, generalizing and functional thinking are related to algebraic thinking (Kieran, 2022). Students will understand that algebra is a language of expression only if they express generalizations (Mason, 1996; Tural-Sonmez, 2019). Studies with patterns have emphasized the link between algebra and generalization to encourage the development of algebraic thinking (e.g., Amit & Neira, 2008; Ramírez et al., 2022). That is, pattern generalization tasks are a powerful and useful tool that supports and develops algebraic thinking (Assmus & Fritzlar, 2022; Mason et al., 2005)

Mathematics can be defined as the science of patterns because of the strong relationships, hierarchy, order and structures in its contents (Steen, 1988). Pattern generalization tasks, on the other hand, are the act of detecting regularities that can be predicted numerically, spatially or logically (Mulligan & Mitchelmore, 2009). Although there are many definitions of patterns related to relationships between art, language, numbers, and items, this study adopts a mathematical understanding of patterns that include numerical, spatial, or logical relationships (Kidd et al., 2019). In some educational contexts, finding a general rule for a data set presented as pairs of independent and dependent data or as ordered data is a typical task in school algebra (Radford, 2018). Therefore, pattern generalization tasks serve as a bridge between students' arithmetic knowledge and their ability to understand symbolic representations (Lannin et al., 2006).

Mathematical patterns are often grouped as "repeating patterns" and "growing patterns" according to their structure (MacKay & De Smedt, 2019; Zazkis & Liljedahl, 2002). It is understood that shapes or numbers are systematically enlarged or reduced in growing patterns (Assmus & Fritzlar, 2022; Lüken et al., 2014). The function type in growing patterns can be linear or non-linear (El Mouhayar & Jurdak, 2015; Gutiérrez et al., 2018; Stacey, 1989; Zazkis & Liljedahl, 2002). In linear pattern tasks, students should observe and use the linear pattern form "$f(n) = an + b$ with $b \neq 0$" (Stacey, 1989). Quadratics are the simplest form of non-linear functions and quadratic relationships play a fundamental role in non-linear function studies (Wilkie, 2022b). In addition, quadratic relationships require higher cognitive demands and are challenging (Ramírez et al., 2022; Wilkie, 2022a).

Patterns can be presented in numerical, geometric/pictorial/figural or computational representations (Rivera & Becker, 2005; Zazkis & Liljedahl, 2002). Geometric patterns consist of objects that convey positions in a structural relationship and are somewhat similar to each other (Rivera & Becker, 2011). When students begin to search for relationships between datasets, students' experience with repetitive or growing patterns can improve their functional thinking (Radford, 2018). In particular, experience with growing geometric patterns is important for initiating and developing algebraic thinking (Gutiérrez et al., 2018). Spatial visualization and generalization of geometric patterns is an accepted way to improve students' understanding of variables in algebra and their functional thinking (Wilkie, 2022a; Wilkie & Clarke, 2016). We integrate into our study growing geometric pattern generalization task in different function type (linear and non-linear).

Rationale and Aim of the Study

General giftedness or mathematical expertise can be predicted by students' pattern skills (Assmus & Fritzlar, 2022; Paz-Baruch et al., 2022). Therefore, it is emphasized that patterns should be included more frequently in studies conducted in the field of gifted education and mathematical giftedness (Eraky et al., 2022; Leikin & Sriraman, 2022). However, studies examining the patterning skills in math of gifted students are limited (e.g., Amit & Neria 2008; Arbona et al., 2019; Assmus & Fritzlar, 2022; Benedicto et al., 2015; Gutiérrez et al. al., 2018). In the context of Turkey, there are very few studies (e.g., Dayan, 2017; Girit-Yıldız & Durmaz, 2021). On the other hand, Leikin et al. (2017) points out the necessity of increasing gifted education and mathematics education studies by integrating them. Studies that deal with mathematics education and gifted education together in Turkey are limited, although they tend to increase in recent years (e.g., Ozturk et al., 2018). This study focuses on the generalization strategies used by gifted students in the growing geometric pattern task. Therefore, the study is important in that it includes both gifted and mathematics education.

In some previous studies, generalization strategies in linear and non-linear pattern tasks of gifted students in pre-algebra level (Amit & Neria, 2008) or secondary school level (Girit-Yıldız & Durmaz, 2021) were reported. The aspect of this study that differs from other studies is that it examines both linear and non-linear pattern generalization strategies of eighth grade students in the last year of middle school. These students will encounter the concept of function and its types in the next education level, high school. Therefore, the study will provide information about gifted students' strategies to generalize non-linear and specifically at quadratics relations beyond linear relations. The findings of the study can contribute to educators and instructional designers to improve their instruction by addressing the individual needs of gifted students in learning environment.

In recent years, mathematics education literature has focused on algebraic thinking and pattern tasks as a way of evaluating knowledge related to generalization skills (Singer & Voica, 2022). It was determined that students' success in generalization of patterns differed according to the pattern representation style, and geometric representations helped students observe functional relationships (Eraky et al., 2022; Lannin et al., 2006; Rivera & Becker, 2011). In this context, most of the studies dealing with growing geometric patterns in the literature are on examining linear relationships (e.g., Chua & Hoyles, 2014a, Friel & Markworth, 2009; Lobato et al., 2013; Markworth, 2010; Montenegro et al., 2018; Radford, 2010; Radford et al., 2007; Rivera & Becker, 2008, 2011; Smith, 2008; Wilkie & Clarke, 2016). However, some studies have examined students' generalization skills in quadritic relationships (Chua & Hoyles, 2014b; Ramírez et al., 2022; Rivera, 2010; Steele, 2008; Wilkie, 2022a, 2022b). Studies that deal with linear and quadritic relationships together are quite limited (e.g., Akkan & Cakıroglu, 2012; El Mouhayar & Jurdak, 2015; Lannin et al., 2006; Wilkie, 2019).

Considering the limited studies, there seems to be a lot to learn about students' processes of discovering relationships in pattern tasks presented in linear and non-linear form. This study focuses on growing geometric pattern generalization task in different function type (linear and non-linear). It is obvious that this study will contribute to the expansion of mathematics education literature. Motivated by the aforementioned concerns, this study aimed to explore the generalization strategies used by gifted students in the growing geometric pattern task. To this end, the study seeks to answer the following question: What are the strategies used by gifted students in the problems of finding the immediate, near, far terms, and the rule of the growing geometric pattern task?

Method

Research Design

In the study, a qualitative approach was adopted and case study design was used. Case study is an in-depth description of a situation or unit of analysis (a limited system) that takes place in real life, a current context or setting (Merriam & Tisdell, 2015; Yin, 2014). In this study, as a limited situation, the strategies used by gifted eighth grade students in the growing geometric pattern task were examined in depth. The analysis unit of the study is five gifted students studying at the eighth grade level determined by the purposeful sampling method.

Participants

The participants of the study are five eighth grade students who were diagnosed with giftedness through diagnostic tests. Gifted students study at both a public secondary school and a Science and Art Center (SAC) in a city center in the Eastern Anatolia Region of Turkey. Participants were determined by criterion sampling, one of the purposeful sampling types. In criterion sampling, the sample for the situations provided beforehand through the determined criteria is taken into account (Patton, 1990). In this context, one of the criteria is that the students are at the eighth grade level. It was determined as a criterion that the eighth grade students should have received education on both generalization of patterns and square root numbers within the scope of the mathematics curriculum. In addition, in line with the opinions of the mathematics and Turkish teachers about the students, it was paid attention to determine the gifted students with good expression skills as participants.

In the findings section, the term "student" will be used instead of "gifted student" due to linguistic fluency. The gifted students' participation in the study was based on their volunteering. 3 of the participants are girls (60%), 2 of them are boys (40%). The first semester mathematics course grade point average of gifted students is in the range of 96-100. Gifted students continue their education at SAC in line with the "Gifted Development Program". The real names of the participants were not given and coded (S1 for the first student).

Instrument

The data of the study were collected with the "Geometric Pattern Task Form" consisting of open-ended problems. The form was developed by the researchers using the pattern generalization types and adaptation of task in the literature (Cai, 2003). The geometric pattern task is of growing nature and includes both linear and non-linear (quadratic) function types. As explained in the literature and rationale sections of the study, we used into our study growing geometric pattern task (GGPT) in different function type (linear and non-linear). The reasons for using both linear and non-linear function types in the task can be summarized under three headings. First, it is emphasized that gifted students prefer to deal with more challenging tasks (e.g., Assmus & Fritzlar, 2022; Nolte & Pamperien, 2017). Second, quadratic relationships require higher cognitive demands and are challenging than linear relationships (Ramírez et al., 2022; Wilkie, 2022a). Third, most of the studies on growing geometric patterns in the literature are on linear relationships (e.g., Chua & Hoyles, 2014a, Montenegro et al., 2018; Wilkie & Clarke, 2016), and studies on pattern tasks presented in non-linear form are limited (e.g., Wilkie, 2022a, 2022b).

Sub-problems belonging to the pattern generalization types frequently used in the literature were assigned to the GGPT (e.g., Amit & Neria 2008; El Mouhayar & Jurdak, 2015; Gutiérrez vd., 2018a; Rivera & Becker, 2011; Stacey, 1989). These sub-problems are for finding immediate, near, far terms and general rule of the pattern. The draft task was submitted to expert opinion (Two lecturers working in the field of mathematics education and three mathematics teachers working with gifted students). The experts evaluated the compliance of the draft task with the following criteria: purpose of the study, pattern structure and representation, language. The experts stated that the draft task was suitable in terms of the specified criteria. Then, a pilot study was conducted with three gifted students who were not participants in the study. In the pilot study, it was aimed to evaluate the feasibility of the task in terms of language, intelligibility and time. Secondly, it is aimed to develop a schema to encode the data. The schema development process is explained in detail in the data analysis section. The pilot study lasted an average of 13 minutes with each student. As a result of the pilot study, no changes were made in the draft task and sub-problems. The GGPT used as a data collection tool in the study is presented Figure 1:

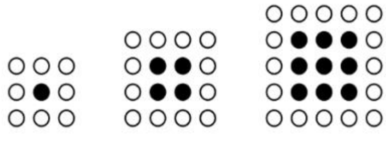

Figure 1. The Growing Geometric Pattern Task in the Form.

Above are the first three shapes of a pattern formed with black and white balls. Examine the pattern. According to this;

- Can you draw the fifth figure? Can you explain how you drew this?
- What is the difference between the numbers of black and white balls in the eleventh figure? Can you explain how you got the answer?

- What is the difference between the numbers of black and white balls in the fifty-first figure? Can you describe how you found it?
- Can you find a rule that gives the difference between the number of black and white balls in any figure of the pattern? Can you explain how you found the rule?

The pattern of black balls in the task is of the non-linear function type and its general rule is $f(n)=V_n = n^2$. The pattern consisting of white balls is of the linear function type and its general rule is $f(n)=V_n = 4n+4$. The sub-problems of the GGPT, on the other hand, are related to a) immediate term, b) near term, c) far term, and d) rule of the pattern.

Data Collection

Data were collected by task-based interview. Task-based interviews, which have their origins in clinical interviews, are used in mathematics education to gather information about students' existing or developing mathematical knowledge structure or problem-solving behaviors (Maher & Sigley, 2014). In the task-based interview, which has an exploratory structure, the student and the interviewer interact in a task environment prepared in accordance with the purpose of the study (Goldin, 2000). In this study, task-based interviews were conducted with gifted students and an interviewer, since it was aimed to explore the generalization strategies exhibited by gifted students in the growing geometric pattern task. The task used in task-based interviewing is the GGPT and its sub-problems. Interviews were conducted by the second author. During the task-based interviews, student responses were not interfered with. However, the interviewer asked the question "Why?, Why not?, Can you explain?" posed such questions. Thus, students were expected to explain their thoughts in more detail. Interviews were conducted in a digital environment, with video and audio recordings. Interview times were planned in advance by meeting with the students. The interview was conducted at a convenient time, in a quiet environment where the participants felt comfortable. Task-based interviews lasted approximately 14 minutes with each student.

Data Analysis

The data sources of the study are task-based interviews and written documents containing the solutions of gifted students. In the data analysis process, firstly, written transcripts of the interview data were made. Then, the data were analyzed by thematic analysis method. Thematic analysis is an ideal method to identify and report patterns/themes in data, either inductive or deductive. In deductive thematic analysis, there is a process of forming themes with theoretical outputs (Braun & Clarke, 2006). In the present study, deductive thematic analysis was used. For data analysis, first of all, the schema was developed during the pilot study process. While developing the scheme, first of all, a list of strategies that students frequently use was created by conducting a literature review (e.g., Amit & Neria 2008; El Mouhayar & Jurdak, 2015; Gutiérrez et al., 2018a; Lannin et al., 2006; Rivera & Becker, 2005; Stacey, 1989; Tanisli & Yavuzsoy-Kose, 2011). Second, the student responses obtained in the pilot study were assigned to the strategies in this list by the two raters. Third, the raters came together and discussed their encodings until they reached a consensus. The list of strategies created as a result of the literature review and the analysis of the student answers obtained in the pilot study is as follows:

Recursive: Obtaining the next figure (term) from the previous figure (term). It is a method of finding the result by continuing the pattern by finding the difference between the terms.

Chunking: It is the method of finding the desired term through arithmetic operations by using the difference between the terms in the pattern and the number of intervals (or the difference between the number of steps of a known term and the number of steps of the desired term).

Contextual: It involves structuring a rule or formula that focuses on the information that provides the situation. Students apply the solution method that they have learned before and are familiar with. This strategy involves a partial understanding of the algebraic structure underlying the pattern. It can be a memorized rule or formula.

Functional: It is for determining the relationship between independent variable (input) and dependent variable (output). This strategy is considered the first step towards determining a function using equations and formulas.

In addition to the strategies, student responses were classified according to the numerical and figural approaches used to generalize the patterns given by geometric representation. In the numerical approach, students transform the geometric pattern into a number pattern and solve the problem through the number pattern. In the figural approach, students use graphical representations of terms to solve the problem. That is, students focus on the structural feature of the shape in geometric patterns.

Trustworthiness

In the study, some precautions were taken in terms of reliability or consistency, internal validity or credibility, external validity or transferability. In the qualitative approach, reliability or consistency is based on the principle that the findings are consistent with the presented data. The "audit technique" can be used to ensure reliability or consistency. In this study, audit technique was used to ensure reliability or consistency (e.g., Merriam & Tisdell, 2015). In the context of this technique, the data collection and data analysis process is presented in detail. The use of a conceptual framework for data analysis is one of the factors that increase the reliability of the study.

Internal validity or credibility of qualitative research is related to capturing the truth or reality. Triangulation technique can be used to increase the credibility of a qualitative research. Triangulation is the joining of two or three measuring points. One type of triangulation is "multiple researchers' participation". It requires the participation of more than one researcher, the presence of two or three people in the data analysis process, and comparing the findings after analyzing the same data independently. Triangulation can also be considered within the scope of the reliability of qualitative research (Merriam & Tisdell, 2015). In this study, "more than one researchers' participation" was used as a type of triangulation. In this direction, the data written in the study was coded by two independent researchers. The inter-rater reliability was calculated as 96%. This result is a sign of the consistency of the encodings. However, researchers have reached a consensus by arguing about the encodings in which the difference occurs.

External validity or transferability in qualitative research is concerned with the generalizability of study results. A "thick description" strategy can be used to increase the portability of study results. This strategy is to describe the setting and participants, and to elaborate the findings with direct quotations from participant interviews and documents (Merriam & Tisdell, 2015). In this study, the qualifications of the participants were described in detail in the context of external validity or transferability. In addition, direct quotations from the interviews and student responses are presented in the findings section.

Results

In the study, all students answered correctly the problem of finding the immediate, near, far terms and the general rule of the pattern. Detailed findings of each sub-problem of the GGPT are presented below.

Strategies Used to Find the Immediate Term

The strategies used by the students to find the immediate term of the GGPT are given in Table 1.

Table 1. Strategies Used to Find the Immediate Term of the GGPT

Pattern type	Approach	Strategy	Student	f
Black ball (non-linear)	Figural	Functional	S3, S1, S4	3
		Recursive	S5	1
	Numerical	Functional	S2	1
White ball (linear)	Figural	Functional	S3, S4	2
		Recursive	S5	1
		Chunking	S1	1
	Numerical	Functional	S2	1

The students were asked to draw the fifth figure of the GGPT. When Table 1 was examined, it was seen that the students categorized the black and white balls separately while drawing the figure. In addition, it was determined that students used figural and numerical approaches while drawing black and white balls.

Students using the figural approach focused on the structure of the figure and thought of the figure as a quadratic system. Recursive, chunking, and functional strategy were used under the figural approach. In the process of drawing both black and white balls, S5, who applied recursive strategy under the figural approach, thought of the figure as a quadratic system. This student stated that in every way, the sides of the quadratic system are formed by increasing by 1. S5 obtained the next shape by making use of the previous figure and used the recursive strategy. S5's explanation is as follows: *Teacher, if I think of the black balls as squares when I examine the figure, it will be 1 on the bottom edge in Figure 1, 2 in Figure 2, 3 in Figure 3, 4 in Figure 4, and 5 in Figure 5. So it would be a 5x5 square. The white balls are also progressing by increasing by 1 in the form of 3, 4, 5. There will be 7 whites on the bottom edge. If we complete it, it becomes a 7x7 square. Black will already have 5x5.*

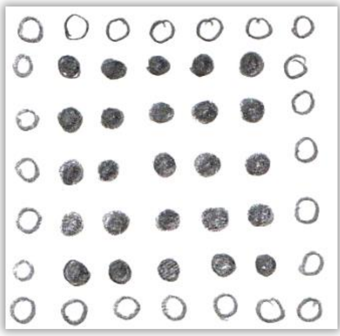

Figure 2. S5's Response to the Immediate Term of the GGPT

S3 and S4, who used the functional strategy under the figural approach, established a relationship between the figure order and the number of balls in the figure while drawing the black and white balls. S3, one of the students who used this strategy, while drawing the figure, stated that the number of black balls is the square of the row number of figure, and the total number of balls in the figure is " the square of 2 more of the row number of figure.". The The dialogue and operations of S3 are given below.

S3: Ok, I can draw. Now when I examine the shapes, Figure 5 is a square area of 7 by 7, its 5x5 area should be painted black.

I: How did you decide it was like that? Can you explain?

S3: When you look at the figure, the whole figure becomes two more than the number of steps.

I: Can you explain a little more?

S3: The 2x2 square in the first figure, the 4x4 square in the second figure, the 5x5 square in the third figüre. Blacks have as many steps as the number of steps. That's 1 in Figure 1 and 2x2 in Figure 2.

Using the functional strategy while drawing the black balls under the figural approach, S1 stated the number of black balls as the square of the row number of figure. However, this student used the chunking strategy under the figural approach for drawing the white balls. In the chunking strategy, firstly, the number of steps from the first step to the desired step is determined. Then, the student added the number of balls in the first term to the number he found and took the square of the result. The explanations of S1 are as follows:

S1: Ummm, I think of the shape as a square. The blacks in the middle are n2 as the square of the number of steps. 1 squared, 2 squared, 3 squared.
I: Yes, what about the white balls?
S1: Ummm, whites are going as 3, 4, 5, that is, increasing. For us to find, I add the number of steps from the first step to 3 and find the number of balls on the outermost edge. If we take the square after finding it, I will have found the number of balls of the whole shape. Then we need to subtract the middle number, the black balls, so that we can find the number of white balls. Umm that's how we do it. For example, there are 4 intervals for step 5. We add 4 to 3 7, the outermost number of sides then becomes 49 (squared by 7). Then we subtract from 25 (the number of black balls) to get 24.
I: What is the reason for adding 4?

S1: After the first step, there are 4 steps until the fifth step. The whole figure was also 49. Black's would also be the square of the number of steps, so if we subtract it, it would be 24. Ummm, first I draw the black while drawing the shape, and then I added the whites.

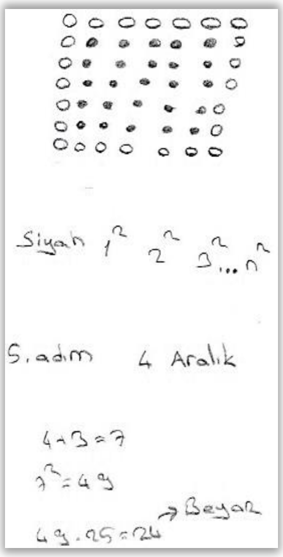

Figure 3. S1's Response to the Immediate Term of the GGPT.

S2, on the other hand, used functional strategy under the numerical approach in drawing both black and white balls. S2 first transformed the geometric pattern into a number pattern. Then, she stated that the number of black balls is the square of the number of steps, and the whole figure is the square of the number of steps 2 more. Using these two pieces of information, he also drew the white ball. The statements of S2 are as follows:

S2: Let me draw the fifth figure, but it will take a while.

I: No problem, you can draw the shape you want.

S2: 7 by 7 something circles. So it will take a while. I drew it.

I: Can you tell me how you drew it?

S2: Teacher, I drew 7 by 7 squares, but I made them black except for the edges.

I: Why 7 by 7?

S2: Because, there are balls as 3 squared in the first figure. Figure 2 is 4 squared, Figure 3 is 5 squared. The figure is the square of 2 more than the number of rows. So (x+2) squared. I also found blacks x^2.

I: How did you find the rule for black balls?

S2: I saw that it goes as 1,4,9. The number of black balls became the square of the row number of figure.

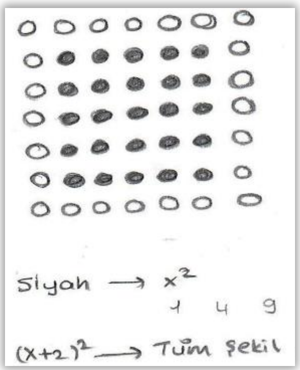

Figure 4. S2's Response to the Immediate Term of the GGPT.

Strategies Used to Find the Near Term

The strategies used by the students to find the near term of the GGPT are presented in Table 2.

Table 2. Strategies Used to Find the Near Term of the GGPT

Pattern type	Approach	Strategy	Student	f
Black ball (non-linear)	Figural	Functional	S1, S3, S5	3
	Numerical	Functional	S2, S4	2
White ball (linear)	Figural	Functional	S3, S5	2
		Chunking	S1	1
	Numerical	Contextual	S4	1
		Functional	S2	1

The students were asked about the difference between the numbers of black and white balls in Figure 11. According to the research findings, it was seen that the students categorized the black and white balls separately. In addition, the students calculated the numbers of black and white balls under the figural and numerical approach.

Students who adopted the figural approach focused on the structure of the figure and used functional and chunking strategies. In the process of finding the number of black balls, S1, S3 and S5, who adopted the figural approach, used the functional strategy. These students expressed the number of black balls as the square of the row number of figure. While calculating the number of white balls, S1 switched to the chunking strategy. S1 first calculated the total number of balls using the chunking strategy. For this, the number of steps from the first step to the desired step was calculated, the number of balls in the first term was added to the number reached and the result reached was squared. Then, she calculated the number of white balls by subtracting the number of black balls from the result he found. While calculating the number of white balls, S3 and S5 continued to use the functional strategy. S3 determined the whole number of balls as the square of two more than the number of shapes. Afterwards, he obtained the number of white balls by subtracting the number of black balls from the result she reached. S5, on the other hand, primarily grouped the shape as top, bottom and side. Then, he determined the number of top and bottom balls as two more than the number of steps, and the number of balls on the sides as the same as the row number of figure. The interview dialogue and procedures of S3, one of these students, are presented below.

I: Can you find the difference between the numbers of black and white balls in figure 11?

S3: I think 48.

I: How did you find it? Can you explain?

S3: Wait a minute, but no. Number of 48 white balls. 121 minus 48 is 73.

I: Can you explain?

S3: The total area minus the black area gives the white balls.

I: What is the total area?

S3: It is the square of two more than the number of steps. So 13 squared is 169. Black is also squared by the number of steps. 11 squared would be 121.169 minus 121 would be the number of white balls. 121 minus 48 is 73.

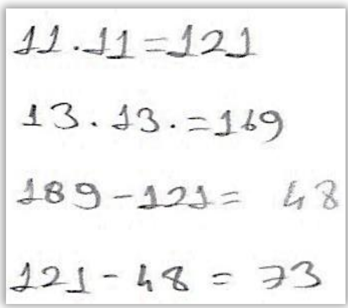

Figure 5. S3's Response to the Near Term of the GGPT.

Students using the numerical approach are S2 and S4. S2, one of these students, used functional strategy in the process of calculating the number of both black and white balls. The student first transformed the geometric pattern into a number pattern according to the number of balls. Then, she correlated the number of each step with the number of steps. Proceeding in this way, she determined the number of black and white balls and used the functional strategy. S2's explanations are as follows: *Eee. Let's write the ball numbers first. In the 11th figure, there are 169 balls from the square of 13, 2 more than 11. There are 121 black balls from the square of 11. Their difference is the number of white balls of 48. The difference in the number of black and white balls is also 73.*

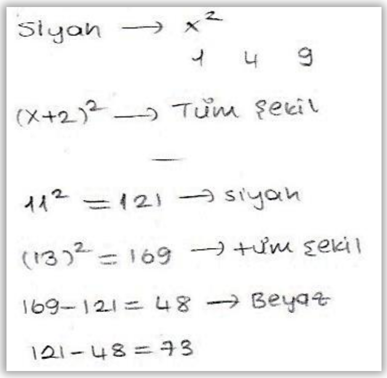

Figure 6. S2's Response to the Near Term of the GGPT.

While S4 adopted the figural approach in the problem of drawing the immediate term, he switched to the numerical approach in the problem of calculating the near term. This student used functional and contextual strategies while calculating the number of black and white balls, respectively. First of all, the student wrote the number pattern containing the black and white ball numbers in the geometric pattern. He stated that the number of black balls is the square of the row number of figure. In the process of calculating the number of white balls, he made use of the increase between terms and partially formed the pattern rule. First of all, he stated that the difference between terms is 4 and the pattern rule should be 4n. By specifying that he should write the number of steps instead of n, he created the rule with a contextual strategy. S4's statements are as follows:

S4: Now, teacher, I must find the white balls first. If I look now, I will count them one by one. Ummm it goes 8, 12, 16. This is 4.n+4.

I: How did you find it?

S4: We should say 4n since there are 4 each. Then we value by the number of steps instead of n to find the other steps. Like 1 for the first step and 2 for the second step. 4.1=4. To be 8, it becomes 4.n+4. The black ball numbers will also go as 1, 4, 9, 16. In step 11, there will be 11 squared. That is, 121 is the square of the shape row. Their difference is also 73.

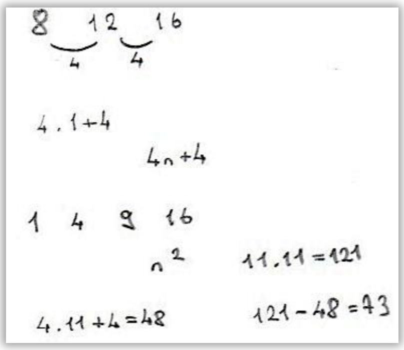

Figure 7. S4's Response to the Near Term of the GGPT

Strategies Used to Find the Far Term

The strategies used by the students to find the far term of the GGPT are shown in Table 3.

Table 3. Strategies Used to Find the Far Term of the GGPT

Pattern type	Approach	Strategy	Student	f
Black ball (non-linear)	Figural	Functional	S1, S3, S5	3
	Numerical	Functional	S2, S4	2
White ball (linear)	Figural	Functional	S3, S5	2
		Chunking	S1	1
	Numerical	Contextual	S4	1
		Functional	S2	1

The students were asked about the difference between the numbers of black and white balls in Figure 51. The findings of the study show that the students continued the approaches and strategies they used in the problem of finding the near term. Accordingly, the students calculated the numbers of black and white balls under figural and numerical approaches.

In the process of finding the number of black and white balls, S3 and S5, who adopted the figural approach, focused on the structure of the figure and used the functional strategy. S3 and S5 determined the number of black balls as the square of the row number of figure. In the process of calculating the number of white balls, S3 first calculated the total number of balls as the square of 2 more than the row number of the figure. Then she subtracted the number of black balls from his result and reached the number of white balls. S5, on the other hand, made the calculations by considering the balls in the figure as upper, lower and side groups. The explanations of S5, one of these students, are as follows: *In figure 51, there are 53 on the top and bottom and 51 on the sides. From 2.51 +2.53 it is 208. Since the number of black balls is also 51st step, it is 2601 from 51.51. Their difference is 2393 from 2601-208.*

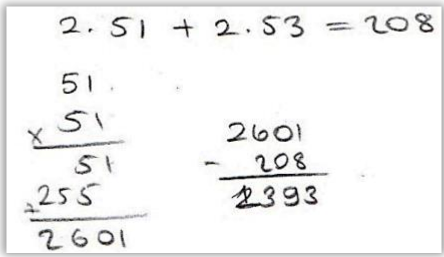

Figure 8. S5's Response to the Far Term of the GGPT.

S1 applied different strategies in the process of finding the number of black and white balls. While calculating the number of black balls, she expressed the number of balls as the square of the row number of figure and displayed a functional strategy. She used the chunking strategy while calculating the number of white balls. First of all, S1 considered the shape as a square structure consisting of white balls as a whole, without distinguishing between white and black. She used the chunking strategy while finding the number of white balls on one side of this square structure. That is, number 1 (first figure) is subtracted from row number of desired figure. She added the number of 3 white balls on one side of the first shape to his result of 50. She squared his result and calculated the total number of balls in the whole figure. Finally, by subtracting the number of black balls from the result he found, she reached the number of white balls. Ö1's explanations are as follows: *The square of 51 is the number of black balls, ummm 2601. As I did in the previous question, I found 51-1=50 and add the number of balls in the first figure to get 53. The square of 53 becomes 2809 the whole figure. Immm 2809 minus 2601 subtracts 208. It becomes the number of white balls. We find the difference of 2601 minus 208, 2393.*

S2, one of the students who showed a numerical approach, displayed a functional strategy while calculating the number of black and white balls. S2 also used the expressions for the rule of the pattern he reached with the functional strategy while drawing the immediate term while calculating the far term. The other student S4, who exhibited the numerical approach, carried out the operations with the strategies he used for the near terms. S4 continued to use the functional strategy when calculating the number of black balls, and the contextual strategy when calculating the number of white balls. S2 and S4's explanations for calculating near terms are given in detail in the section above. The students' explanations of strategies for calculating far terms are also parallel.

Strategies Used to Find the General Rule of Pattern

The strategies used by the students to find the general rule of the GGPT are given in Table 4.

Table 4. Strategies Used to Find the General Rule of the GGPT

Pattern type	Approach	Strategy	Student	f
Black ball (non-linear)	Figural	Functional	S1, S3, S5	3
	Numerical	Functional	S2, S4	2
White ball (linear)	Figural	Functional	S1, S3, S5	3
	Numerical	Contextual	S4	1
		Functional	S2	1

The students were asked whether there was a rule that gave the difference between the number of black and white balls in any step of the pattern. When Table 5 is examined, it is seen that the students use the contextual or functional strategy under the figural and numerical approach. S1, S3, S5 focused on the square structures of the figures and used the functional strategy to determine the rule for both white and black ball numbers. These students expressed the rule by associating the row number of the figures with the figures. A remarkable finding is related to S1's strategy transition. Using the chunking strategy in the problems of finding the immediate, near, and far terms to calculate the number of white balls, S1 switched to the functional strategy to find the rule of the pattern. S1's statements are as follows: *The square of the number of steps is the number of black balls. For whites, we can find it like this, the square of 2 more than the rows number of steps becomes the whole shape. Subtract the blacks from the whole figure (ummm) and we get the whites. When we subtract them again, we find the difference.*

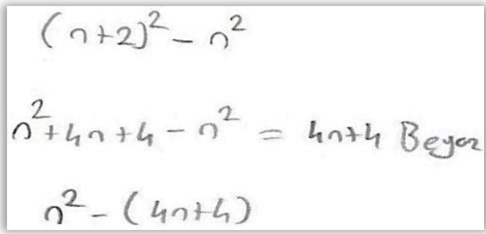

Figure 9. S1's Response to the General Rule of the GGPT

S2, one of the students, determined the rule of black and white ball numbers by using functional strategy under the numerical approach. First of all, S2 wrote the number pattern including the total number of balls and black balls in the geometric pattern. Afterwards, he expressed the desired rule by associating the number of balls in each step with the number of steps. The explanations of S2 are below.

S2: I will find the numbers of white and black balls and try to find a rule from their difference. It's not like that either x^2, sorry. Now the white ball numbers are 4x+4.

I: Can you explain how you found it?

S2: The whole number of balls is found by adding 2 to the number of steps and squaring it. That is, the number of black balls is x^2, since the total number of balls is $(x+2)^2$. So, the number of black balls is the square of the number of steps. I noticed the difference of two squares rule when subtracting from each other. When he made the transactions, the number of white balls became 4x+4. Blacks are x^2. Ummm, their difference is x^2-4x-4.

Another student who adopts the numerical approach is S4. S4 found the rules for the number of black and white balls with functional and contextual strategies, respectively. S4 first expressed the numbers of black and white balls in the geometric pattern as a number pattern. While finding the rule that gives the number of black balls, he related the step order and the number in the step. While finding the rule that gives the number of white balls, he said that the difference between terms is 4 and the pattern rule should be 4n. Then, he reached the rule by stating that he should write the number of steps instead of n. S4's explanations are as follows: *I already explained the rules in the previous problems. We were finding whites from 4.n+4. Also, we found n squares in black ball numbers. We find it by subtracting from each other.*

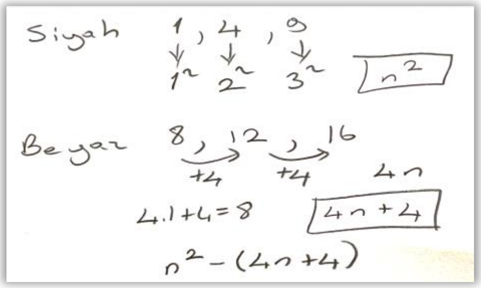

Figure 10. S4's Response to the General Rule of the GGPT.

Conclusion and Discussion

Students' patterning skills are one of the current research topics in gifted education and mathematical giftedness studies, because general giftedness or mathematical expertise can be predicted by students' patterning skills (Assmus & Fritzlar, 2022; Paz-Baruch et al., 2022). In this study, generalization strategies exhibited by gifted students in the GGPT were examined. It is seen that studies integrating gifted and mathematics education are insufficient (Leikin et al., 2017), in addition, studies examining the patterning skills of gifted students are limited (e.g., Amit & Neria 2008; Arbona et al., 2019; Assmus & Fritzlar, 2022; Benedicto et al., 2015; Eraky et al., 2022; Gutiérrez et al., 2018). Based on this situation, the results of the study were also discussed with the results of the study conducted with students who were not diagnosed as gifted (non-gifted). Thus, a richer perspective is expected to be presented

The results of the study show that gifted students exhibit figural and numerical approaches to the GGPT. This result is consistent with the results of studies that previously reported that gifted students showed a figural and numerical approach when working with geometric patterns (e.g., Gutiérrez et al., 2018). In addition, studies conducted with non-gifted students reported similar results (e.g., Rivera & Becker, 2005). However, in geometric patterns, students are expected to analyze the figural aspects of the pattern structure rather than the numerical aspects (Wilkie, 2022a). However, according to the results of the study, it was determined that some gifted students responded the pattern problems with the numerical approach. The fact that gifted students pay attention to the numerical aspects of the patterns is an indication that they have a superficial understanding of the relationships in the pattern structure (Rivera & Becker, 2011). Besides, Paz-Baruch et al. (2022) states that mathematically gifted students have high levels of noticing patterns and visual competiencies. Eraky et al. (2022) also emphasizes that observing geometric patterns plays an important role in developing functional thinking skills of gifted students. Despite this information, the reason why some gifted students resorted to numerical approach in the study may be that they have more experience with number patterns and solve problems individually. Montenegro et al.'s (2018) statements support this view. Montenegro et al. (2018) stated in their study that middle school students could not automatically detect the spatial characteristics of geometric patterns individually.

According to the study findings, in the process of drawing the immediate term of the GGPT, the majority of the students used the functional strategy (four students) while drawing the black balls (non-linear). However, only one student applied the recursive strategy. While drawing the white balls (linear), three students made use of the functional strategy, and one student each made use of the recursive and chunking strategy. These results are consistent with the results of studies showing that gifted students use functional strategy by focusing on the structure of the figure in immediate terms of geometric patterns (e.g., Amit & Neria, 2008; Gutiérrez et al., 2018). It has also been reported in studies that gifted students (Amit & Neria, 2008) or non-gifted students (Lannin et al., 2006; Syawahid et al., 2020) apply to recursive strategy to find immediate term.

The students reached the answer by using the same strategies in the problems of finding near and far terms. Accordingly, all of the students used the functional strategy in the process of finding the number of black balls (non-linear). The literature has highlighted that students have difficulty understanding quadratic concepts and representations (equations, tables, and graphs) (Lobato et al., 2012; Wilkie, 2022a, 2022b). Therefore, quadratics, the simplest type of non-linear functions used in this study, can be seen as challenging tasks with high cognitive demand for students (Ramírez et al., 2022; Wilkie, 2022b). In this study, it was seen that gifted students performed successfully in pattern problems in non-linear form. This result of the study supports the results of the study revealing that gifted students prefer to deal with more challenging tasks (e.g., Assmus & Fritzlar, 2022; Nolte & Pamperien, 2017).

In the problem of finding the number of white balls (linear) in the near and far terms, three students applied the functional strategy, while one student each benefited from the chunking and contextual strategies. The results of the study are consistent with the results of the study showing that gifted students reach the answer by using the functional strategy correctly in geometric pattern problems (e.g., Amit & Neria, 2008; Arbona et al., 2019; Gutiérrez et al., 2018). However, the use of chunking and contextual strategies by some gifted students is an indication that these students cannot see the input-output relationship. This situation can be associated with the fact that gifted students have had experiences that ignore the focus on the input-output relationship in the process of generalizing the patterns.

In the problem of finding the general rule of the GGPT, all students found the rule that gives the black ball number (non-linear) with the functional strategy. The students, who reached the rule of the black ball number (non-linear) with the figural-functional strategy, also found the rule of the white ball number (linear) with the same strategy. A remarkable finding is that a student (S1), who used the chunking strategy to find the number of white balls (linear) in immediate, near, and far term problems, shifted to the functional strategy while finding the pattern rule. This result supports the results of Amit and Neria's (2008) study, which determined that gifted students are flexible enough to shift from local approaches to global approaches while transitioning from near generalization situations to far generalization situations. This student used functional strategy in all problems related to the number of black balls (non-linear).

S4, who used functional strategy in all problems related to the number of black balls (non-linear), used contextual strategy in the problems of finding the near, far terms, and the rule of the pattern related to the number of white balls (linear). These findings show that gifted students are flexible in their strategy choices while solving problems. Previous studies support this result (e.g., Amit & Neria, 2008; Assmus & Fritzlar, 2022; Greenes, 1981; Gutiérrez et al., 2018). Assmus and Fritzlar (2022) suggested that gifted students show flexibility in mental processes in mathematical activities. Greenes (1981) explained that gifted students are flexible in organizing data. Gutiérrez et al. (2018), on the other hand, stated that gifted students quickly move from one strategy to another, which they think is more useful and beneficial.

The problem of finding the general rule of the pattern, that is, generalizing the pattern, requires more cognitive demand for students (Ramírez et al., 2022; Ureña et al., 2022). Therefore, the transition of gifted students to functional strategy supports the findings of the study showing that these students spend more mental effort on complex situations (Gutiérrez et al., 2018; Leikin et al., 2017). Problems of finding the near term may not be difficult enough for gifted students. Because of this situation, students may have responded the desired problems with strategies (e.g., recursive, contextual) that do not require focusing on the general structure of the pattern and seeing the input-output relationship.

The results of the study revealed that gifted students mostly apply to functional strategy in the problems of finding near, far, and the rule of the pattern. This result supports the findings of the studies showing that gifted students frequently use functional strategies in generalization tasks (e.g., Amit & Neria, 2008; Gutiérrez et al., 2018). According to the findings of the study, gifted students are successful in generalizing growing geometric patterns in both linear and non-linear forms. This result coincides with the results of the studies, which revealed that gifted students were successful in generalizing the patterns (e.,g., Amit & Neria, 2008; Benedicto et al., 2015, Eraky et al., 2022; Paz-Baruch et al., 2022). For example, Eraky et al. (2022) concluded that gifted students are more successful in determining the relationships between quantities and quantities in geometric patterns than in number patterns. Paz-Baruch et al. (2022) showed that mathematical gifted students have high visual competencies and pattern generalization skills. In addition, this result of the study is consistent with studies revealing that geometric representation in patterns helps the development of linear (e.g., Chua & Hoyles, 2014a, Friel & Markworth, 2009; Lobato et al., 2013; Markworth, 2010; Montenegro et al., 2018; Radford, 2010; Radford et al., 2007; Rivera & Becker, 2008, 2011; Smith, 2008) or non-linear (e.g., Chua & Hoyles, 2014b; Ellis, 2011; Ramírez et al., 2022; Rivera, 2010; Steele, 2008; Wilkie, 2022a, 2022b) generalization skills of non-gifted students.

Limitations and Implications

According to the results of the study, gifted students used functional strategy in all problems of finding near, far terms and general rule for the number of black balls (non-linear). However, in the problems of finding the number of white balls, different strategies than the functional strategy were also used. In this context, pattern tasks involving non-linear relationships may be centralized in the development of functional thinking and generalization skills of gifted students in classroom practices.

In the study, it was observed that some students used recursive, chunking or contextual strategies that limited functional thinking. This may have resulted from the experiences students encountered in their classroom environment. Therefore, it is important that mathematics teachers who teach gifted students have sufficient knowledge of pattern generalization strategies. As a matter of fact, the literature emphasizes that teachers have a role in students' understanding of the mathematical structure of patterns (e.g., Wilkie, 2021). In line with this emphasis, studies may be designed to determine and improve the pattern knowledge of teachers working with gifted students.

Study results in the literature show that gifted students perform at different levels in pattern tasks presented in different representations (Eraky et al., 2022). However, one of the limitations of this study is the geometric representation of the growing pattern. In future studies, the strategies used by gifted students in pattern tasks presented in different representations such as a graphs or daily life context may be examined and compared.

Another limitation of the study is that it works with gifted students at the eighth grade level. However, the literature emphasizes the necessity of starting algebra from an early age (Türkmen &Tanışlı, 2019). In the context of linear and non-linear patterns with earlier gifted students studies have been done (e.g., Amit and Neria's (2008) study with grades 6–7, Gutiérrez et al.'s (2018) study with third-grade (9 year-old)). In future studies, it may be examined how the strategies used by gifted students at different grade levels in the process of generalizing the patterns change according to the grade level.

It has been revealed by the results of previous research that geometric patterns are a concept related to mathematical giftedness and mathematical creativity (Asmuss & Fritzlar, 2022). In future studies, creativity skills of gifted students in the process of working with linear or non-linear forms of growing geometric patterns may be investigated.

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