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Research Article

Statistically order compact operators on Riesz spaces

Abdullah Aydın

Department of Mathematics, Muş Alparslan University, Muş, Turkey

Abstract

This research paper introduces and establishes the concept of compact operators in the context of Riesz spaces, specifically considering statistical order convergence. We define statistical order compact operators as operators that map statistical order bounded sequences to sequences with statistical order convergent subsequences. Additionally, we define statistical M-weakly compact operators. By utilizing these non-topological concepts, we derive some new results pertaining to these operators.

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1. Introduction

Fast [11] and Steinhaus [19] independently introduced the concept of statistical convergence in 1951. Thereafter, it garnered the attention of numerous mathematicians and became the focus of their studies (cf. [3, 4, 11, 14, 19]). On the other hand, Riesz initially introduced the concept of vector lattice (or Riesz space) in 1921 [18], which has found numerous applications in diverse disciplines such as economics, operator theory, and measure theory (cf. [1–6, 13, 22]). Convergences in Riesz spaces, such as order convergence and statistical order convergence, are not topological in general (cf. [12, Thm.2]). However, certain types of continuous operators, such as statistical order continuous operators, have been defined with respect to order convergence (cf. [6, 8, 16]). The aim of this study is to introduce the notion of statistically compact operators on Riesz spaces, as there is currently no comprehensive study of compact operators in the theory of Riesz space with respect to statistical convergence.

Recall that an ordered vector space is referred to as a *Riesz space* if the infimum and supremum of all pairs x and y exist. In this paper, unless stated otherwise, Riesz spaces are denoted by the letters E and F. In Riesz spaces, a sequence (x_n) is called;

- *order bounded* if $|x_n| \leq u$ holds for each $n \in \mathbb{N}$ and for some positive elements $\theta \leq u$.
- *order Cauchy* sequence if the inequality $|x_{n+k} - x_n| \leq q_n$ holds for all $n, k \in \mathbb{N}$ and for some sequences $q_n \downarrow \theta$.

Email address: aaydin.aabdullah@gmail.com
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- *order convergent* to a vector x if the inequality $|x_n - x| \leq q_n$ holds for a sequence $q_n \downarrow \theta$, denoted as $x_n \xrightarrow{o} x$.

Throughout this paper, operators are assumed to be linear, and vector spaces are considered real. The notation $\mathcal{L}(E, F)$ represents the collection of all operators from E to F. Let $T \in \mathcal{L}(E, F)$ be an operator. The following definitions are used:

(a) T is termed *sequentially order compact* if the image of an order bounded sequence possesses an order convergent subsequence.
(b) T is referred to as an *order bounded operator* if the range of each order bounded set is order bounded.
(c) If $x_n \xrightarrow{o} x$ implies $Tx_n \xrightarrow{o} Tx$, then T is called a *σ-order continuous operator*.
(d) T is called an *order continuous operator* if $Tx_\alpha \xrightarrow{o} Tx$ holds for every $x_\alpha \xrightarrow{o} x$.

The collection $\mathcal{L}_b(E, F)$, denoting all order bounded operators between E and F, forms a vector space. Moreover, it follows from [2, Thm.1.18] that $\mathcal{L}_b(E, F)$ is also a Dedekind complete Riesz space whenever F possesses the Dedekind completeness property, i.e., every order bounded subset has a supremum and infimum.

Let K be a subset of natural numbers. The *asymptotic density* of K, denoted as $\delta(K)$, is defined as the limit (if it exists):

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{K}(\{k \leq n : k \in K\}),$$

where $\mathcal{K}$ represents the cardinality. Furthermore, a sequence (x_k) in a Riesz space is said to be *statistically convergent* to x if the following limit (if it exists) holds:

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{K}(\{k \leq n : |x_k - x| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

Consider a sequence (x_n) in a Riesz space. The following definitions are introduced (refer to [3, 4, 20] for further details):

- *Statistically order decreasing* (denoted as $x_n \downarrow^{st_o} \theta$) to θ if there exists a set $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that $x_k \downarrow \theta$ on K.
- *Statistically order convergent* (denoted as $x_n \xrightarrow{st_o} x$) to $x \in E$ if $|x_k - x| \leq q_k$ holds for a sequence $q_n \downarrow^{st_o} \theta$ with a set $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and for all $k \in K$.
- *Statistically order bounded* (denoted as st_o-bounded) if there exists a positive vector $\theta \leq u \in E_+$ and an index set $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $|x_k| \leq u$ for every $k \in K$.

It should be noted that every statistically order convergent sequence and order bounded sequence is st_o-bounded. However, the converse is not generally true. Additionally, the following notions from [8] are worth recalling. An operator T between Riesz spaces is considered:

- *Statistically σ-order continuous* if $Tx_n \xrightarrow{st_o} Tx$ holds in F whenever $x_n \xrightarrow{st_o} x$ in E.
- *Statistically order bounded* if it maps statistically order bounded sequences to statistically order bounded sequences.

Lastly, it is worth mentioning that a positive vector $u \geq 0$ in a Riesz space is termed an *atom* if $x \wedge y$ implies either $x = \theta$ or $y = \theta$ for each pair $x, y \in [\theta, u]$.

2. Statistical order compact operators

Recall that an operator T defined between normed spaces is called *compact* if the image of the unit ball under T is relatively compact, or equivalently, if every norm-bounded sequence has a subsequence whose image under T converges. Now, let's define compact operators with respect to statistical order convergence among Riesz spaces.

Definition 2.1. Let $T \in \mathcal{L}(E,F)$. T is called *statistically order compact* (or st_o-compact) if every statistically order bounded sequence (x_n) in E has a subsequence (x_{j_n}) such that

$$T(x_{j_n}) \xrightarrow{\mathrm{st_o}} z$$

holds in F for some $z \in F$.

Note that T is st_o-compact if and only if there exists a further subsequence $(x_{j_{n_k}})$ of (x_{j_n}) such that $T(x_{j_{n_k}}) \xrightarrow{o} z$. In this paper, we denote $\mathcal{L}_{st_o}(E,F)$ as the set of all st_o-compact operators between E and F.

Remark 2.2. It can be easily observed that a sequentially order compact operator is statistically order compact because an order-convergent sequence is statistically order convergent.

Since a statistically order convergent sequence is not order convergent in general (see for example [20, Exam.3]), the converse of Remark 2.2 does not hold in general. Now, let's prove that $\mathcal{L}_{st_o}(E,F)$ is a vector space.

Lemma 2.3. *The set $\mathcal{L}_{st_o}(E,F)$ is a vector space.*

Proof. Suppose $S,T \in \mathcal{L}_{st_o}(E,F)$ and (x_n) is an st_o-bounded sequence in E. Then, there exist subsequences $(x_i)_{i\in I}$ and $(x_j)_{j\in J}$ of (x_n) with subsets $\delta(I) = \delta(J) = 1$ such that $T(x_i) \xrightarrow{\mathrm{st_o}} y$ and $S(x_j) \xrightarrow{\mathrm{st_o}} z$ for some $y,z \in F$. By applying [20, Thm.6], we have $(T+S)(x_m) \xrightarrow{\mathrm{st_o}} y+z$ for $m \in M := I \cap J$, where $\delta(M) = 1$. Thus, we have shown the st_o-compactness of $T+S$. Similarly, we can prove the case of multiplication by a scalar. □

Proposition 2.4. *Take three operators $R,S \in L(E)$ and $T \in \mathcal{L}_{st_o}(E)$.*

(i) $S \circ T \in \mathcal{L}_{st_o}(E)$ holds whenever S is statistical σ-order continuous.
(ii) $T \circ P \in \mathcal{L}_{st_o}(E)$ holds if P is st_o-bounded.

Proof. (i) Assume that S is a statistically σ-order continuous operator and (x_n) is an st_o-bounded sequence. Then, by using the st_o-compactness of T, we have a subsequence $(x_k)_{k\in K}$ of (x_n) such that $T(x_k) \xrightarrow{\mathrm{st_o}} y$ for some $y \in E$ and some index set $\delta(K) = 1$. Now, by applying st_o-continuity of S, we have $S(T(x_k)) \xrightarrow{\mathrm{st_o}} S(y)$, i.e., $(S \circ T)(x_k) \xrightarrow{\mathrm{st_o}} S(y)$. So, we get the desired result.

(ii) Suppose that P is an st_o-bounded operator and (x_n) is an st_o-bounded sequence. Then, (Px_n) is also an st_o-bounded sequence. So, we have a subsequence $(Px_k)_{k\in K}$ of (Px_n) such that $T(Rx_k) \xrightarrow{\mathrm{st_o}} y$ for some $y \in E$ and for an index set $\delta(K) = 1$ because T is st_o-compact operator. Therefore, we obtain that $T \circ P$ is an st_o-compact operator. □

Proposition 2.5. *Every operator $T \in \mathcal{L}(E,F)$ is st_o-compact if T is st_o-bounded and F is an atomic KB-space.*

Proof. Assume (x_n) is an st_o-bounded sequence in E. Since T is st_o-bounded, $T(x_n)$ is an st_o-bounded sequence in F. Thus, there exists an index set K with $\delta(K) = 1$ such that $|T(x_k)| \leq u$ for all $k \in K$ and for some $u \in F_+$, i.e., $T(x_n)$ has an order bounded subsequence $(T(x_k))_{k\in K}$ with $\delta(K) = 1$. According to [7, Rem.6], there exists a further subsequence $(T(x_m))_{m\in M}$ of $(T(x_k))_{k\in K}$ such that $T(x_m) \xrightarrow{o} y$ for the same vector $y \in F$. Since $\delta(M) = 1$, we conclude that $T \in \mathcal{L}_{st_o}(E,F)$. □

Theorem 2.6. *Every statistically order compact operator is statistically order bounded.*

Proof. Assume $T \in \mathcal{L}_{st_o}(E,F)$ is not statistically order bounded. By contradiction, we can find a sequence (x_n) that is statistically order bounded in E, but (Tx_n) is not statistically order bounded in F. Since every order bounded sequence is statistically order bounded, this implies that (Tx_n) is not order bounded. Therefore, for all positive elements w in $F_+ := \{x \in F : \theta \leq x\}$, there exist some indexes $n_w \in \mathbb{N}$ such that

$$|Tx_{n_w}| \nleq w.$$

On the other hand, using the st_o-compactness of T, we have a subsequence $(x_k)_{k\in K}$ of (x_n) such that $T(x_k) \xrightarrow{\mathrm{st_o}} z$ for some $z \in F$ because (x_n) is a statistically order bounded sequence in E. Therefore, there exists a further sequence $q_k \downarrow^{\mathrm{st_o}} \theta$ in F with an index subset $J \subseteq K$ such that $\delta(J) = 1$ and

$$|Tx_j - z| \leq q_j$$

for each $j \in J$. This implies that (Tx_j) is order bounded in F since $q_j \downarrow \theta$ on J. However, there exists $u \in F_+$ such that

$$|Tx_j| \nleq u$$

for some $j_u \in J$, which contradicts the assumption. Hence, every statistically order compact operator must be statistically order bounded. □

In general, the converse of Theorem 2.6 may not hold. The following example illustrates that both st_o-bounded and statistically σ-order continuous operators may not be st_o-compact.

Example 2.7. Consider the Riesz space $L_1[0,1]$. It can be observed that the identity operator I on $L_1[0,1]$ is both statistically order bounded and statistically σ-order continuous. However, I is not statistically order compact. To see this, let (r_n) be the sequence of Rademacher functions defined on $[0,1]$ as $r_n(t) := \mathrm{sgn}(\sin(2^n\pi t))$ for each $n \in \mathbb{N}$ and $t \in [0,1]$. Since $|r_n| = 1$ for every $n \in \mathbb{N}$, (r_n) is statistically order bounded. Suppose that $(r_k)_{k\in K}$ is a subsequence of (r_n) such that $r_k \xrightarrow{\mathrm{st_o}} f$ for some f in $L_1[0,1]$. This implies that there exists a further subsequence $(r_j)_{j\in J}$ of $(r_k)_{k\in K}$ with $\delta(J) = 1$ and $r_j \xrightarrow{\mathrm{o}} f$. Let $j_0 \in J$ be fixed. Then, for every $j \geq j_0$, we have $\int_0^1 r_{j_0} r_j d\mu = 0$. Since $r_{j_0} r_j \xrightarrow{\mathrm{o}} r_{j_0} r$, we have the following convergence:

$$\int_0^1 r_j r d\mu \to \int_0^1 r r d\mu = \int_0^1 r^2 d\mu.$$

By utilizing the order continuity of the integral, we conclude that $\int_0^1 r_j r d\mu = 0$. Thus, we obtain $\int_0^1 r^2 d\mu = 0$. However, this contradicts the fact that $|r| = 1$. Hence, we demonstrate that (r_n) does not have any st_o-convergent subsequence, and therefore, I is not st_o-compact.

Example 2.8. Not every statistically order compact operator is order bounded. An example illustrating this fact can be found in [10, Exam.6], where the given operator is sequentially order compact. Thus, by applying Remark 2.2, it is also statistically order compact. However, it is not an order bounded operator.

Consider [15, Exam.4.2] for the following example.

Example 2.9. A statistically order compact operator may not be statistically σ-order continuous. Consider any ultrafilter $\mathcal{U}$ on the Boolean algebra $\mathcal{B}$ consisting of the Borel subsets of $[0,1]$ modulo null sets. The operator $T_{\mathcal{U}}$ from $L_\infty[0,1]$ to $\mathbb{R}$ defined by

$$T_{\mathcal{U}}(S) := \lim_{A\in\mathcal{U}} \frac{1}{\mu(A)} \int_A S d\mu$$

is statistically order compact but not statistically σ-order continuous.

3. More results of st_o-compact operators

We remind that a lattice norm on Riesz spaces as a norm $\|\cdot\|$ that satisfies $\|x\| \leq \|y\|$ for all vectors $|x| \leq |y|$.

Theorem 3.1.

(i) If $T \in \mathcal{L}(E,F)$ is a compact operator, where E is a normed lattice and F is a Banach lattice, then T is st_o-compact.

(ii) If $T \in \mathcal{L}_{st_o}(E,F)$ is an operator, where E is an AM-space (i.e., $x \wedge y = 0$ implies $\|x \vee y\| = \|x\| \vee \|y\|$) with a strong norm unit and F is an order continuous normed lattice, then T is compact.

Proof. (i) Let (x_n) be an st_o-bounded sequence in E. This means there exists an index set K and some positive element $u \in E_+$ such that $|x_k| \leq u$ for all $k \in K$, which implies that $(x_k)_{k\in K}$ is order bounded. Since E is a normed lattice, we have $\|x_k\| \leq \|u\|$ for all $k \in K$. Thus, $(x_k)_{k\in K}$ is a norm bounded sequence in E. By the compactness of T, we have $T(x_m) \xrightarrow{\|\cdot\|} y$ for some subsequence $(x_m)_{m\in M}$ of $(x_k)_{k\in K}$ and $y \in F$. Applying a result from [21, Thm.VII.2.1], there exists a further subsequence $(x_i)_{i\in I}$ of $(x_m)_{m\in M}$ such that $T(x_i) \xrightarrow{o} y$ because F is a Banach lattice. Therefore, since $(x_k)_{k\in K}$ is order bounded and the subsequence $(x_i)_{i\in I}$ of $(x_k)_{k\in K}$ satisfies $T(x_i) \xrightarrow{o} y$, we obtain $\delta(I) = 1$, which means T is st_o-compact.

(ii) Consider an arbitrary norm bounded sequence (x_n) in E. Since E satisfies the AM-property with a strong norm unit, according to [22, p.490], (x_n) is also order bounded. Thus, (x_n) is an st_o-bounded sequence. By using the st_o-compactness of T, we have $Tx_k \xrightarrow{\mathrm{st_o}} y$ for some subsequence $(x_k)_{k\in K}$ of (x_n) with $\delta(K) = 1$ and for some $y \in F$. This implies that there is a further subsequence $(x_m)_{m\in M}$ of $(x_k)_{k\in K}$ satisfying $Tx_m \xrightarrow{o} y$ with the index set $\delta(M) = 1$. The convergence of $Tx_m \xrightarrow{\|\cdot\|} y$ can be obtained from the order continuous lattice norm. Therefore, T is compact. □

Consider an operator T defined on E as $T(x) = f(x)u$, where x belongs to E, f is an order bounded linear functional on E, and u is a fixed vector in F. This defines an operator T belonging to $\mathcal{L}_b(E,F)$, which is referred to as a *rank one operator* (cf. [2, p.64]).

Theorem 3.2. *Every st_o-bounded finite rank operator is st_o-compact.*

Proof. Let u be a fixed vector in F. Without loss of generality, assume that T is defined as $T(x) = f(x)u$ for all $x \in E$, where $f : E \to \mathbb{R}$ is an st_o-bounded functional. For an arbitrary st_o-bounded sequence (x_n) in E, $f(x_n)$ is st_o-bounded in $\mathbb{R}$ because f is an st_o-bounded functional. Hence, there exists a subsequence $(x_k)_{k\in K}$ of (x_n) such that $f(x_k)$ is bounded in $\mathbb{R}$ and $\delta(K) = 1$. By applying the Bolzano-Weierstrass Theorem, we obtain a further subsequence $(x_m)_{m\in M}$ of $(x_k)_{k\in K}$ such that $f(x_m) \to \alpha$ for some $\alpha \in \mathbb{R}$, where $\delta(M) = 1$. Thus, we observe the following inequality:

$$|T(x_m) - \alpha u| = |f(x_m)u - \alpha u| = |f(x_m) - \alpha||u| \to 0.$$

This is due to the Archimedean property of F and the fact that $|f(x_m) - \alpha| \to 0$. Therefore, we conclude that T is st_o-compact. □

Example 3.3. The space $\mathcal{L}_{st_o}(E,F)$ is not necessarily order closed. To illustrate this, consider an operator T defined as $T(e_n) = (r_n)^+$, where e_n denotes the standard unit basis in ℓ_1 and (r_n) represents the Rademacher function as given in Example 2.7. Let π_n be a projection and (S_n) be a sequence defined by $S_n(x) := (T \circ \pi_n)(x) = \sum_{k=1}^{n} x_k r_k^+$ for all $x \in \ell_1$. It is evident that each S_n is a finite rank one operator. By applying Theorem 3.2, we deduce that (S_n) is a sequence in $\mathcal{L}_{st_o}(\ell_1, L_1)$. However, it is clear that $S_n \xrightarrow{o} T$. Example 2.7 shows that (r_n) does not possess any st_o-convergent subsequence. Hence, we conclude that T is not st_o-compact.

Remind that an operator $T \in \mathcal{L}(E,F)$ satisfying the property $T(x \vee y) = T(x) \vee T(y)$ for each pair x and y in E is called *Riesz homomorphism* (or *lattice homomorphism*) (cf. [1, Def.1.30]).

Proposition 3.4. *Let T be a Riesz homomorphism in $\mathcal{L}_{st_o}(E,F)$, where F is an order complete Riesz space. If $\theta \leq S \leq T$ holds for some operator $S \in \mathcal{L}(E,F)$, then S is st_o-compact.*

Proof. Suppose that (x_n) is an st_o-bounded sequence in E. By selecting a subsequence, we assume that $(Tx_k) \xrightarrow{\text{st}_\text{o}} y$ for some $y \in F$, and consider a subsequence $(x_{k_n})_{k_n \in K}$ of (x_n) with $\delta(K) = 1$. By utilizing [13, Thm.18.2], we have the following inequality:

$$|Sx_{k_n} - Sx_{k_m}| \leq S(|x_{k_n} - x_{k_m}|) \leq T(|x_{k_n} - x_{k_m}|) = |Tx_{k_n} - Tx_{k_m}|$$

for all $k_n \in K$, as T is a Riesz homomorphism and S is a positive operator. Consequently, we deduce from the last inequality that $(Sx_{k_n})_{k_n \in K}$ is a statistical order Cauchy sequence due to $(Tx_{k_n} - Tx_{k_m}) \xrightarrow{\text{st}_\text{o}} \theta$. By utilizing the order completeness of F, we obtain $Sx_{k_n} \xrightarrow{\text{o}} z$ for some $z \in F$. Applying [20, p.7], we further conclude that $Sx_{k_n} \xrightarrow{\text{st}_\text{o}} z$ since $\delta(K) = 1$. Hence, S is an st_o-compact operator, which completes the proof. □

Theorem 3.5. *Let E be a Dedekind complete Riesz space, and T be a positive, statistically σ-order continuous, and st_o-compact operator on E. Then, $T \circ S_n \xrightarrow{\text{st}_\text{o}} T \circ S$ holds for any sequence (S_n) of statistically σ-order continuous and decreasing operators on E with $S_n \xrightarrow{\text{st}_\text{o}} S$ for some operator S on E.*

Proof. Suppose that T and (S_n) satisfy the assumptions of the theorem. It follows from the statistically σ-order continuity of T and S for each n that $T \circ S_n$ is a statistically σ-order continuous operator for every n. Moreover, since $S_n \xrightarrow{\text{st}_\text{o}} S$ holds for some operator S on E, we have a subsequence $(S_k)_{k \in K}$ of (S_n) with $\delta(K) = 1$ such that $S_k \xrightarrow{\text{o}} S$. Thus, by considering [21, Thm.VIII.2.3], we obtain $S_k x \xrightarrow{\text{o}} Sx$ for all vectors x in E. Therefore, $S_n x \xrightarrow{\text{st}_\text{o}} Sx$ for every $x \in E$. This implies that $T(S_n x) \xrightarrow{\text{st}_\text{o}} T(Sx)$ or $(T \circ S_n)(x) \xrightarrow{\text{st}_\text{o}} (T \circ S)(x)$ holds for each $x \in E$ since T is statistically σ-order continuous. Additionally, since (S_n) is decreasing and T is positive, then $(T \circ S_k)$ forms a decreasing sequence. Therefore, by applying [21, Thm.VIII.2.4], we obtain $T \circ S_k \xrightarrow{\text{st}_\text{o}} T \circ S$, i.e., $T \circ S_n \xrightarrow{\text{st}_\text{o}} T \circ S$. □

Theorem 3.6. *Let (T_j) be a sequence of statistically σ-order compact operators in $\mathcal{L}_b(E,F)$, where F is Dedekind complete. If (T_j) is statistically σ-order convergent to $T \in \mathcal{L}_b(E,F)$, then $T \in \mathcal{L}\sigma st_o(E,F)$.*

Proof. Assume that (x_n) is a statistically σ-order bounded sequence in E. This means that there exist an index set I and a positive element $u \geq 0$ in E such that $|x_i| \leq u$ holds for each $i \in I$. Using a standard diagonal argument, we can find a subsequence $(x_m)_{m \in M}$ of (x_n) such that $T_j x_m \xrightarrow{\text{st}_\text{o}} y_j$ for any $j \in \mathbb{N}$ and for some $y_j \in F$ with $\delta(M) = 1$ as $m \to \infty$, because T_j is statistically σ-order compact for each j. Since (T_j) is statistically σ-order convergent to T, we have a subsequence $(T_{j_k})_{j_k \in K}$ of (T_j) with index set $\delta(K) = 1$ such that $Tj_k \xrightarrow{\text{o}} T$ as $j_k \to \infty$. It can be observed that $T_{j_k} x_m \xrightarrow{\text{o}} y_{j_k}$ as $m \to \infty$ for each $j_k \in K$.

Now, we will show that $(y_{j_k})_{j_k \in K}$ is a statistically σ-order Cauchy sequence in F. Firstly, we observe the following inequality:

$$\begin{aligned} |y_{j_n} - y_{j_k}| &= |y_{j_n} - T_{j_n} x_m + T_{j_n} x_m - T_{j_k} x_m + T_{j_k} x_m - y_{j_k}| \\ &\leq |y_{j_n} - T_{j_n} x_m| + |T_{j_n} x_m - T_{j_k} x_m| + |T_{j_k} x_m - y_{j_k}|. \quad (*) \end{aligned}$$

for every $j_n, j_k \in K$. Then, we obtain that both the first and third terms in the last inequality converge to zero in the statistical σ-order as $j_n \to \infty$ and $j_k \to \infty$, respectively.

Using the Dedekind completeness of F, we can apply Theorem 1.18 from [2], and thus we have

$$|T_{j_n}x_m - T_{j_k}x_m| \leq |T_{j_n} - T_{j_k}|(|x_m|) \leq |T_{j_n} - T_{j_k}|(u)$$

for all $m \in I$. Considering [21, Thm.VIII.2.3], it follows from $T_{j_k} \xrightarrow{o} T$ in $L_b(E,F)$ that $|T_{j_n} - T_{j_k}|(u) \xrightarrow{o} 0$ in F as $j_k \to \infty$. Thus, it follows from $(*)$ that $|y_{j_n} - y_{j_k}| \xrightarrow{o} 0$ in F as $j_n, j_k \to \infty$. Therefore, (y_{j_k}) is a statistically σ-order Cauchy sequence in F. Using Remark 7.2 from [17], we know that Dedekind completeness implies order completeness. Therefore, F is order complete, and thus (y_{j_k}) is order convergent to some element y in F as $j_k \to \infty$. Hence, using [2, Thm.1.14], we have

$$\begin{aligned}|Tx_m - y| &\leq |Tx_m - T_{j_k}x_m + T_{j_k}x_m - y_{j_k} + y_{j_k} - y| \\ &\leq |T_{j_k} - T|(|x_m|) + |T_{j_k}x_m - y_{j_k}| + |y_{j_k} - y| \\ &\leq |T_{j_k} - T|(u) + |T_{j_k}x_m - y_{j_k}| + |y_{j_k} - y|.\end{aligned}$$

Fixing j_k, we take $m \to \infty$ to obtain

$$\limsup_{m\to\infty} |Tx_m - y| \leq |T_{j_k} - T|(u) + |y_{j_k} - y|.$$

Since j_k is arbitrary, we have $\limsup_{m\to\infty} |Tx_m - y| = 0$. Therefore, we have $|Tx_m - y| \xrightarrow{o} 0$, i.e., $Tx_n \xrightarrow{\mathrm{st}_o} y$. Thus, T is statistically σ-order compact as desired. □

Theorem 3.7. *Let $T \in \mathcal{L}_{\sigma st_o}(E,F)$ and $S \in \mathcal{L}(E,F)$ be a Riesz homomorphism. If F is an order continuous Banach lattice, then $S \circ T \in \mathcal{L}_{\sigma st_o}(E,F)$.*

Proof. Suppose that (x_n) is a statistically σ-order bounded sequence in E. Then, there exists a subsequence $(x_k)_{k\in K}$ of (x_n) with $\delta(K) = 1$ such that $T(x_k) \xrightarrow{\mathrm{st}_o} y$ for some $y \in F$, because $T \in \mathcal{L}_{\sigma st_o}(E,F)$. Thus, there exist a further sequence $q_k \downarrow^{st_o} \theta$ in F such that

$$|T(x_m) - y| \leq q_m$$

for each $m \in M$, where M is an index set with $\delta(M) = 1$. Since $q_m \downarrow 0$ holds and F has an order continuous lattice norm, it follows that $\|q_m\| \downarrow 0$. On the other hand, by applying [2, Thm.4.3], we have $\|S(q_m)\| \downarrow$ because every Riesz homomorphism is a positive operator. Now, by using [21, Thm.VII.2.1], there exists a subsequence $(q_j)_{j\in J}$ of $(q_m)_{m\in M}$ such that $S(q_j) \xrightarrow{o} 0$, i.e., $S(q_j) \downarrow 0$ because of the positivity of S, where $\delta(J) = 1$ by our assumption in this paper. Therefore, by considering [2, Thm.2.14], we have

$$|(S \circ T)(x_j) - S(y)| = S(|T(x_j) - y|) \leq S(q_j)$$

for all $j \in J$. Hence, we have $(S \circ T)(x_j) \xrightarrow{o} S(y)$, i.e., $(S \circ T)(x_n) \xrightarrow{\mathrm{st}_o} S(y)$. Therefore, we obtain the statistically σ-order compactness of $S \circ T$. □

Proposition 3.8. *Let E be a Banach lattice and F be a σ-order continuous Banach lattice. Then an st_o-compact operator from E to F is norm bounded.*

Proof. Suppose that T is not norm bounded. So, there exists a norm bounded sequence (x_n) in E such that $\|x_n\| \leq \frac{1}{3^n}$ and (Tx_n) is not norm bounded in F. It is clear that (x_n) is also st_o-bounded. Then, it follows from st_o-compactness of T that (x_n) has a subsequence $(x_k)_{k\in K}$ with $\delta(K) = 1$ such that $Tx_k \xrightarrow{\mathrm{st}_o} y$ for some $y \in F$. Thus, there exists a further subsequence $(x_m)_{m\in M}$ with $\delta(M) = 1$ of $(x_k)_{k\in K}$ such that $Tx_k \xrightarrow{o} y$. It follows from σ-order continuity norm on F that $Tx_k \xrightarrow{\|\cdot\|} y$, which is contradicting with $\|Tx_n\| \to \infty$. Thus, T is norm bounded. □

Proposition 3.9. *Let $T \in \mathcal{L}_{st_o}(E,F)$ and G be a regular, majorizing and order complete sublattice of F. If $T(E)$ is a subspace of G, then $T : E \to G$ is st_o-compact.*

Proof. Assume that (x_n) is an st_o-bounded sequence in E. Then, there exists a subsequence $(x_k)_{k\in K}$ of (x_n) with $\delta(K)=1$ such that $T(x_k)\xrightarrow{\mathrm{st_o}}y$ for some $y\in F$ because of $T\in \mathcal{L}st_o(E,F)$. Also, it follows from Theorem 2.6 that $T:E\to F$ is an st_o-bounded operator. Thus, since (x_n) is st_o-bounded in E, (Tx_n) is an st_o-bounded sequence in F. Then, (x_n) has a further subsequence $(x_m)_{m\in M}$ of (x_n) with $\delta(M)=1$ such that (Tx_m) is order bounded in F. Moreover, the subsequence (Tx_m) is order bounded in G because G is majorizing and $T(E)$ is a subspace of G. Now, by applying [9, Lem.27], we obtain that $Tx_m\xrightarrow{\mathrm{o}}y$ in G, i.e., $Tx_n\xrightarrow{\mathrm{st_o}}y$ in G. Thus, we get the desired result. □

4. Statistical M-weakly compact operators

Remind that any two elements x and y in Riesz spaces are called disjoint whenever $|x|\wedge|y|=0$. A norm-bounded operator from a normed lattice to a normed space is said to be M-weakly compact if the image of each disjoint norm-bounded sequence under this operator is norm convergent to zero. Motivated by this, we give the following notions.

Definition 4.1. An operator $T\in\mathcal{L}(E,F)$ is said to be *statistical M-weakly compact* (or shortly *st-M_w-compact*) if $Tx_n\xrightarrow{s}0$ holds for all disjoint st_o-bounded sequences (x_n).

Proposition 4.2. *A statistically σ-order continuous operator is st-M_w-compact.*

Proof. Suppose that $T\in\mathcal{L}(E,F)$ is a statistically σ-order continuous operator. Take a disjoint st_o-bounded sequence (x_n) in E. Then, (x_n) has an order-bounded subsequence $(x_k)_{k\in K}$ with $\delta(K)=1$. It follows from [6, Rem.10] that $x_k\xrightarrow{o}0$ because (x_k) is also a disjoint subsequence. Thus, we have $x_n\xrightarrow{s}0$ in E. By using the st_o-continuity of T, we have $Tx_n\xrightarrow{s}0$ in F. Therefore, T is *st-M_w*-compact. □

In the following work, we show that the domination property holds for *st-M_w*-compact operators.

Proposition 4.3. *If $S,T\in\mathcal{L}(E,F)$ satisfy $0\le S\le T$ and T is st-M_w-compact, then S is st-M_w-compact.*

Proof. Suppose that (x_n) is a disjoint st_o-bounded sequence in E. By applying the *st-M_w*-compactness of T, we have $Tx_n\xrightarrow{s}0$ in F. Thus, there exists a subsequence $(x_k)_{k\in K}$ of (x_n) with $\delta(K)=1$ such that $Tx_k\xrightarrow{o}0$. Since $0\le S|x_k|\le T|x_k|$ for all $k\in K$, we have $Sx_k\xrightarrow{o}0$ because the inequality $|Sx_k|\le S|x_k|$ holds for each $k\in K$. Therefore, $Sx_n\xrightarrow{s}0$, and thus, S is *st-M_w*-compact. □

Proposition 4.4. *Every st_o-bounded and M-weakly compact operator from a σ-order continuous normed lattice to an atomic normed lattice is st-M_w-compact.*

Proof. Suppose that $T\in\mathcal{L}(E,F)$ satisfies the conditions of the proposition. Let (x_n) be a disjoint st_o-bounded sequence in E. Then, it has an order-bounded subsequence $(x_k)_{k\in K}$ with $\delta(K)=1$. Thus, $(x_k)_{k\in K}$ is norm-bounded in E. Therefore, $\lim\limits_{k\to\infty}|Tx_k|=0$ because T is M-weakly compact and $(x_k)_{k\in K}$ is a disjoint sequence. On the other hand, since $(x_k)_{k\in K}$ is order-bounded, it is also st_o-bounded. Therefore, $(Tx_k)_{k\in K}$ is an st_o-bounded sequence in F because T is an st_o-bounded operator. Hence, it has an order-bounded subsequence $(Tx_m)_{m\in M}$ with $\delta(M)=1$. Take any atom $a\in F$. Then, we have the following inequality:

$$|f_a(Tx_m)|\le\|f_a\|\|Tx_m\|\to 0$$

Thus, we obtain $Tx_m\xrightarrow{o}0$ because F is atomic, and therefore, we have $Tx_n\xrightarrow{s}0$. Consequently, T is *st-M_w*-compact. □

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