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THE FELL APPROACH STRUCTURE

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Abstract. In the present paper we construct a new approach structure called Fell approach structure. We define the new structure by means of lower regular function frames and prove that the Top-coreflection of this new structure is the ordinary Fell topology. We also give analogue result for the extended Fell topology and investigate some properties of Fell approach structure.

1. Introduction

Hyperspaces of topological spaces were initiated by Felix Hausdorff (1868) and Leopold Vietoris (1891). The theory occupy an important place in the applications of convex analysis, optimization theory and the theory of Banach spaces. Hyperspaces of topological spaces are an important way of obtaining information on the structure of a topological space X. Although the most important and well-studied hyperspace topologies on $CL(X)$ are the Wijsman topology, the Hausdorff metric topology and the hit and miss topologies. These topologies are investigated in [6]. Lowen and Wuyts [16] investigated the corresponding approach structures of the Vietoris topology and the other are investigated by Lowen and Sioen in [11, 14]. In most of cases they obtained the well known hyperspace topologies as the Top-coreflections of their new constructed approach structures.

The Fell topology is also known as a useful construct in terms of applications, especially in convex analysis, probability theory and its applications to optimization [1, 2]. In this paper we construct a new approach structure in the setting of hyperspaces and we prove that its Top-coreflection is the well known Fell topology. We also investigate some properties of this new structure in the setting of approach theory.

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We refer to R.Lowen [12, 17] for extensive literature to study on approach spaces and we refer to G.Beer [4] for more information on hyperspace topologies.

2. Preliminaries

Throughout this work, given a nonempty set X, 2^X denotes the set of all subsets of X, $2^{(X)}$ denotes the set of all finite subsets of X. Given a topological space (X,τ) by $CL(X)$ we denote the set of all closed subsets of X and $K(X)$ represents the set of all compact subsets of X, in addition $\mathcal{W} = CL(X) \cup \{\emptyset\}$. The hit and miss sets of a subset A in X are defined as

$$A^- := \{B \in CL(X) \mid B \cap A \neq \emptyset\} \text{ and } A^+ := \{B \in CL(X) \mid B \subset A\},$$

respectively. We also consider $\mathbb{P}$:=[0,∞] with its usual order and complete lattice structure as an additive semigroup. For any $A \subset X$, the indicator of A is defined as

$$\begin{aligned} \theta_A : X &\longrightarrow \mathbb{P} \\ x &\longmapsto \theta_A(x) = \begin{cases} 0 & ,\ \mathrm{x} \in A \\ \infty & ,\ \mathrm{x} \notin A. \end{cases} \end{aligned}$$

For a Hausdorff space (X,τ), the lower-Vietoris topology τ_V^- and the upper-Vietoris topology τ_V^+ on $CL(X)$ are generated by the subbasis $\{V^- \mid V \in \tau\}$ and the basis $\{W^+ \mid W \in \tau\}$, respectively. The Vietoris topology is simply the supremum of its upper part and lower part, i.e. $\tau_V = \tau_V^- \bigvee \tau_V^+$ [6].

The upper-Fell topology τ_{Fell}^+ on $CL(X)$ is generated by the basis

$$\{W^+ \mid W \in \tau, W^c \in K(X)\}$$

and the **Fell topology** τ_{Fell} on $CL(X)$ is generated by the subbasis

$$\{V^- \mid V \in \tau\} \cup \{W^+ \mid W \in \tau, W^c \in K(X)\} \text{ [6].}$$

Approach spaces can be described in terms of several equivalent mathematical structures; such as distance, limit operator, gauge, approach system, upper hull operator and lower regular function frame. Now we recall the definition of lower regular function frame.

A **lower regular function frame** [17] is a collection of functions $\mathcal{L} \subseteq \mathbb{P}^X$ with the following properties:

(LR1) $\forall \mathfrak{K} \subseteq \mathcal{L} : \bigvee \mathfrak{K} \in \mathcal{L}$,

(LR2) $\forall \mathfrak{K} \subseteq \mathcal{L}$ such that $\mathfrak{K}$ is finite : $\bigwedge \mathfrak{K} \in \mathcal{L}$ (that is stable for finite infima),

(LR3) $\forall \mu \in \mathcal{L}$, $\forall \alpha \in \mathbb{P} : \mu + \alpha \in \mathcal{L}$ (that is translation invariant),

(LR4) $\forall \mu \in \mathcal{L}$, $\forall \alpha \in [0, \inf \mu] : \mu - \alpha \in \mathcal{L}$.

A basis for a lower regular function frame $\mathcal{L}$ is a collection $\mathcal{B} \subset \mathcal{L}$ such that any function in $\mathcal{L}$ can be obtained as a supremum of functions in $\mathcal{B}$. In addition while Lowen and Wuyts [16] introducing the Vietoris approach structure, they gave a notion of a basis and a subbasis for a lower regular function frame. If the collection $\mathfrak{B} \subset \mathbb{P}^X$ is stable for finite infima then $\mathfrak{B}$ is a subbasis for a lower regular function frame defined on X and if the subbasis $\mathfrak{B}$ is translation invariant then $\mathfrak{B}$ is a basis for a lower regular function frame on X. If $\mathfrak{B} \subset \mathbb{P}^X$, the smallest lower regular function frame containing $\mathfrak{B}$ (or the lower regular function frame generated by $\mathfrak{B}$) is defined as

$$[\mathfrak{B}] = \{\sup_{j\in J} \inf_{k\in K_j} \mu_{j,k} \mid \forall j \in J, \forall k \in K_j : K_j \text{ finite }, \mu_{j,k} \in \mathfrak{B}\}, \tag{1}$$

and in this case we call $\mathfrak{B}$ a subbasis of $[\mathfrak{B}]$, if moreover $\mathfrak{B}$ is closed for finite infima we call $\mathfrak{B}$ a basis for $[\mathfrak{B}]$.

In [12] it is proved that a lower regular function frame, a distance and an approach system are equivalent mathematical structures. In addition for a given lower regular function frame $\mathcal{L}$ the corresponding distance is defined as

$$\delta(x, A) = \sup\{\rho(x) \mid \rho \in \mathcal{L}, \rho_{|A=0}\} \tag{2}$$

and for a given distance δ the corresponding approach system $\mathcal{A}$ is defined as

$$\mathcal{A}(x) = \{\psi \in \mathbb{P}^X \mid \forall A \subset X : \inf_{y\in A} \psi(y) \leq \delta(x, A)\} \tag{3}$$

If (X, τ) is topological space, then

$$\mathcal{L}_\tau = \{\mu \in \mathbb{P}^X \mid \mu \text{ lower semicontinuous}\}$$

is a lower regular function frame on X. On the other hand; if there exists a topology τ on X such that $\mathcal{L} = \mathcal{L}_\tau$, then $(X, \mathcal{L})$ is called a **topological approach space** [12]. A function $f : (X, \mathcal{L}) \to (X^{'}, \mathcal{L}^{'})$ between approach spaces is called a contraction if for all $\nu \in \mathcal{L}^{'}$, $\nu \circ f \in \mathcal{L}$. The category whose objects are approach spaces and morphisms are contractions is denoted by **App**. App is a topological category and **Top** is embedded as a concretely coreflective subcategory of **App**. For any approach space $(X, \mathcal{L})$, Top-coreflection $\tau_{\mathcal{L}}^{tc}$ determined by $\mathcal{L}$ is the topology associated with the following topological closure operator:

$$cl_{\mathcal{L}}(A) = \Big\{x \in X \mid \sup_{\substack{\rho\in\mathcal{L} \\ \rho_{|A=0}}} \rho(x) = 0\Big\} \ ; \ A \subset X. \tag{4}$$

Note that the equality (4) can be written as

$$cl_{\mathcal{L}}(A) = \bigcap_{\substack{\rho\in\mathcal{L} \\ \rho_{|A}=0}} \{\rho = 0\} \tag{5}$$

Before describing the construction process of the Fell approach structure, let us give the definition of Vietoris approach structure investigated in [16] by means of regular function frames. If $\mu \in \mathbb{P}^X$, then the functions $\mu^\wedge$ and $\mu^\vee$ are defined as

$$\begin{aligned} \mu^\wedge : CL(X) &\longrightarrow \mathbb{P} \\ A &\longmapsto \mu^\wedge(A) = \inf_{x\in A} \mu(x) \end{aligned} \qquad \begin{aligned} \mu^\vee : CL(X) &\longrightarrow \mathbb{P} \\ A &\longmapsto \mu^\vee(A) = \sup_{x\in A} \mu(x). \end{aligned}$$

Lowen and Wuyts obtained in [16] that for the function θ_A of $A \in CL(X)$, $\theta_A^{\wedge} = \theta_{A^-}$ and $\theta_A^{\vee} = \theta_{A^+}$ and for a subcollection $\mathcal{A}$ in $CL(X)$,

$$\theta_{\cap\mathcal{A}} = \sup_{A\in\mathcal{A}} \theta_A \text{ and } \theta_{\cup\mathcal{A}} = \inf_{A\in\mathcal{A}} \theta_A. \tag{6}$$

Given an approach space $(X, \mathcal{L})$, $\mathcal{L}^{\bigwedge} = \{\mu^{\wedge} \mid \mu \in \mathcal{L}\}$ is a basis for a lower regular function frame. The corresponding lower regular function frame is

$$\mathcal{L}_V^{\bigwedge} = \{\sup_{j\in J} \mu_j^{\wedge} \mid \forall j \in J : \mu_j \in \mathcal{L}\}.$$

This approach structure is called **Vietoris $\bigwedge$-structure**. Moreover, $\mathcal{L}^{\bigvee} = \{\mu^{\vee} | \mu \in \mathcal{L}\}$ is a subbasis for a lower regular function frame. The corresponding lower regular function frame is

$$\mathcal{L}_V^{\bigvee} = \{\sup_{j\in J} \inf_{k\in I_j} \mu_{j,k}^{\vee} \mid J \neq \emptyset, \forall j,k: \ I_j \subset J \ finite, \ \mu_{j,k} \in \mathcal{L}\}.$$

This approach structure is called **Vietoris $\bigvee$-structure**. Finally, the **Vietoris approach structure** is a lower regular function frame with the subbasis $\mathcal{L}^{\bigwedge} \cup \mathcal{L}^{\bigvee}$.

We denote the expression “ such that ” by “ s.t. ” briefly.

3. The Fell Approach Structure

In this section we construct a new approach structure corresponding to the Fell topology and investigate its properties. Here, $CL(X)$ and $K(X)$ denote the families of the closed and the compact subsets, respectively, of the Top-coreflection $\tau_{\mathcal{L}}^{tc}$ of the approach structure $\mathcal{L}$. To construct Fell approach structure, we modify the function $\mu^{\wedge}$ defined by Lowen and Wuyts [16] using compact sets. Let $\mu \in \mathbb{P}^X$ and B∈ K(X), we define the function

$$\mu_B^{\wedge} : CL(X) \longrightarrow \mathbb{P}$$
$$A \longmapsto \mu_B^{\wedge}(A) = \inf_{x\in A\cap B} \mu(x).$$

In the sequel the considered approach spaces are assumed to be Hausdorff approach spaces [15] $(X, \mathcal{L})$, that are the approach spaces such that their Top-coreflections are Hausdorff. In the following result we proved that being a Hausdorff approach space can be characterized by means of lower regular function frames.

Proposition 1. *For an approach space $(X, \mathcal{L})$, the following properties are equivalent.*

(i) $(X, \tau_{\mathcal{L}}^{tc})$ *is Hausdorff.*

(ii) $x \neq y \Longrightarrow (\exists \rho, \mu \in \mathcal{L} \ni \rho(x) > 0, \rho(y) = 0$ *and* $\mu(x) = 0, \mu(y) > 0)$.

Proof. Let $(X, \tau_{\mathcal{L}}^{tc})$ be a Hausdorff space. Then

$$\exists W, G \in \tau_{\mathcal{L}}^{tc} \text{ s.t. } x \in W, y \in G \text{ and } W \cap G = \emptyset$$

Since $W \in \tau_{\mathcal{L}}^{tc}$, we know that $y \in X - W = cl_{\mathcal{L}}(X - W)$ and $x \notin X - W$. Thus by (4) it is clear that

$$\forall \rho \in \mathcal{L} \text{ s.t. } \rho_{|X-W=0} : \rho(y) = 0$$

and

$$\exists \rho' \in \mathcal{L} \text{ s.t. } \rho'_{|X-W=0} : \rho'(x) > 0.$$

Similarly, since $G \in \tau_{\mathcal{L}}^{tc}$, $x \in X - G$ and $y \notin X - G$, one can obtain that

$$\forall \mu \in \mathcal{L} \text{ s.t. } \mu_{|X-G=0} : \mu(x) = 0$$

and

$$\exists \mu' \in \mathcal{L} \text{ s.t. } \mu'_{|X-G=0} : \mu'(y) > 0.$$

On the other hand, when (ii) holds we have three posibilities: If $\rho(x) < \mu(y)$, then by Proposition 2.2.8 in [17], $y \in \mu^{-1}(]\rho(x), +\infty[) \in \tau_{\mathcal{L}}^{tc}$ and $x \in \mu^{-1}([0, \rho(x)[) \in \tau_{\mathcal{L}}^{tc}$. Moreover,

$$\mu^{-1}(]\rho(x), +\infty[) \cap \mu^{-1}([0, \rho(x)[) = \emptyset$$

Hence $(X, \tau_{\mathcal{L}}^{tc})$ is Hausdorff. Similarly, if $\mu(y) < \rho(x)$ one can easily obtain the same fact. And if $\mu(y) = \rho(x)$, by the assumption since we have that both $\mu(y)$, $\rho(x)$ are positive, there exist a real number r such that $0 < r < \mu(y)$. Then $x \in \mu^{-1}([0, r[) \in \tau_{\mathcal{L}}^{tc}$ and $y \in \mu^{-1}(]r, +\infty[) \in \tau_{\mathcal{L}}^{tc}$. Moreover,

$$\mu^{-1}([0, r[) \cap \mu^{-1}(]r, +\infty[) = \emptyset$$

which completes the proof. □

Remark 1. *In [7] and [8] Baran and Qasim gave different definitions of T_0 and T_1 approach spaces. We hope that the characterization given in Proposition 1 will lead a way to give an analogue definition of T_2 spaces (Hausdorff spaces).*

The following result gives some basic properties of the modified function given in the beginning of this chapter.

Proposition 2. *If $(X, \mathcal{L})$ is an approach space, then the following statements are valid.*

(i) $\forall A, C \in CL(X)$, $\forall \mathcal{B} \subset K(X)$ *s.t.* $|\mathcal{B}| < \infty$:

$$\min_{B \in \mathcal{B}} \inf_{x \in B \cap C} \theta_A(x) = \inf_{x \in (\cup \mathcal{B}) \cap C} \theta_A(x),$$

(ii) $\forall A \in CL(X), \forall B \in K(X) : (\theta_A)^{\wedge}_B = \theta_{(A \cap B)^-}$,

(iii) $\forall \mathcal{A} \subset CL(X)$, $\forall B \in K(X) : (\theta_{\cup \mathcal{A}})^{\wedge}_B = \inf_{A \in \mathcal{A}} (\theta_A)^{\wedge}_B$,

(iv) $\forall A \in CL(X)$, $\mathcal{B} \subset K(X)$ *and* $|\mathcal{B}| < \infty : (\theta_A)^{\wedge}_{\cup \mathcal{B}} = \min_{B \in \mathcal{B}} (\theta_A)^{\wedge}_B$.

Proof. **(i)** It is straightforward. **(ii)** Let $C \in CL(X)$. Since $(\theta_A)^{\wedge}_B$ can only take on two values, ∞ or 0, we must consider two possible cases. Whenever $(\theta_A)^{\wedge}_B(C) = \infty$ then it means that $B \cap C = \emptyset$ or $A \cap B \cap C = \emptyset$. In both cases, clearly $C \notin (A \cap B)^-$ and thus $\theta_{(A\cap B)^-}(C) = \infty$. Also, whenever $\theta_{(A\cap B)^-}(C) = \infty$, one can easily show that $(\theta_A)^{\wedge}_B(C) = \infty$. For the second possibility, $(\theta_A)^{\wedge}_B(C) = 0$ iff $A \cap B \cap C \neq \emptyset$ which means that $C \in (A \cap B)^-$. Hence $\theta_{(A\cap B)^-}(C) = 0$.
(iii) Let $C \in CL(X)$. By (6) we obtain

$$(\theta_{\cup\mathcal{A}})^{\wedge}_B(C) = \inf_{x\in B\cap C} \theta_{\cup\mathcal{A}}(x) = \inf_{x\in B\cap C} \inf_{A\in\mathcal{A}} \theta_A(x)$$

and so if $B \cap C = \emptyset$, then

$$\inf_{A\in\mathcal{A}} (\theta_A)^{\wedge}_B(C) = \inf_{A\in\mathcal{A}} \inf_{x\in B\cap C} \theta_A(x) = \infty.$$

If $B \cap C \neq \emptyset$, then by (6) we obtain

$$\begin{aligned}(\theta_{\cup\mathcal{A}})^{\wedge}_B(C) &= \inf_{x\in B\cap C} \theta_{\cup\mathcal{A}}(x)\\ &= \inf_{x\in B\cap C} \inf_{A\in\mathcal{A}} \theta_A(x)\\ &= \inf_{A\in\mathcal{A}} \inf_{x\in B\cap C} \theta_A(x)\\ &= \inf_{A\in\mathcal{A}} (\theta_{\mathcal{A}})^{\wedge}_B(C).\end{aligned}$$

(iv) Let $C \in CL(X)$. For the finite subcollection $\mathcal{B} \subset K(X)$, if $(\cup\mathcal{B}) \cap C = \emptyset$, then

$$(\theta_A)^{\wedge}_{\cup\mathcal{B}}(C) = \inf_{x\in(\cup\mathcal{B})\cap C} \theta_A(x) = \infty$$

and

$$\min_{B\in\mathcal{B}} (\theta_A)^{\wedge}_B(C) = \min_{B\in\mathcal{B}} \inf_{x\in B\cap C} \theta_A(x) = \infty.$$

If $(\cup\mathcal{B}) \cap C \neq \emptyset$, then by (i) we obtain

$$\begin{aligned}\min_{B\in\mathcal{B}} (\theta_A)^{\wedge}_B(C) &= \min_{B\in\mathcal{B}} \inf_{x\in B\cap C} \theta_A(x)\\ &= \inf_{x\in(\cup B)\cap C} \theta_A(x)\\ &= (\theta_A)^{\wedge}_{\cup\mathcal{B}}(C).\end{aligned}$$

□

Proposition 3. *In an approach space $(X, \mathcal{L})$, the collection*

$$\mathcal{L}^{\bigwedge_{Fell}} = \{\mu^{\wedge}_B \mid \mu \in \mathcal{L}, B \in K(X)\}$$

is a subbasis for a lower regular function frame on $CL(X)$ and the corresponding lower regular function frame is

$$\mathcal{L}^{\bigwedge}_{Fell} = \left\{ \sup_{j\in J} \inf_{\substack{\mu\in\mathcal{L}_j\\ B\in K_j}} \mu^{\wedge}_B \mid J \neq \emptyset, \mathcal{L}_j \subset \mathcal{L}, K_j \subset K(X), \mathcal{L}_j \text{ and } K_j \text{ are finite} \right\}.$$

Proof. For all $\mu \in \mathcal{L}, B \in K(X)$, $\alpha > 0$ and $A \in CL(X)$, clearly

$$(\mu_B^{\wedge} + \alpha)(A) = \mu_B^{\wedge}(A) + \alpha = \inf_{x \in A \cap B} \mu(x) + \alpha = \inf_{x \in A \cap B} (\mu + \alpha)(x) = (\mu + \alpha)_B^{\wedge}(A).$$

Since $\mathcal{L}$ is translation invariant, we have $(\mu+\alpha)_B^{\wedge} \in \mathcal{L}^{\bigwedge_{Fell}}$ thus $\mathcal{L}^{\bigwedge_{Fell}}$ is translation invariant. Therefore, $\mathcal{L}^{\bigwedge_{Fell}}$ is a subbasis for a lower regular function frame. Thus we obtaine the following family;

$$\left\{ \inf_{\substack{\mu \in \mathcal{L}_j \\ B \in K_j}} \mu_B^{\wedge} \mid \mathcal{L}_j \subset \mathcal{L}, K_j \subset K(X), \mathcal{L}_j \textit{ and } K_j \textit{ are finite} \right\},$$

which is a basis for a lower regular function frame, and the lower regular function frame generated by this basis is

$$\mathcal{L}_{Fell}^{\bigwedge} = \left\{ \sup_{j \in J} \inf_{\substack{\mu \in \mathcal{L}_j \\ B \in K_j}} \mu_B^{\wedge} \mid J \neq \emptyset, \mathcal{L}_j \subset \mathcal{L}, K_j \subset K(X), \mathcal{L}_j \textit{ and } K_j \textit{ are finite} \right\}.$$

□

We call the approach structure $\mathcal{L}_{Fell}^{\bigwedge}$ as **Fell $\bigwedge$- approach structure**.

Theorem 1. *The collection $\mathcal{L}^{\bigvee} \cup \mathcal{L}^{\bigwedge_{Fell}}$ is a subbasis for the lower regular function frame;*

$$\mathcal{L}_{Fell} = \left\{ \sup_{j \in J} \Big(\inf_{\substack{\mu \in \mathcal{L}_j \\ B \in K_j}} \mu_B^{\wedge} \bigwedge \inf_{\mu \in \mathcal{L}_{t_j}} \mu^{\vee} \Big) \mid \mathcal{L}_j, \mathcal{L}_{t_j} \subset \mathcal{L}, K_j \subset K(X), \right.$$
$$\left. \mathcal{L}_j, \mathcal{L}_{t_j} \textit{ and } K_j \textit{ are finite} \right\}.$$

Proof. Since $\mathcal{L}^{\bigvee}$ and $\mathcal{L}^{\bigwedge_{Fell}}$ are both translation invariant, so $\mathcal{L}^{\bigvee} \cup \mathcal{L}^{\bigwedge_{Fell}}$ is. Thus this union is a subbasis for a lower regular function frame. Hence $\mathcal{L}^{\bigvee} \cup \mathcal{L}^{\bigwedge_{Fell}}$ generates a lower regular function frame (see (1)) which coincides with $\mathcal{L}_{Fell}$. □

We call the approach structure $\mathcal{L}_{Fell}$ as **Fell approach structure**. Now we should point out that this generalization is meaningful by introducing its relation with the ordinary Fell topology.

If $\mathcal{L}$ is a lower regular function frame, then it was shown in [16] that

$$\forall \mu \in \mathcal{L} : \{\mu = 0\}^{+} = \{\mu^{\vee} = 0\}. \tag{7}$$

The following lemma gives analogue equalities for our modified functions $\mu_B^{\wedge}$ whenever $\mu \in \mathcal{L}$, $B \in K(X)$.

Lemma 1. *In an approach space $(X, \mathcal{L})$, the following holds*
(i) $\forall \mu \in \mathcal{L}, \forall B \in K(X)$: $\{\mu_B^{\wedge} = 0\} = \big(\{\mu = 0\} \bigcap B\big)^{-}$,

(ii) *For all* $K \in K(X)$,

$$\left(\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}\right)^{-}=\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}^{-}=\bigcap_{\rho\in\mathcal{J}}\{\rho_K^{\wedge}=0\}$$

where $\mathcal{J}=\{\rho\in\mathcal{L}\mid \rho_{|_K}=0\}$.

Proof. **(i)** To prove the equality we shall show that $\mu:(X,\tau_{\mathcal{L}}^{tc})\to\mathbb{P}$ is lower semicontinuous. For an arbitrary $\alpha>0$ if $x\notin\{\mu\le\alpha\}$ since we can consider the mapping $\rho:=(\mu-\alpha)\bigvee 0$ that lies in $\mathcal{L}$, $x\notin\{\rho=0\}$. Thus by (5)

$$x\notin\bigcap_{\substack{\rho\in\mathcal{L}\\ \rho_{|_{\{\mu\le\alpha\}}}=0}}\{\rho=0\}=cl_{\tau}\{\mu\le\alpha\}.$$

Therefore $\{\mu\le\alpha\}$ is a closed subset in the Top-coreflection $\tau_{\mathcal{L}}^{tc}$ of $\mathcal{L}$. Hence μ is lower semicontinuous. Now let us consider the claimed equality:

$$A\in\{\mu_B^{\wedge}=0\}\iff\inf_{x\in A\cap B}\mu(x)=0.$$

By the fact that a lower semicontinuous mapping takes on its infimum value on a compact set [9], we obtain

$$\begin{aligned}\inf_{x\in A\cap B}\mu(x)=0 &\iff \exists x\in A\cap B:\mu(x)=0\\ &\iff A\cap B\cap\{\mu=0\}\neq\emptyset\\ &\iff A\in(\{\mu=0\}\cap B)^{-}.\end{aligned}$$

(ii) Let $K\in K(X)$, then

$$\begin{aligned}A\in\left(\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}\right)^{-} &\iff A\bigcap\left(\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}\right)\neq\emptyset\\ &\iff \exists a\in A\text{ and }a\in\{\rho=0\}\text{ for all }\rho\in\mathcal{J}\\ &\iff A\cap\{\rho=0\}\neq\emptyset\text{ for all }\rho\in\mathcal{J}\\ &\iff A\in\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}^{-}.\end{aligned}$$

For the second equality

$$\begin{aligned}A\in\bigcap_{\rho\in\mathcal{J}}\{\rho=0\}^{-} &\implies \forall\rho\in\mathcal{J}:A\in\{\rho=0\}^{-}\\ &\implies \forall\rho\in\mathcal{J}:\exists a\in A\text{ s.t. }\rho(a)=0\\ &\implies \forall\rho\in\mathcal{J}:\inf_{a\in A}\rho(a)=0.\end{aligned}$$

Moreover, since $K \in CL(X)$, by (5) we know that

$$K = cl_{\mathcal{L}}(K) = \bigcap_{\rho \in \mathcal{J}} \{\rho = 0\}.$$

Thus, if $A \in K^-$, then the first equality provides that

$$\forall \rho \in \mathcal{J} : \inf_{a \in A \cap K} \rho(a) = 0$$

thus

$$\forall \rho \in \mathcal{J} : A \in \{\rho_K^{\wedge} = 0\}.$$

Consequently, we obtain that $A \in \bigcap_{\rho \in \mathcal{J}} \{\rho_K^{\wedge} = 0\}$. Conversely, if $A \in \bigcap_{\rho \in \mathcal{J}} \{\rho_K^{\wedge} = 0\}$, then

$$\forall \rho \in \mathcal{J} : \inf_{a \in A \cap K} \rho(a) = 0.$$

By the lower semicontinuity of ρ and compactness of $A \cap K$,

$$\exists a \in A \cap K \text{ s.t. } \rho(a) = 0 \text{ for all } \rho \in \mathcal{J}.$$

Therefore

$$\forall \rho \in J : A \cap \{\rho = 0\} \neq \emptyset.$$

Hence, $A \in \bigcap_{\rho \in \mathcal{J}} \{\rho = 0\}^-$ which completes the proof.

□

Remark 2. *Lowen and Wuyts [16] proved that if $(X, \mathcal{L})$ is a topological approach space, then $(CL(X), \mathcal{L}_V^{\bigwedge}), (CL(X), \mathcal{L}_V^{\bigvee})$ and $(CL(X), \mathcal{L}_V)$ are topological approach spaces.*

With the following theorem we investigate the analogue fact for our new structures.

Theorem 2. *Whenever $\mathcal{L}$ is a topological approach structure on X, $\mathcal{L}_{Fell}^{\wedge}$ and $\mathcal{L}_{Fell}$ are topological approach structures on $CL(X)$.*

Proof. By Proposition 2.1.2 (5) in [17] (page 93-94), it suffices to prove that $\theta_{\{\mu_B^{\wedge}=0\}} \in \mathcal{L}_{Fell}^{\wedge}$ for all $\mu \in \mathcal{L}$ and $B \in K(X)$. With respect to the same theorem (i) we know that $\theta_{\{\mu \leq \varepsilon\}} \in \mathcal{L}$ for all $\varepsilon > 0$. Thus $(\theta_{\{\mu \leq \varepsilon\}})_B^{\wedge} \in \mathcal{L}_{Fell}^{\wedge}$ for all $\varepsilon > 0$ and $B \in K(X)$. By Proposition 2 (ii) we have $(\theta_{\{\mu \leq \varepsilon\}})_B^{\wedge} = \theta_{(\{\mu \leq \varepsilon\} \cap B)^-}$. Therefore by (LR1); in order to complete the proof it is sufficient to show that $\theta_{\{\mu_B^{\wedge}=0\}} = \sup_{\varepsilon > 0} \theta_{(\{\mu \leq \varepsilon\} \cap B)^-}$. Since the indicator function takes on only two values, we shall consider both of the possibilities. Let $A \in CL(X)$

$$\begin{aligned} \theta_{\{\mu_B^{\wedge}=0\}}(A) = 0 &\iff A \in \{\mu_B^{\wedge} = 0\} \\ &\iff \forall \varepsilon > 0 : \exists x_\varepsilon \in A \cap B \text{ s.t. } \mu(x_\varepsilon) \leq \varepsilon \\ &\iff \forall \varepsilon > 0 : A \cap B \cap \{\mu \leq \varepsilon\} \neq \emptyset \\ &\iff \forall \varepsilon > 0 : A \in (B \cap \{\mu \leq \varepsilon\})^- \end{aligned}$$

$$\begin{aligned} &\Longleftrightarrow \quad \forall \varepsilon > 0 : \theta_{(B \cap \{\mu \le \varepsilon\})^-}(A) = 0 \\ &\Longleftrightarrow \quad \sup_{\varepsilon > 0} \theta_{(B \cap \{\mu \le \varepsilon\})^-}(A) = 0. \end{aligned}$$

In addition

$$\begin{aligned} \sup_{\varepsilon > 0} \theta_{(\{\mu \le \varepsilon\} \cap B)^-}(A) = \infty \quad &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } \theta_{(\{\mu \le \varepsilon\} \cap B)^-}(A) = \infty \\ &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } A \notin (\{\mu \le \varepsilon\} \cap B)^- \\ &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } A \cap (\{\mu \le \varepsilon\} \cap B) = \emptyset \\ &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } A \cap B \subset \{\mu > \varepsilon\} \\ &\Longrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } \inf_{x \in A \cap B} \mu(x) \ge \varepsilon \\ &\Longrightarrow \quad \theta_{\{\mu_B^\wedge = 0\}}(A) = \infty. \end{aligned}$$

On the other hand if $\theta_{\{\mu_B^\wedge = 0\}}(A) = \infty$, then

$$\begin{aligned} A \notin \{\mu_B^\wedge = 0\} \quad &\Longrightarrow \quad \inf_{x \in A \cap B} \mu(x) > 0 \\ &\Longrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } \inf_{x \in A \cap B} \mu(x) > \varepsilon \\ &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } A \cap B \cap (\{\mu \le \varepsilon\}) = \emptyset \\ &\Longleftrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } A \notin (\{\mu \le \varepsilon\} \cap B)^- \\ &\Longrightarrow \quad \exists \varepsilon > 0 \text{ s.t. } \theta_{(\{\mu \le \varepsilon\} \cap B)^-}(A) = \infty \\ &\Longrightarrow \quad \sup_{\varepsilon > 0} \theta_{(\{\mu \le \varepsilon\} \cap B)^-}(A) = \infty \end{aligned}$$

Hence $\mathcal{L}_{Fell}^\wedge$ is a topological approach structure. Since $\mathcal{L}_{Fell}^\wedge$ and $\mathcal{L}_V^\vee$ are topological approach structures [16], one can obtain easily that $\mathcal{L}_{Fell}$ is a topological approach structure . □

The facts given in the following lemma are expressed by Lowen and Wuyts in [16] (see page 288 line 23). There $\mathcal{L}$ is expressed as a regular function frame and in [17] that structure is renamed as lower regular function frame.

Lemma 2. *Let $\mathcal{L}$ be a lower regular function frame on X then*
(i) *If $\mathcal{B}$ is a basis for $\mathcal{L}$, then $\mathcal{C} := \{\{\rho = 0\} \mid \rho \in \mathcal{B}\}$ is a basis for the collection of closed subsets of $\tau_{\mathcal{L}}^{tc}$.*
(ii) *If $\mathcal{S}$ is a subbasis for $\mathcal{L}$, then $\mathcal{T} := \{\{\mu = 0\} \mid \mu \in \mathcal{S}\}$ is a subbasis for the collection of closed subsets of $\tau_{\mathcal{L}}^{tc}$ [16].*

Remark 3. *It was proved by Lowen and Wuyts in [16] that Top-coreflections of $\mathcal{L}_V^{\bigwedge}, \mathcal{L}_V^{\bigvee}$ and $\mathcal{L}_V$ are τ_V^+, τ_V^- and τ_V, respectively. Lowen and Wuyts also showed that if $(X, \mathcal{L})$ is any approach space, Top-coreflection of $\mathcal{L}_V^{\bigvee}$ coincides with $(\tau_{\mathcal{L}}^{tc})_V^-$, whereas there is no relation between Top-coreflections of $\mathcal{L}_V^{\bigwedge}, \mathcal{L}_V$ and $(\tau_{\mathcal{L}}^{tc})_V^+, (\tau_{\mathcal{L}}^{tc})_V$, respectively on the whole of CL(X). Nevertheless, the following equalities hold only on $K(X)$, that is*

$$\tau^{tc}_{\mathcal{L}^{\wedge}_V} = (\tau^{tc}_{\mathcal{L}})^{+}_V \text{ and } \tau^{tc}_{\mathcal{L}_V} = (\tau^{tc}_{\mathcal{L}})_V.$$

Now we show that the Top-coreflection $\tau^{tc}_{\mathcal{L}^{\wedge}_{Fell}}$ of the approach structure $\mathcal{L}^{\wedge}_{Fell}$ is the upper Fell topology on CL(X) and Top-coreflection $\tau^{tc}_{\mathcal{L}_{Fell}}$ of the Fell approach structure $\mathcal{L}_{Fell}$ is the Fell topology on CL(X).

The following theorem is the main result of this paper.

Theorem 3. *For a lower regular function frame $\mathcal{L}$ on X, the following properties hold.*

(i) $\tau^{tc}_{\mathcal{L}^{\wedge}_{Fell}} = (\tau^{tc}_{\mathcal{L}})^{+}_{Fell}$,

(ii) $\tau^{tc}_{\mathcal{L}_{Fell}} = (\tau^{tc}_{\mathcal{L}})_{Fell}$.

Proof. **(i)** Since $\mathcal{L}^{\wedge_{Fell}} = \{\mu^{\wedge}_B | \mu \in \mathcal{L}, B \in K(X)\}$ is a subbasis for $\mathcal{L}^{\wedge}_{Fell}$, by Lemma 2 (ii) we obtain that the family

$$\mathcal{S} = \left\{ \{\mu^{\wedge}_B = 0\} \mid \mu \in \mathcal{L}, B \in K(X) \right\}$$

is a subbasis for the collection $\mathcal{F}_{\mathcal{L}^{\wedge}_{Fell}}$ of closed subsets of $(CL(X), \tau^{tc}_{\mathcal{L}^{\wedge}_{Fell}})$. Moreover,

$$\mathcal{B} = \{K^{-} \mid K \in K(X)\}$$

is a basis for the collection $\mathcal{F}^{+}_{\mathcal{L}_{Fell}}$ of closed subsets of $(CL(X), (\tau^{tc}_{\mathcal{L}})^{+}_{Fell})$. It is sufficient to prove that $\mathcal{S} \subset \mathcal{B}$ in order to obtain $\mathcal{F}_{\mathcal{L}^{\wedge}_{Fell}} \subset \mathcal{F}^{+}_{\mathcal{L}_{Fell}}$. Thus let $\mathcal{A} \in \mathcal{S}$, then

$$\exists \mu \in \mathcal{L}, \exists B \in K(X) \text{ s.t. } \mathcal{A} = \{\mu^{\wedge}_B = 0\}.$$

Therefore by Lemma 1 (i), $\mathcal{A} = \{\{\mu = 0\} \cap B\}^{-}$ and by the lower semicontinuity of μ, we obtain $\{\mu = 0\} \cap B \in K(X)$ and then $\mathcal{F}_{\mathcal{L}^{\wedge}_{Fell}} \subset \mathcal{F}^{+}_{\mathcal{L}_{Fell}}$. On the other hand, by the fact that $K(X) \subset CL(X)$ and by (5)

$$\begin{aligned} \mathcal{A} \in \mathcal{B} &\implies \exists K \in K(X) : \mathcal{A} = K^{-} \\ &\implies \mathcal{A} = \left(\bigcap_{\rho \in \mathcal{J}} \{\rho = 0\} \right)^{-}, \text{ where } \mathcal{J} = \{\rho \in \mathcal{L} \mid \rho_{|K=0}\}. \end{aligned}$$

Moreover, by Lemma 1 (ii) and since $\{\rho^{\wedge}_K = 0\} \in \mathcal{S}$ for all $\rho \in \mathcal{J}$ we obtain that $\mathcal{F}^{+}_{\mathcal{L}_{Fell}} \subset \mathcal{F}_{\mathcal{L}^{\wedge}_{Fell}}$.

(ii) We know that $\mathcal{L}^{\wedge_{Fell}} \bigcup \mathcal{L}^{\vee} = \{\mu^{\wedge}_B \mid \mu \in \mathcal{L}, B \in K(X)\} \bigcup \{\nu^{\vee} \mid \nu \in \mathcal{L}\}$ is a subbasis for $\mathcal{L}_{Fell}$. By Lemma 2 (ii),

$$\mathcal{S}_1 = \left\{ \{\eta = 0\} \mid \eta \in \mathcal{L}^{\wedge_{Fell}} \bigcup \mathcal{L}^{\vee} \right\}$$

is a subbasis for the collection $\mathcal{F}_{\mathcal{L}_{Fell}}$ of the closed sets of $(CL(X), \tau^{tc}_{\mathcal{L}_{Fell}})$. In addition

$$\mathcal{S}_2 = \Big\{F^+ \mid F \in CL(X)\Big\} \bigcup \Big\{K^- \mid K \in K(X)\Big\},$$

is a subbasis for the collection $(\mathcal{F}_{\mathcal{L}})_{Fell}$ of closed subsets of $(CL(X), (\tau^{tc}_{\mathcal{L}})_{Fell})$. Now we shall prove that $\mathcal{S}_1 \subset \mathcal{S}_2$ in order to obtain that $\mathcal{F}_{\mathcal{L}_{Fell}} \subset (\mathcal{F}_{\mathcal{L}})_{Fell}$. Let $\mathcal{A} \in \mathcal{S}_1$, then

$$\exists \eta \in \mathcal{L}^{\wedge_{Fell}} \bigcup \mathcal{L}^{\vee} : \mathcal{A} = \{\eta = 0\}.$$

Thus we have two possibilities. If $\eta \in \mathcal{L}^{\wedge_{Fell}}$, then

$$\exists \mu \in \mathcal{L}, \exists B \in K(X) : \mathcal{A} = \{\mu^{\wedge}_B = 0\}$$

Lemma 1 (i) provides that $\mathcal{A} \in \mathcal{S}_2$. If $\eta \in \mathcal{L}^{\vee}$, then

$$\exists \mu \in \mathcal{L} \ni \mathcal{A} = \{\mu^{\vee} = 0\}$$

and by (7) $\mathcal{A} \in (\mathcal{F}_{\mathcal{L}})_{Fell}$. On the other hand when $\mathcal{A} \in \mathcal{S}_2$ we have two posibilities. If there exists $F \in CL(X)$ s.t. $\mathcal{A} = F^+$, by (5) and (7) we obtain

$$\mathcal{A} = \Bigg(\bigcap_{\substack{\rho \in \mathcal{L} \\ \rho_{|F} = 0}} \{\rho = 0\}\Bigg)^+ = \bigcap_{\substack{\rho \in \mathcal{L} \\ \rho_{|F} = 0}} \{\rho = 0\}^+ = \bigcap_{\substack{\rho \in \mathcal{L} \\ \rho_{|F} = 0}} \{\rho^{\vee} = 0\} \in (\tau_{\mathcal{L}_{Fell}})^c.$$

If there exists $K \in K(X) \subset CL(X)$ for which $\mathcal{A} = K^-$, then respectively (5) and Lemma 1 (ii) provides that

$$\mathcal{A} = K^- = \Bigg(\bigcap_{\substack{\rho \in \mathcal{L} \\ \rho_{|K} = 0}} \{\rho = 0\}\Bigg)^- = \bigcap_{\substack{\rho \in \mathcal{L} \\ \rho_{|K} = 0}} \{\rho^{\wedge}_K = 0\} \in \mathcal{F}_{\mathcal{L}_{Fell}}.$$

Hence it follows that $(\mathcal{F}_{\mathcal{L}})_{Fell} \subset \mathcal{F}_{\mathcal{L}_{Fell}}$. □

The following example gives rise to observe how one shall construct the members of the subbasis of the Fell approach structure step by step. To make it more clear we considered the topological case.

Example 1. *Let $\mathcal{L}_{\mathcal{U}}$ be the induced frame on $\mathbb{R}$, where $\mathcal{U}$ is the usual topology on $\mathbb{R}$. In this case, of course, $\tau^{tc}_{\mathcal{L}_{\mathcal{U}}} = \mathcal{U}$. Now we shall consider $CL(\mathbb{R})$ with its Fell approach structure. Here $\mathcal{L}^{\vee} = \{\mu^{\vee} | \mu : (\mathbb{R}, \mathcal{U}) \to \mathbb{P}$ lower semi continuous$\}$ and $\mathcal{L}^{\wedge_{Fell}} = \{\mu^{\wedge}_B | \mu : (\mathbb{R}, \mathcal{U}) \to \mathbb{P}$ lower semi continuous and $B \in K(\mathbb{R})\}$. If we let the lower semi continuous mapping $\mu : \mathbb{R} \longrightarrow \mathbb{P}$ defined as*

$$\mu(x) = \begin{cases} x^2, & x > 1 \\ 1 - x^2, & x \leq 1 \end{cases},$$

then $\mu^{\vee} \in \mathcal{L}^{\vee}$ *and* $\mu_B^{\wedge} \in \mathcal{L}^{\wedge_{Fell}}$. *For* $A \in CL(\mathbb{R})$, $\mu_B^{\wedge}(A) = \infty$ *whenever* $A \cap B = \emptyset$ *and if* $A \cap B \neq \emptyset$ *there exist* $x_0 \in A \cap B$ *such that* $\mu_B^{\wedge}(A) = \mu(x_0)$. *Particularly,* $\theta_A \in \mathcal{L}_{\mathcal{U}_{Fell}}$ *for each* $A \in CL(\mathbb{R})$. *Because if we let* $B = [0,1]$, *then by Proposition 2.1.2 (3) in [17],* $(\theta_{\{\mu=0\}})_B^{\wedge} \in \mathcal{L}_{\mathcal{U}_{Fell}}$. *And one can easily see that* $(\theta_{\{\mu=0\}})_B^{\wedge} = \theta_A$.

Now we construct an approach structure corresponding to the extended Fell topology. Extended Fell topology τ_{eFell} is a topology on $\mathcal{W} = CL(X) \cup \{\emptyset\}$ with subbasis

$$\{V^-|V \in \tau\} \bigcup \{W^+|W \in \tau, W^c \in K(X)\}$$

where W^+ is considered as the set of subsets of W which belongs to $\mathcal{W}$. While constructing the extended Fell approach space, the domains of $\mu_B^{\wedge}$ and $\mu^{\vee}$ are assumed to be $\mathcal{W}$ instead of $CL(X)$.

Proposition 4. *If* $(X, \mathcal{L})$ *is an approach space, then*

$$\mathcal{L}^{\bigwedge_{eFell}} = \{\mu_B^{\wedge} \mid \mu \in \mathcal{L}, B \in K(X)\}$$

is a subbasis for a lower regular function frame and the corresponding frame is

$$\mathcal{L}^{\bigwedge}_{eFell} = \left\{ \sup_{j \in J} \inf_{\substack{\mu \in \mathcal{L}_j \\ B \in K_j}} \mu_B^{\wedge} \mid J \neq \emptyset, \mathcal{L}_j \subset \mathcal{L}, K_j \subset K(X), \mathcal{L}_j \textit{ and } K_j \textit{ are finite} \right\}.$$

Proof. The proof goes along the same lines in Proposition 3. □

Theorem 4. *The collection* $\mathcal{L}^{\bigvee} \cup \mathcal{L}^{\bigwedge_{eFell}}$ *is a subbasis for a lower regular function frame. The corresponding lower regular function frame is*

$$\mathcal{L}_{eFell} = \left\{ \sup_{j \in J} \Big(\inf_{\substack{\mu \in \mathcal{L}_j \\ B \in K_j}} \mu_B^{\wedge} \bigwedge \inf_{\mu \in \mathcal{L}_{t_j}} \mu^{\vee} \Big) \mid \mathcal{L}_j, \mathcal{L}_{t_j} \subset \mathcal{L}, K_j \subset K(X), \right.$$
$$\left. \mathcal{L}_j, \mathcal{L}_{t_j} \textit{ and } K_j \textit{ are finite} \right\}$$

Proof. The proof goes along the same lines in Theorem 1. □

The approach structures $\mathcal{L}^{\bigwedge}_{eFell}$ and $\mathcal{L}_{eFell}$ are called **extended Fell $\bigwedge$-approach structure** and **extended Fell approach structure**, respectively. In the following result we give the fact that the Top-coreflection of extended Fell approach structure is the extended Fell topology on $\mathcal{W}$.

Theorem 5. *For a lower regular function frame* $\mathcal{L}$ *on X, the following properties hold*

(i) $\tau^{tc}_{\mathcal{L}^{\wedge}_{eFell}} = (\tau^{tc}_{\mathcal{L}})^{+}_{eFell}$,

(ii) $\tau^{tc}_{\mathcal{L}_{eFell}} = (\tau^{tc}_{\mathcal{L}})_{eFell}$.

Proof. The proof goes along the same lines in Theorem 3. □

In [16] the measure of compactness of an approach space $(X,\mathcal{L})$ is given as

$$\chi_c(X) = \sup_{\mathcal{F}\in F(X)} \inf_{x\in X} \sup_{F\in\mathcal{F}} \sup_{\substack{\rho\in\mathcal{L}\\ \rho_{|F}=0}} \rho(A)$$

where $F(X)$ is the set of all filters on X. If an approach space has an index of compactness equal to zero, then in [10] it is said to be 0-compact. Lowen and Wuyts [16] proved that the index of compactness of X can be reformulated in terms of FS-sets; that is a subset $\mathcal{B}$ of $\mathcal{L}$ such that $\inf_{\mu\in\mathcal{C}}\mu = 0$ for each finite subcollection $\mathcal{C}$ of $\mathcal{B}$. For a subbasis $\mathcal{B}$ of $\mathcal{L}$, if an FS-set is contained in $\mathcal{B}$ we say it is an FS-set in $\mathcal{B}$. The set of all FS-sets in $\mathcal{B}$ is denoted by $B_s(\mathcal{B})$ and the following equality holds.

$$\chi_c(X) = \sup_{\mathcal{I}\in B_s(\mathcal{B})} \inf_{x\in X} \bigvee \mathcal{I}(x)$$

Here, for clarity we shall write $\chi_c(X_{\mathcal{L}})$ instead of $\chi_c(X)$. In the following theorem we show that extended Fell $\wedge$-approach space is 0-compact and then it gives a result which mentions that the compactness index of $(\mathcal{W},\mathcal{L}_{eFell})$ is zero.

Theorem 6. *$\chi_c(CL(X_{\mathcal{L}^{\wedge}_{Fell}})) = 0$ for any approach space $(X,\mathcal{L})$.*

Proof. Consider the subbasis $\mathcal{L}^{\bigwedge_{Fell}}$ for $\mathcal{L}^{\bigwedge}_{Fell}$. We shall prove that

$$\forall\mathcal{I}\in B_s(\mathcal{L}^{\bigwedge_{Fell}}) : \inf_{A\in CL(X)} \bigvee \mathcal{I}(A) = 0$$

Let $\mathcal{I}\in B_s(\mathcal{L}^{\bigwedge_{Fell}})$, i.e $\mathcal{I}$ is an FS-set in $\mathcal{L}^{\wedge}_{Fell}$, then for $\{\mu^{\wedge}_K\}\in 2^{(I)}$ where $\mu\in\mathcal{L}$ and $K\in K(X)$ we obtain that

$$\inf_{A\in CL(X)} \mu^{\wedge}_K(A) = 0$$

Clearly for all $A\in CL(X)$, $A\cap K\subset X$ and so $\mu^{\wedge}_K(X)\leq\mu^{\wedge}_K(A)$. Then

$$\mu^{\wedge}_K(X)\leq \inf_{A\in CL(X)} \mu^{\wedge}_K(A)$$

Therefore $\mu^{\wedge}_K(X)=0$. Since $K\in K(X)$ is arbitrary it follows that

$$\bigvee \mathcal{I}(X) = \sup_{\mu^{\wedge}_B\in\mathcal{I}} \mu^{\wedge}_B(X) = 0.$$

Consequently $\inf_{A\in CL(X)}\bigvee\mathcal{I}(A) = 0$. □

Corollary 1. *$\chi_c(\mathcal{W}_{\mathcal{L}_{eFell}}) = 0$ for any approach space $(X,\mathcal{L})$.*

Proof. The compactness of $(\mathcal{W},\tau_{eFell})$ is given in [4] and we know that an approach space with a compact topological coreflection is 0-compact [17]. By these two facts, Theorem 5 provides that the compactness index of the extended Fell approach space is zero. □

Proposition 5. *For a lower regular function frame $\mathcal{L}$ on X, the following properties hold.*

(i) If $\rho \in \mathcal{L}_{eFell}$ s.t. $\rho_{|_{\mathcal{D}}} = 0$ whenever $\mathcal{D} \subset CL(X)$, then $\rho_{|_{CL(X)}} \in \mathcal{L}_{Fell}$ and $(\rho_{|_{CL(X)}})_{|_{\mathcal{D}}} = 0$,

(ii) If $\nu \in \mathcal{A}_{eFell}(B)$, then $\nu_{|_{CL(X)}} \in \mathcal{A}_{Fell}(B)$ for an arbitrary $B \in CL(X)$.

Proof. **(i)** If $\rho \in \mathcal{L}_{eFell}$ s.t. $\rho_{|_{\mathcal{D}}} = 0$, then by the definition of $\mathcal{L}_{Fell}$ and the facts about restriction of a function, clearly $\rho_{|_{CL(X)}} \in \mathcal{L}_{Fell}$ and $(\rho_{|_{CL(X)}})_{|_{\mathcal{D}}} = 0$.
(ii) By (2) and (3) it is clear that

$$\mathcal{A}_{eFell}(B) = \left\{ \phi \in \mathbb{P}^{\mathcal{W}} \mid \forall \mathcal{D} \subset \mathcal{W} : \inf_{D \in \mathcal{D}} \phi(D) \leq \sup_{\substack{\rho \in \mathcal{L}_{eFell} \\ \rho_{|_{\mathcal{D}}=0}}} \rho(B) \right\}.$$

If $\nu \in \mathcal{A}_{eFell}(B)$, then

$$\forall \mathcal{D} \subset \mathcal{W} : \inf_{D \in \mathcal{D}} \nu(D) \leq \sup_{\substack{\rho \in \mathcal{L}_{eFell} \\ \rho_{|_{\mathcal{D}}=0}}} \rho(B). \tag{8}$$

Thus (8) is also true for an arbitrary subfamily $\mathcal{D}$ of $CL(X)$. In addition, for all $D \in \mathcal{D} \subset CL(X)$, it is obvious that $\nu_{|_{CL(X)}}(D) = \nu(D)$. Therefore

$$\inf_{D \in \mathcal{D}} \nu_{|_{CL(X)}}(D) = \inf_{D \in \mathcal{D}} \nu(D) \leq \sup_{\substack{\rho \in \mathcal{L}_{eFell} \\ \rho_{|_{\mathcal{D}}=0}}} \rho(B).$$

To complete the proof we shall prove that

$$\sup_{\substack{\rho \in \mathcal{L}_{eFell} \\ \rho_{|_{\mathcal{D}}=0}}} \rho(B) \leq \sup_{\substack{\mu \in \mathcal{L}_{Fell} \\ \mu_{|_{\mathcal{D}}}=0}} \mu(B).$$

For an arbitrary $\alpha > 0$ suppose that

$$\forall \mu \in \mathcal{L}_{Fell} \text{ s.t. } \mu_{|_{\mathcal{D}}} = 0 : \mu(B) < \alpha \tag{9}$$

If $\rho \in \mathcal{L}_{eFell}$ s.t. $\rho_{|_{\mathcal{D}}} = 0$, then by **(i)** it is clear that $\rho_{|_{CL(X)}} \in \mathcal{L}_{Fell}$ and $(\rho_{|_{CL(X)}})_{|_{\mathcal{D}}} = 0$. Thus by (9) $\sup_{\substack{\rho \in \mathcal{L}_{eFell} \\ \rho_{|_{\mathcal{D}}}=0}} \rho(B) \leq \alpha$ which completes the proof. □

An approach space $(X, \mathcal{L})$ is said to be LC1 iff its Top-coreflection is locally compact [13]. By using Corollary 5.1.4 in [4], we obtain the following result as an analogue of the same Corollary by means of approach theory.

Theorem 7. *If $(X, \mathcal{L})$ is a LC1-Hausdorff approach space, then $(\mathcal{W}, \mathcal{L}_{eFell})$ is a 0-compact Hausdorff approach space and $(CL(X), \mathcal{L}_{Fell})$ is a LC1 Hausdorff approach space.*

Proof. It was first observed in [5] that $(\mathcal{W}, \tau_{eFell})$ is a Hausdorff topological space. In Theorem 5 we proved that the Top-coreflection of $(\mathcal{W}, \mathcal{L}_{eFell})$ is $(\mathcal{W}, \tau_{eFell})$. Then by these two facts and Corollary 1 it is clear that $(\mathcal{W}, \mathcal{L}_{eFell})$ is a 0-compact Hausdorff space. Since $(X, \mathcal{L})$ is LC1, we know that $(X, \tau_{\mathcal{L}}^{tc})$ is locally compact. Then $(CL(X), (\tau_{\mathcal{L}}^{tc})_{Fell})$ is locally compact by Corollary 5.1.4 in [4]. Consequently $(CL(X), \mathcal{L}_{Fell})$ is LC1 by Theorem 3 (ii) and definition of the property LC1, respectively. Then the proof is completed since $(X, \tau_{\mathcal{L}}^{tc})$ is locally compact. Because, in [3], it is said that being locally compact provides that $(\mathcal{W}, \tau_{eFell})$ is Hausdorff and so the subhyperspace $(CL(X), \tau_{Fell})$ is. Thus by Theorem 3 clearly $(CL(X), \mathcal{L}_{Fell})$ is a Hausdorff approach space. In addition it can be easily seen by Proposition 1, whenever $(\mathcal{W}, \mathcal{L}_{eFell})$ is assumed to be a Hausdorff approach space. Let $A, B \in CL(X)$ and $A \neq B$, then

$$\exists \rho, \mu \in \mathcal{L}_{eFell} \ni \rho(A) > 0, \rho(B) = 0 \text{ and } \mu(A) = 0, \mu(B) > 0.$$

By Proposition 5 (i), we know that $\rho_{|_{CL(X)}}, \mu_{|_{CL(X)}} \in \mathcal{L}_{Fell}$. Therefore by the fact that, $\rho_{|_{CL(X)}}(A) = \rho(A)$ and $\mu_{|_{CL(X)}}(A) = \mu(A)$ for each $A \in CL(X)$, we obtain

$$\rho_{|_{CL(X)}}(A) > 0, \rho_{|_{CL(X)}}(B) = 0 \text{ and } \mu_{|_{CL(X)}}(A) = 0, \mu_{|_{CL(X)}}(B) > 0.$$

Hence $(CL(X), \mathcal{L}_{Fell})$ is a Hausdorff approach space. □

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