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FIXED POINTS OF ENRICHED CONTRACTION AND ALMOST ENRICHED CRR CONTRACTION MAPS WITH RATIONAL EXPRESSIONS AND CONVERGENCE OF FIXED POINTS

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Abstract. We define enriched Jaggi contraction map, enriched Dass and Gupta contraction map and almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$ in Banach spaces and prove the existence and uniqueness of fixed points of these maps. Further, we show that the sequence of fixed points of the corresponding enriched contraction maps converges to the fixed point of the uniform limit operator of these enriched contraction maps.

1. Introduction

Generalization of contraction conditions and finding the existence of fixed points play an important role in the development of fixed point theory. There are many works where the notion of fixed point play some role, apparently, in different context. For instance, we refer Mustafa, Hakan and Turkoglu [5], Mustafa, Hakan and Sadullah [6] and the references cited in these papers. Further, there are several generalizations of Banach contraction maps, one among them is contraction conditions involving rational expressions. Dass and Gupta [3] initiated and introduced contraction condition with rational expression as follows:

Let (X, d) be a metric space and $T : X \to X$. There exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and T satifies

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y) \tag{1.1}$$

for all $x, y \in X$. Dass and Gupta [3] proved that if $T : X \to X$, X complete metric space, satisfies the inequality (1.1) and if T is continuous then T has a unique fixed point in X.

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In 1977, Jaggi [4] introduced a different rational type contraction condition independent that of contraction condition (1.1), i.e., there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y) \tag{1.2}$$

for all $x, y \in X, x \neq y$, and proved that every map $T : X \to X, X$ complete metric space, that satifies (1.2) has a unique fixed point in X, provided T is continuous.

A map T that satisfies (1.2) is said to be a Jaggi contraction map.

On the other hand, Berinde and Păcurar [1], introduced a larger class of mappings, namely, enriched contraction mappings in normed linear spaces which are more general than contraction maps.

Definition 1.1. (*Berinde and Păcurar* [2]) *Let* $(X, \|\cdot\|)$ *be a normed linear space. Let* $T : X \to X$*. If there exist* $k \in [0, +\infty)$ *and* $a \in [0, k + 1)$ *such that*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\|, \tag{1.3}$$

for all $x, y \in X$*, then we say that* T *is a* $(k, a)-$*enriched contraction.*

Theorem 1.1. (*Berinde and Păcurar* [2]) *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T : X \to X$ *be a* $(k, a)-$*enriched contraction. Let* $x_0 \in X$ *and* $\lambda \in (0, 1]$*. Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, n \geq 0, \tag{1.4}$$

converges to p *(say) in* X *and* p *is the unique fixed point of* T*.*

On further extensions of $(k, a)-$enriched contractions, we refer (Berinde and Păcurar [2]).

Definition 1.2. (*Berinde and Păcurar* [2]) *Let* $(X, \|\cdot\|)$ *be a normed linear space. Let* $T : X \to X$*. If there exist* $k \in [0, +\infty)$ *and* $a, b \geq 0$*, satisfying* $a + 2b < 1$ *such that*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|), \tag{1.5}$$

for all $x, y \in X$*, then we say that* T *is a* $(k, a, b)-$*enriched Ciric-Reich-Rus contraction map .*

Here onwards, we call these maps by $(k, a, b)-$enriched CRR contraction maps. If $a = 0$ in (1.5) then T is said to a $(k, b)-$enriched Kannan mapping [2].

Theorem 1.2. (*Berinde and Păcurar* [2]) *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T : X \to X$ *be a* $(k, a, b)-$*enriched CRR contraction map. Let* $x_0 \in X$ *and* $\lambda \in (0, 1]$*. Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, n \geq 0, \tag{1.6}$$

converges to u *(say) in* X *and* u *is the unique fixed point of* T*.*

In Section 2 of this paper, we define enriched Jaggi contraction map, enriched Dass and Gupta contraction map in Banach spaces and prove the existence and uniqueness of fixed points.

In Section 3, we define almost $(k, a, b, \lambda)-$enriched CRR contraction maps with $\lambda = \frac{1}{k+1}$ in Banach spaces and prove the existence and uniqueness of fixed points.

In Section 4, we prove that, if the sequence of enriched contraction maps converges uniformly to an operator with a unique fixed point then the corresponding sequence of fixed points of sequence of enriched contraction maps also converges to the fixed point of the limit operator in Banach spaces.

2. Fixed point results on enriched contraction maps with rational expressions

Let $(X, \|\cdot\|)$ be a normed linear space and $T : X \to X$. For any $\lambda \in [0,1)$, we denote

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \ x \in X. \tag{2.1}$$

Definition 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $\alpha, \beta \in [0,1)$ with $\alpha+\beta < 1$ and $k \in [0,+\infty)$ such that for $\lambda = \frac{1}{k+1}$, T satisfies the inequality*

$$\|k(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Tx\|\|y-T_\lambda y\|}{\|x-y\|} \tag{2.2}$$

for all $x, y \in X$ and $x \neq y$, then we say that T is an enriched Jaggi contraction map.

Here we note that every Jaggi contraction is a special case of enriched Jaggi contraction when $k = 0$. But, every enriched Jaggi contraction need not be a Jaggi contraction. The following example illustrates this fact.

Example 2.1. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by $Tx = 1 - \frac{3}{2}x, x \in \mathbb{R}$. We choose $k = 2$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. We now consider*

$$\begin{aligned}|k(x-y) + Tx - Ty| &= |2(x-y) + 1 - \tfrac{3}{2}x - 1 + \tfrac{3}{2}y| \\ &= \tfrac{1}{2}|x-y| \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|\frac{5}{2}x-1||\frac{5}{6}y-\frac{1}{3}|}{|x-y|} \\ &= \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x+\frac{3}{2}x-1||y-\frac{1}{6}y-\frac{1}{3}|}{|x-y|} \\ &= \alpha|x-y| + \beta\tfrac{|x-Tx||y-T_\lambda y|}{|x-y|},\end{aligned}$$

so that T satisfies the inequality (2.2) with $\alpha + \beta < 1$.
Hence T is an enriched Jaggi contraction map.
Now, by choosing $x = 0, y = \frac{2}{5}$, we have

$$\begin{aligned}|Tx - Ty| = |T0 - T(\tfrac{2}{5})| = \tfrac{3}{5} \nleq \alpha \cdot \tfrac{2}{5} + \beta \cdot 0 &= \alpha|0 - \tfrac{2}{5}| + \beta\tfrac{|0-T0|\cdot|\frac{2}{5}-T(\frac{2}{5})|}{|0-\frac{2}{5}|} \\ &= \alpha|x-y| + \beta\tfrac{|x-Tx||y-Ty|}{|x-y|},\end{aligned}$$

for any $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta < 1$.
Hence T is not a Jaggi contraction map.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be continuous. Assume that T is an enriched Jaggi contraction map. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, ...,$ converges to s (say) in X, and s is the unique fixed point of T_λ. Further, s is the unique fixed point of T.*

Proof. Let $x_0 \in X$. We consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, ...$.
For $\lambda = \frac{1}{k+1} < 1$, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (2.2) becomes
$\|(\frac{1}{\lambda} - 1)(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Tx\|\|y-T_\lambda y\|}{\|x-y\|}$ for all $x, y \in X$ for $x \neq y$.
i.e.,
$\|(1-\lambda)(x-y) + \lambda Tx - \lambda Ty\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|\lambda x-\lambda Tx\|\|y-T_\lambda y\|}{\|x-y\|}$, $x \neq y$ and hence

$$\|T_\lambda x - T_\lambda y\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|x-T_\lambda x\|\|y-T_\lambda y\|}{\|x-y\|} \text{ for all } x, y \in X \text{ and } x \neq y. \tag{2.3}$$

By taking $x = x_{n-1}$ and $y = x_n$ in (2.3), we get
$\|T_\lambda x_{n-1} - T_\lambda x_n\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\frac{\|x_{n-1}-T_\lambda x_{n-1}\|\|x_n - T_\lambda x_n\|}{\|x_{n-1}-x_n\|}$, *i.e.*,
$\|x_n - x_{n+1}\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\frac{\|x_{n-1}-x_n\|\|x_n-x_{n+1}\|}{\|x_{n-1}-x_n\|}$. This implies that
$\|x_n - x_{n+1}\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\|x_n - x_{n+1}\|$, so that
$\|x_n - x_{n+1}\| \leq \eta\|x_{n-1} - x_n\|$ for $n = 1, 2, ...$, where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Hence, inductively, it follows that
$\|x_n - x_{n+1}\| \leq \eta^n\|x_0 - x_1\|$ for $n = 1, 2, ...$.
Therefore it is easy to see that the sequence $\{x_n\}$ is Cauchy.
Since X is complete, we have $\lim\limits_{n\to\infty} x_n = s$ (say), $s \in X$.
Since T is continuous on X, we have T_λ is so and hence
$s = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} T_\lambda x_n = T_\lambda \lim\limits_{n\to\infty} x_n = T_\lambda s.$
Therefore s is a fixed point of T_λ.
Let t be another fixed point of T_λ and $s \neq t$. Now, from the inequality (2.3), we have

$$
\begin{aligned}
0 < \|s - t\| &= \|T_\lambda s - T_\lambda t\| \\
&\leq \alpha\lambda\|s - t\| + \beta\frac{\|s-T_\lambda s\|\|t-T_\lambda t\|}{\|s-t\|},
\end{aligned}
$$

which implies that
$0 < \|s - t\| \leq \alpha\lambda\|s - t\|$,
a contradiction.
Therefore $t = s$, and T_λ has a unique fixed point s.
Thus, it follows that T has a unique fixed point s in X. □

Remark. *If $k = 0$ and $\beta = 0$ in the inequality (2.2), then T is a contraction and in this case, contraction principle follows as a corollary to Theorem 2.1.*

Example 2.2. *Let $X = \mathbb{R}$ with the usual norm and we define $T : X \to X$ by $Tx = -2x - 3, x \in \mathbb{R}$. We choose $k = \frac{3}{2}$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. We now consider*

$$
\begin{aligned}
|k(x - y) + Tx - Ty| &= |\tfrac{3}{2}(x - y) - 2x - 3 - (-2y - 3)| \\
&= \tfrac{1}{2}|x - y| \\
&\leq \tfrac{1}{2}|x - y| + \tfrac{1}{3}\frac{|x-(-2x-3)||y-(-\frac{1}{5}y-\frac{6}{5})|}{|x-y|} \\
&= \alpha|x - y| + \beta\frac{|x-Tx||y-T_\lambda y|}{|x-y|}.
\end{aligned}
$$

Therefore T satisfies the inequality (2.2) of Theorem 2.1 with $\alpha + \beta < 1$ and '$-\frac{1}{3}$' is the unique fixed point of T.

Here we observe that T is not a contraction. So contraction mapping principle is not applicable.

For any positive integer p, we denote T^p, the composition of p number of selfmaps T. Here we note that $T^1 = T$. Also we denote $T^0 = I$, I the identity map of X. In this case, $T^0_\lambda = I$ for every $\lambda \in [0, 1]$.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$. Assume that T is an enriched Jaggi contraction map. Let $x_0 \in X$. If T^p is continuous for some positive integer p, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ by $x_{n+1} = T^p_\lambda x_n, n = 0, 1, 2, ...$.
Then by applying Theorem 2.1 to T^p_λ, we get that the sequence $\{x_n\}$ converges to s, and $T^p_\lambda(s) = s$, and this s is unique.
We now show that $T_\lambda(s) = s$.
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Then $\{x_{n_k}\}$ also converges to s. Now

$T_\lambda^p(s) = T_\lambda^p(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} T_\lambda^p x_{n_k} = \lim_{k\to\infty} x_{n_k+1} = s.$
Let r be the smallest positive integer such that $T_\lambda^r(s) = s$. Then $T_\lambda^i(s) \neq s$ for all $i = 1, 2, ..., r-1$.
For $i \in \{1, 2, ..., r-1, r\}$, we have

$$\begin{aligned}\|T_\lambda^i(s) - T_\lambda^{i-1}(s)\| &= \|T_\lambda(T_\lambda^{i-1}(s)) - T_\lambda(T_\lambda^{i-2}(s))\|\\ &\leq \alpha\lambda\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\| + \beta\tfrac{\|T_\lambda^{i-1}(s)-T_\lambda^i(s)\|\|T_\lambda^{i-2}(s)-T_\lambda^{i-1}(s)\|}{\|T_\lambda^{i-1}(s)-T_\lambda^{i-2}(s)\|}\\ &= \alpha\lambda\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\| + \beta\|T_\lambda^{i-1}(s) - T_\lambda^i(s)\|\end{aligned}$$

$$\|T_\lambda^i(s) - T_\lambda^{i-1}(s)\| \leq (\frac{\alpha\lambda}{1-\beta})\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\|. \tag{2.4}$$

If $r > 1$, then

$$\begin{aligned}\|T_\lambda(s) - s\| &= \|T_\lambda s - T_\lambda^r(s)\|\\ &= \|T_\lambda s - T_\lambda(T_\lambda^{r-1}(s))\|\\ &\leq \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\tfrac{\|s-T_\lambda(s)\|\|T_\lambda^{r-1}(s)-T_\lambda^r(s)\|}{\|s-T_\lambda^{r-1}(s)\|}\\ &= \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\tfrac{\|s-T_\lambda(s)\|\|T_\lambda^{r-1}(s)-s\|}{\|s-T_\lambda^{r-1}(s)\|}\\ &= \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\|s - T_\lambda(s)\| \text{ which implies that}\end{aligned}$$

$(1-\beta)\|s - T_\lambda(s)\| \leq \alpha\lambda\|s - T_\lambda^{r-1}(s)\|$. Therefore

$$\|s - T_\lambda(s)\| \leq (\frac{\alpha\lambda}{1-\beta})\|s - T_\lambda^{r-1}(s)\|. \tag{2.5}$$

Also, by (2.4) with $i = r$, we have

$$\begin{aligned}\|s - T_\lambda^{r-1}(s)\| &= \|T_\lambda^r(s) - T_\lambda^{r-1}(s)\|\\ &\leq (\tfrac{\alpha\lambda}{1-\beta})\|T_\lambda^{r-1}(s) - T_\lambda^{r-2}(s)\|.\end{aligned}$$

On repeated application of the inequality (2.4), we get

$$\begin{aligned}\|s - T_\lambda^{r-1}(s)\| = \|T_\lambda^r(s) - T_\lambda^{r-1}(s)\| &\leq (\tfrac{\alpha\lambda}{1-\beta})\|T_\lambda^{r-1}(s) - T_\lambda^{r-2}(s)\|\\ &\vdots\\ &\leq (\tfrac{\alpha\lambda}{1-\beta})^{r-1}\|T_\lambda(s) - T_\lambda^0(s)\|, \text{ and hence}\end{aligned}$$

$$\|s - T_\lambda^{r-1}(s)\| \leq (\frac{\alpha\lambda}{1-\beta})^{r-1}\|T_\lambda(s) - s\|, \text{ since } T_\lambda^0 \text{ is the identity map.} \tag{2.6}$$

From (2.5) and (2.6), we have
$\|s - T_\lambda(s)\| \leq (\frac{\alpha\lambda}{1-\beta})^r\|s - T_\lambda(s)\|,$
a contradiction, since $\frac{\alpha\lambda}{1-\beta} < 1$.
Therefore $T_\lambda s = s$.
Uniqueness of fixed point of T_λ follows as in the proof of Theorem 2.1.
Thus s is the unique fixed point of T. □

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$. Assume that there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and $k \in [0, \infty)$ such that for $\lambda = \frac{1}{k+1}$, and for some positive integer q, T satisfies*

$$\|k(x-y) + T^q x - T^q y\| \leq \alpha\|x - y\| + \beta\frac{\|x - T^q x\|\|y - T_\lambda^q y\|}{\|x - y\|} \tag{2.7}$$

for all $x, y \in X$ and $x \neq y$; where $T_\lambda^q(x) = (1-\lambda)x + \lambda T^q x$.

If T^q is continuous then T has a unique fixed point in X.

Proof. By Theorem 2.1, T_λ^q has a unique fixed point s (say) in X. Then $T_\lambda(s) = T_\lambda(T_\lambda^q(s)) = T_\lambda^q(T_\lambda(s))$. Hence $T_\lambda(s)$ is also a fixed point of T_λ^q.
Now, by the uniqueness of fixed point of T_λ^q, we have $T_\lambda(s) = s$.
Since T_λ^q has a unique fixed point s, it follows that s is a unique fixed point of T_λ.
Hence it follows that s is the unique fixed point of T. □

The following example shows that Theorem 2.3 is more general than Theorem 2.1.

Example 2.3. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{1}{3} & \text{if } x \in [0,\infty) \\ -x & \text{if } x \in (-\infty, 0) \end{cases}.$$

Then $T^2x = \frac{1}{3}$ for all $x \in \mathbb{R}$ so that T^2 is continuous on X. Indeed, inequality (2.7) of Theorem 2.3 holds with $q = 2$, $k = \frac{1}{2}, \alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. For, for any $x \in [0,\infty)$, $y \in (-\infty, 0)$, we have

$$\begin{aligned} |k(x-y) + T^2x - T^2y| &= |\tfrac{1}{2}(x-y) + \tfrac{1}{3} - \tfrac{1}{3}| \\ &= \tfrac{1}{2}|x-y| \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x-\frac{1}{3}||y-\frac{1}{3}|}{|x-y|} \\ &= \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x-T^2x||y-T_\lambda^2 y|}{|x-y|} \\ &= \alpha|x-y| + \beta\tfrac{|x-T^2x||y-T_\lambda^2 y|}{|x-y|}. \end{aligned}$$

Thus T^2 satisfies the hypotheses of Theorem 2.3 and '$\frac{1}{3}$' is the unique fixed point of T. Here we observe that T is not continuous and so Theorem 2.1 is not applicable.

Definition 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and $k \in [0,+\infty)$ such that for $\lambda = \frac{1}{k+1}$, T satisfy the inequality*

$$\|k(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|y-Ty\|(1+\|x-T_\lambda x\|)}{1+\|x-y\|} \tag{2.8}$$

for all $x, y \in X$, then we say that T is an enriched Dass and Gupta contraction map.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be continuous. Assume that T is an enriched Dass and Gupta contraction map. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, \ldots$ converges to q (say) in X, and q is the unique fixed point of T.*

Proof. The proof of this theorem is similar to that of Theorem 2.1.

□

Definition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let $S, T : X \to X$. If there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and $k \in [0,+\infty)$ such that for $\lambda = \frac{1}{k+1}$, S and T satisfy the inequality*

$$\|k(x-y) + Sx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Sx\|\|y-T_\lambda y\|}{\|x-y\|} \tag{2.9}$$

for all $x, y \in X$ and $x \neq y$ then we say that the pair (S, T) is an enriched Jaggi contraction pair of maps. Here we note that if $S = T$ in the inequality (2.9), then T is an enriched Jaggi contraction map.

In the following, we extend Theorem 2.1 to a pair of selfmaps.

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a Banach space. Let $S, T : X \to X$. Suppose that the pair (S, T) is an enriched Jaggi contraction pair of maps. Let $x_0 \in X$. We define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$x_n = \begin{cases} S_\lambda x_{2m-1}, & if\ n = 2m,\ m = 1, 2, ... \\ T_\lambda x_{2m}, & if\ n = 2m+1,\ m = 0, 1, 2... \ . \end{cases}$$

Then $\{x_n\}$ converges to u (say) in X, and u is the unique common fixed point of S and T, provided S and T are continuous.

Proof. Let $\lambda = \frac{1}{k+1} < 1$. In this case, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (2.9) becomes
$\|(\frac{1}{\lambda} - 1)(x-y) + Sx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Sx\|\|y-T_\lambda y\|}{\|x-y\|}$ for all $x, y \in X, x \neq y$. *i.e.,*
$\|(1-\lambda)(x-y) + Sx - Ty\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|\lambda x-\lambda Sx\|\|y-T_\lambda y\|}{\|x-y\|}$. *i.e.,*
$\|S_\lambda x - T_\lambda y\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|x-S_\lambda x\|\|y-T_\lambda y\|}{\|x-y\|}$ for any $x, y \in X$ and $x \neq y$.
Case (i) $n = 2m$. In this case, we consider

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{2m+1} - x_{2m}\| \\ &= \|T_\lambda x_{2m} - S_\lambda x_{2m-1}\| \\ &= \|S_\lambda x_{2m-1} - T_\lambda x_{2m}\| \\ &\leq \alpha\lambda\|x_{2m-1} - x_{2m}\| + \beta\frac{\|x_{2m-1}-S_\lambda x_{2m-1}\|\|x_{2m}-T_\lambda x_{2m}\|}{\|x_{2m-1}-x_{2m}\|} \\ &= \alpha\lambda\|x_{2m-1} - x_{2m}\| + \beta\frac{\|x_{2m-1}-x_{2m}\|\|x_{2m}-x_{2m+1}\|}{\|x_{2m-1}-x_{2m}\|}\end{aligned}$$

$(1-\beta)\|x_{2m} - x_{2m+1}\| \leq \alpha\lambda\|x_{2m-1} - x_{2m}\|$. Thus, we have
$\|x_{2m+1} - x_{2m}\| \leq \eta\|x_{2m} - x_{2m-1}\|$ where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Case (ii) $n = 2m+1$. In this case, we consider

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{2m+2} - x_{2m+1}\| \\ &= \|S_\lambda x_{2m+1} - T_\lambda x_{2m}\| \\ &\leq \alpha\lambda\|x_{2m+1} - x_{2m}\| + \beta\frac{\|x_{2m+1}-S_\lambda x_{2m+1}\|\|x_{2m}-T_\lambda x_{2m}\|}{\|x_{2m+1}-x_{2m}\|} \\ &= \alpha\lambda\|x_{2m+1} - x_{2m}\| + \beta\frac{\|x_{2m+1}-x_{2m+2}\|\|x_{2m}-x_{2m+1}\|}{\|x_{2m+1}-x_{2m}\|}\end{aligned}$$

$(1-\beta)\|x_{2m+2} - x_{2m+1}\| \leq \alpha\lambda\|x_{2m+1} - x_{2m}\|$
That is
$\|x_{2m+2} - x_{2m+1}\| \leq \eta\|x_{2m+1} - x_{2m}\|$ where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Thus from Case (i) and Case (ii), it follows that
$\|x_{n+1} - x_n\| \leq \eta\|x_n - x_{n-1}\|$ for all $n = 1, 2, 3, ...$.
Now, inductively, it follows that
$\|x_{n+1} - x_n\| \leq \eta^n\|x_1 - x_0\|$ for all $n = 1, 2, ...$.
Thus the sequence $\{x_n\}$ is Cauchy.
Since X is complete, we have $\lim_{n\to\infty} x_n = u$ (say), $u \in X$.
Suppose that S is continuous. So S_λ is continuous on X.
$u = \lim_{m\to\infty} x_{2m} = \lim_{m\to\infty} S_\lambda x_{2m-1} = S_\lambda \lim_{m\to\infty} x_{2m-1} = S_\lambda u$.
Therefore u is a fixed point of S_λ.
Suppose that T is continuous. So T_λ is continuous on X.
$u = \lim_{m\to\infty} x_{2m+1} = \lim_{m\to\infty} T_\lambda x_{2m} = T_\lambda \lim_{m\to\infty} x_{2m} = T_\lambda u$.
Therefore u is a common fixed point of T_λ and S_λ, and hence u is a common fixed point of S and T.

Uniqueness of this common fixed point follows trivially from the inequality (2.9). □

Remark. *Theorem 2.1 follows by choosing $S = T$ in Theorem 2.5.*

3. Fixed points of almost $(k, a, b, \lambda)-$enriched CRR contraction maps

Definition 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $k \in (0, +\infty), L \geq 0$ and $a, b \geq 0$ satisfying $a + 2b < 1$ such that*

$$\|k(x-y)+Tx-Ty\| \leq a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|) + L\min\{\|y-T_\lambda x\|, \tfrac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\} \tag{3.1}$$

for all $x, y \in X$ with $\lambda = \frac{1}{k+1}$, then we say that T is an almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be an almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, ...$ converges to p (say) in X, and p is the unique fixed point of T.*

Proof. Let $x_0 \in X$. We consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, ...$.
For $\lambda = \frac{1}{k+1} < 1$, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (3.1) becomes

$$\|(\tfrac{1}{\lambda}-1)(x-y)+Tx-Ty\| \leq a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|) + L\min\{\|y-T_\lambda x\|, \tfrac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}$$

for all $x, y \in X$. Therefore

$$\|(1-\lambda)(x-y)+\lambda Tx-\lambda Ty\| \leq \lambda a\|x-y\| + b(\|\lambda x-\lambda Tx\| + \|\lambda y-\lambda Ty\|) + \lambda L\min\{\|y-T_\lambda x\|, \tfrac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}.$$

That is

$$\|T_\lambda x-T_\lambda y\| \leq \lambda a\|x-y\| + b(\|x-T_\lambda x\| + \|y-T_\lambda y\|) + \lambda L\min\{\|y-T_\lambda x\|, \frac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}. \tag{3.2}$$

By taking $x = x_{n-1}$ and $y = x_n$ in (3.2), we get

$$\|T_\lambda x_{n-1}-T_\lambda x_n\| \leq \lambda a\|x_{n-1}-x_n\| + b(\|x_{n-1}-T_\lambda x_{n-1}\| + \|x_n-T_\lambda x_n\|) + \lambda L\min\{\|x_n-T_\lambda x_{n-1}\|, \tfrac{\|x_{n-1}-T_\lambda x_{n-1}\|[1+\|x_{n-1}-T_\lambda x_n\|]}{1+\|x_{n-1}-x_n\|}\}$$

which implies that

$$\begin{aligned}\|x_n-x_{n+1}\| &\leq \lambda a\|x_{n-1}-x_n\| + b(\|x_{n-1}-x_n\| + \|x_n-x_{n+1}\|) + \lambda L\min\{\|x_n-x_n\|, \tfrac{\|x_n-x_{n+1}\|[1+\|x_{n-1}-x_{n+1}\|]}{1+\|x_{n-1}-x_n\|}\} \\ &\leq a\|x_{n-1}-x_n\| + b(\|x_{n-1}-x_n\| + \|x_n-x_{n+1}\|) + \lambda L\min\{0, \tfrac{\|x_n-x_{n+1}\|[1+\|x_{n-1}-x_{n+1}\|]}{1+\|x_{n-1}-x_n\|}\}\end{aligned}$$

so that
$(1-b)\|x_n-x_{n+1}\| \leq (a+b)\|x_{n-1}-x_n\|$
$\|x_n-x_{n+1}\| \leq \delta\|x_{n-1}-x_n\|$ where $\delta = \frac{a+b}{1-b} < 1$.
Inductively, it follows that
$\|x_n-x_{n+1}\| \leq \delta^n\|x_0-x_1\|$ for $n = 1, 2, ...$.
Therefore $\{x_n\}$ is Cauchy. Since X is complete, we have $\lim_{n\to\infty} x_n = p$ (say), $p \in X$.
Now we show that p is the fixed point of T_λ.
We consider

$$\begin{aligned}\|p-T_\lambda p\| &\leq \|p-x_{n+1}\| + \|x_{n+1}-T_\lambda p\| \\ &= \|p-T_\lambda x_n\| + \|T_\lambda x_n-T_\lambda p\| \\ &\leq \|p-T_\lambda x_n\| + \lambda a\|x_n-p\| + b(\|x_n-T_\lambda x_n\| + \|p-T_\lambda p\|) + \lambda L\min\{\|p-T_\lambda x_n\|, \tfrac{\|x_n-T_\lambda x_n\|[1+\|x_n-T_\lambda p\|]}{1+\|x_n-p\|}\}.\end{aligned}$$

On letting $n \to \infty$, we get
$\|p-T_\lambda p\| \leq \|p-p\| + a\|p-p\| + b(\|p-p\| + \|p-T_\lambda p\|) + L\min\{\|p-p\|, \frac{\|p-p\|[1+\|p-T_\lambda p\|]}{1+\|p-p\|}\}$
$\leq b\|p-T_\lambda p\|$ so that
$(1-b)\|p-T_\lambda p\| \leq 0$. Since $(1-b) > 0$, it follows that
$\|p-T_\lambda p\| = 0$ and hence $T_\lambda p = p$.
Therefore p is a fixed point of T_λ.
Let q be another fixed point of T_λ and $q \neq p$. Then
$0 < \|p-q\| = \|T_\lambda p - T_\lambda q\| \leq a\|p-q\| + b(\|p-T_\lambda p\| + \|q-T_\lambda q\|) + \lambda L\min\{\|q-T_\lambda p\|, \frac{\|p-T_\lambda p\|[1+\|p-T_\lambda q\|]}{1+\|p-q\|}\}$
so that
$\|p-q\| \leq a\|p-q\|$,
a contradiction.
Therefore $p = q$.
Therefore T_λ has a unique fixed point. Thus, it follows that T has a unique fixed point in X. □

Remark. *Theorem 3.1 extends Theorem 1.2 to the case of almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.*

Example 3.1. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{8}, & 0 \leq x < 2 \\ 0, & (-\infty, 0) \cup [2, \infty) \end{cases}.$$

We choose $k = \frac{1}{2}, a = \frac{1}{2}$ and $b = \frac{1}{5}$ with $a + 2b < 1$.
Let $x \in [0, 2), y \in [2, \infty)$ We now consider

$$\begin{aligned} |k(x-y) + Tx - Ty| &= |\tfrac{1}{2}(x-y) + \tfrac{x}{8} - 0| \\ &= |\tfrac{1}{2}(x-y) + \tfrac{x}{8}| \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{7}{40}x \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{7}{40}x + \tfrac{1}{5}|y| + L\min\{|y - \tfrac{5}{12}x|, \tfrac{|x-\frac{1}{3}y|}{1+|x-y|}\} \\ &= \tfrac{1}{2}|x-y| + \tfrac{1}{5}(|x - \tfrac{x}{8}| + |y-0|) + L\min\{|y - \tfrac{5}{12}x|, \tfrac{|x-\frac{1}{3}y|}{1+|x-y|}\} \\ &= a|x-y| + b(|x-Tx| + |y-Ty|) + L\min\{|y - T_\lambda x|, \tfrac{|x-T_\lambda y|}{1+|x-y|}\}. \end{aligned}$$

Therefore inequality (3.1) holds for any $L \geq 0$. Hence T is an almost $(\frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{2}{3})-$enriched CRR contraction map on $\mathbb{R}$. So T satisfies the hypotheses of Theorem 3.1 and '0' is the unique fixed point of T.

4. Convergence of sequence of fixed points of enriched contraction maps

In the following, $\mathbb{Z}^+$ denotes the set of all natural numbers.

Theorem 4.1. *Let $\{T_n\}$ be a sequence of $(k, a)-$enriched contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...$, which exists by Theorem 1.1. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - a$.*

Proof. First suppose that $u_n \to u$ as $n \to \infty$. Assume that $Tu \neq u$.
Let $\epsilon = \|Tu - u\| > 0$. Then there exists $N_1 \in \mathbb{Z}^+$ such that $\|u_n - u\| < \frac{\epsilon}{2(1+a+k)}$ for all $n \geq N_1$.

Since $T_n \to T$ uniformly, we have, there exists $N_2 \in \mathbb{Z}^+$ such that $\|T_n u - Tu\| < \frac{\epsilon}{2}$ for all $n \geq N_2$ and for all $u \in X$.
Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have

$$\begin{aligned}0 < \epsilon = \|u - Tu\| &\leq \|u - u_n\| + \|u_n - T_n u\| + \|T_n u - Tu\| \\ &= \|u_n - u\| + \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\ &= \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\ &\leq \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\| \\ &\leq \|u_n - u\| + a\|u_n - u\| + k\|u_n - u\| + \|T_n u - Tu\| \\ &= (1 + a + k)\|u_n - u\| + \|T_n u - Tu\| \\ &< (1 + a + k)\frac{\epsilon}{2(1+a+k)} + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

a contradiction.
Therefore $Tu = u$.
Conversely, assume that $Tu = u$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $\|T_n u - Tu\| < \frac{\epsilon}{c}$ for all $n \geq N$ and for all $u \in X$, where $c = \frac{1}{1-a-k} > 0$.
Let $n \geq N$. Then

$$\begin{aligned}\|u_n - u\| &= \|T_n u_n - Tu\| \\ &\leq \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\ &= \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\ &\leq a\|u_n - u\| + k\|u_n - u\| + \|T_n u - Tu\| \\ &= (a + k)\|u_n - u\| + \|T_n u - Tu\|\end{aligned}$$

$$(1 - a - k)\|u_n - u\| \leq \|T_n u - Tu\|$$

$$\|u_n - u\| \leq c\|T_n u - Tu\| < c.\frac{\epsilon}{c} = \epsilon.$$

Therefore $u_n \to u$ as $n \to \infty$.

Hence the theorem follows. □

Theorem 4.2. *Let $\{T_n\}$ be a sequence of enriched Jaggi contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...,$ which exists by Theorem 2.1. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - \alpha$.*

Proof. Follows as that of Theorem 4.1.

□

Theorem 4.3. *Let $\{T_n\}$ be a sequence of enriched Dass and Gupta contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...,$ which exists by Theorem 2.4. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - \alpha$.*

Proof. Suppose that $u_n \to u$ as $n \to \infty$.
Now, we consider

$$\begin{aligned}\|u - Tu\| &\leq \|u - u_n\| + \|u_n - T_n u\| + \|T_n u - Tu\| \\ &= \|u_n - u\| + \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\ &= \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\ &\leq \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\| \\ &\leq \|u_n - u\| + \alpha\|u_n - u\| + \beta\frac{\|u - T_n u\|(1 + \|u_n - (T_n)_\lambda u_n\|)}{1 + \|u - u_n\|} + k\|u_n - u\| \\ &\qquad + \|T_n u - Tu\| \\ &= \|u_n - u\| + \alpha\|u_n - u\| + \beta\frac{\|u - T_n u\|}{1 + \|u_n - u\|} + k\|u_n - u\| + \|T_n u - Tu\|, \text{ since}\end{aligned}$$

$$\|u_n - (T_n)_\lambda u_n\| = 0$$
$$\leq (1+\alpha+k)\|u_n - u\| + \beta\|u - T_n u\| + \|T_n u - Tu\|$$
$$\leq (1+\alpha+k)\|u_n - u\| + \beta[\|u - Tu\| + \|Tu - T_n u\|] + \|T_n u - Tu\|.$$

Therefore

$(1-\beta)\|u - Tu\| \leq (1+\alpha+k)\|u_n - u\| + (1+\beta)\|T_n u - Tu\|$, and hence

$\|u - Tu\| \leq \frac{1+\alpha+k}{1-\beta}\|u_n - u\| + \frac{1+\beta}{1-\beta}\|T_n u - Tu\| \to 0$ as $n \to \infty$, since $\{T_n\}$ converges to T uniformly.

Therefore $Tu = u$.

Conversely, we assume that $Tu = u$. We consider

$$\|u_n - u\| = \|T_n u_n - Tu\|$$
$$\leq \|T_n u_n - T_n u\| + \|T_n u - Tu\|$$
$$= \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\|$$
$$\leq \alpha\|u_n - u\| + \beta\frac{\|u - T_n u\|(1 + \|u_n - (T_n)_\lambda u_n\|)}{1 + \|u - u_n\|} + k\|u_n - u\| + \|T_n u - Tu\|$$
$$\leq \alpha\|u_n - u\| + \beta\|u - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\|$$
$$\leq \alpha\|u_n - u\| + \beta[\|u - Tu\| + \|Tu - T_n u\|] + k\|u_n - u\| + \|T_n u - Tu\|$$
$$= (\alpha + k)\|u_n - u\| + (1+\beta)\|T_n u - Tu\|, \text{ and hence}$$

$(1-\alpha-k)\|u_n - u\| \leq (1+\beta)\|T_n u - Tu\|$. Therefore

$\|u_n - u\| \leq c\|T_n u - Tu\| \to 0$ as $n \to \infty$, where $c = \frac{1+\beta}{1-\alpha-k}$ is a positive constant.

Therefore $u_n \to u$ as $n \to \infty$.

Hence the theorem follows. □

5. Conclusion

In this paper, we defined enriched Jaggi contraction map, enriched Dass and Gupta contraction map and almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$ in Banach spaces. It is noted that every Jaggi contraction is an enriched Jaggi contraction but its converse is not true (Example 2.1) so that enriched Jaggi contraction maps are more general than Jaggi contraction maps. We proved the existence and uniqueness of fixed points of enriched Jaggi contraction map (Theorem 2.1). We provided an example in support of Theorem 2.1 and we observed that T is not a contraction and contraction mapping principle is not applicable. Hence Theorem 2.1 generalizes contraction mapping principle. Further, we extended Theorem 2.1 in which T^p is continuous for some positive integer p (Theorem 2.2). Also, we extended Theorem 2.1 for the map T^q for some positive integer q (Thereoem 2.3). An example (Example 2.3) is provided where T^q is satisfies the inequality (2.7), but T is not continuous. Since T is not continuous, Theorem 2.1 is not applicable. Also, it is easy to see that we can extend Theorem 2.1 to enriched Dass and Gupta contraction map. Further, enriched Jaggi contraction is extended to a pair of selfmaps and proved the existence and uniqueness of common fixed points. Also, we proved the existence and uniqueness of fixed points of almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.

Also, we proved that the sequence of fixed points $\{u_n\}$ of the corresponding enriched contraction maps $\{T_n\}$ converges to the fixed point u of the uniform limit operator T of these enriched contraction maps $\{T_n\}$. Conversely, if u is a fixed point of T then $\{u_n\}$ converges to u under certain assumption. Further, we extended this technique to a sequence of enriched Jaggi contraction maps and enriched Dass and Gupta contraction maps.

In the direction of future research, we would like to suggest the following:

1) Some new fixed point results can be investigated by introducing more general

enriched contraction conditions.
2) Some new fixed point results for multi-valued contractions can be investigated.

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