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A Note on Quasi-Metrizable Spaces

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Abstract. In a previous paper on quasi-uniformizable spaces related to statistical metric spaces [2] it was determined three conditions to obtain a first countable T1-topology. In this paper, we determine two conditions to define a quasi-metrizable topological space. Namely, we determine the conditions to obtain both a quasi-metric and a quasi-uniform topology coincides with a quasi-metrizable topological space.

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1. Introduction

Throughout this paper X is a nonempty set, D is the diagonal set, namely, $D = \{(x, x) : x \in X\}$ and $P(X)$ is the collection of all subset of X. Recall that for $J, K \in P(X \times X)$,

$$J^{-1} = \{(s, p) \mid (p, s) \in J\},$$

$$J \circ K = \{(p, s) : \text{there exists } \ r \in X \ \text{ such that } \ (p, r) \in K \ \text{ and } \ (r, s) \in J\}$$

and that a sub-family $\Im$ of $P(X)$ is said to be a filter on X if the following are satisfied:

(i) $\varnothing \notin \Im$,

(ii) The intersection of finitely many elements of $\Im$ belongs to $\Im$,

(iii) Any element of $P(X)$ containing an element of $\Im$ belongs to $\Im$.

The notion of "semi-uniform space" was introduced by Nachbin in 1948 [5]. In 1960, it was called as "quasi-uniform space" by Császár [1].

Recall that a filter $\Im$ on $X \times X$ is said to be a quasi-uniformity, if each element of $\Im$ contains the diagonal and for each $J \in \Im$, there exists $K \in \Im$ satisfying $K \circ K \subseteq J$. In this case, the couple $(X, \Im)$ is called a quasi-uniform space. It is well-known that if $\Im$ is a quasi-uniformity on X then the collection $\tau_{\Im} = \{R \subseteq X : \text{for each } r \in R \text{ there exists } J \in \Im \text{ such that } J(r) \subseteq R\}$ is a topology on X generated by $\Im$, where $J(r) = \{s \in X : (r, s) \in J\}$. The first direct topological

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proof of the converse, that is, there is a quasi-uniformity for a given topology compatible with the topology, was proved by Pervin [6] by using a result of Kelley [4].

Definition 1.1. A subcollection $\mathfrak{E}$ of $\mathfrak{I}$ in a quasi-uniform space $(X, \mathfrak{I})$ is said to be a basis for $\mathfrak{I}$, if for any $J \in \mathfrak{I}$ there exists $K \in \mathfrak{E}$ such that $K \subseteq J$.

Theorem 1.2 ([3]). *Let $\mathfrak{E}$ be a subcollection of $P(X \times X)$. Then, there exists a quasi-uniformity $\mathfrak{I}$ having $\mathfrak{E}$ as a basis if and only if $\mathfrak{E}$ is a filter basis for which each element of it contains D, and for each element J of $\mathfrak{E}$ there exists $K \in \mathfrak{E}$ satisfying $K \circ K \subseteq J$.*

At this point we state some basic notions of statistical metric spaces.

Definition 1.3. A function $\phi : [-\infty, \infty] \to \mathbb{R}$ is said to be a distribution function if the following two conditions are satisfied;
(i) ϕ is monotone increasing,
and
(ii) $\phi(-\infty) = 0$ and $\phi(\infty) = 1$.
If in addition $\phi(0) = 0$, then ϕ is called a distance function.

Example 1.4. The function from $[0, \infty]$ to [0,1]

$$\varrho_0(r) = \begin{cases} 0 & : r \leq 0 \\ 1 & : r > 0 \end{cases}$$

is a left-continuous distance function which is called an unit step function.

By Ω and Ω_L, we denote the collection of all distance functions and all left-continuous distance functions respectively.
Recall that, a function $\varpi : X \times X \to [0, \infty[$ is said to be a quasi-metric, if $\varpi(r, r) = 0$ and $\varpi(r, s) \leq \varpi(r, t) + \varpi(t, s)$ for all $r, s, t \in X$.
Remark that if ϖ is a quasi-metric on X, then the collection

$$\{G \subseteq X \mid \text{ for each } \ r \in G \quad \text{there exists } \ \mu > 0 \ \text{ such that } \ B_\mu(r) \subseteq G\}$$

is a topology on X, where $B_\mu(r) = \{s \in X : \ \varpi(r, s) < \mu\}$.

Definition 1.5. Let ϖ be a quasi-metric on X. Then, we say that X is quasi-metrizable if the collection $S(r) = \{B_\eta(r) : \eta > 0\}$ is a local basis at each $r \in X$.

The main goal of this paper is to determine the conditions to obtain a quasi-metric and a quasi-uniform topology coincides with quasi-metrizable topological space.
For the terminology of quasi-uniform spaces and statistical metric spaces we refer to [3] and [7] respectively.

2. Results

Let $F : X \times X \to \Omega$ be a function and $\delta : [0, 1] \times [0, 1] \to [0, 1]$ a function satisfying $\delta \geq \delta_0$, where $\delta_0(u, v) = max\{u + v - 1, 0\}$, and $\delta(u_1, v_1) \leq \delta(u_2, v_2)$ for $u_1 \leq u_2$ and $v_1 \leq v_2$ for all $u, v, u_1, v_1, u_2, v_2 \in [0, 1]$. We consider the space (X, F, δ).
Let $\eta > 0$. Put $A_\eta = \{(u, v) \in X \times X : F_{uv}(\eta) > 1 - \eta\}$, where F_{uv} denotes the value of F at (u, v). We also consider the function $\rho : X \times X \to \mathbb{R}$, defined by, for $u, v \in X$

$$\rho(u, v) = inf\{1 - F_{uv}(\eta) + \eta : \eta > 0\},$$

Proposition 2.1. *Let $F : X \times X \to \Omega$ be a function. Then, for each $u, v \in X$ and $\eta, \eta_1, \eta_2 > 0$.*
(i) *$(u, v) \in A_\eta \implies \rho(u, v) < 2\eta$,*
(ii) *$\rho(u, v) < \eta \implies (u, v) \in A_\eta$,*
(iii) *If $\eta_1 \leq \eta_2$, then $A_{\eta_1} \subseteq A_{\eta_2}$.*

Proof. **(i)** Let $(u, v) \in X$ and $\eta > 0$. If $(u, v) \in A_\eta$, then $\rho(u, v) \leq 1 - F_{uv}(\eta) + \eta < 2\eta$.
(ii) Suppose that $\rho(u, v) < \eta$. Then, there exists $\mu > 0$ such that $1 - F_{uv}(\mu) + \mu < \eta$ and, $0 \leq F_{uv}(\mu) \leq 1$, $\mu < \eta$. It follows from here that, since F_{uv} is monotone increasing, $F_{uv}(\eta) \geq F_{uv}(\mu) > 1 - (\eta - \mu) > 1 - \eta$. Thus, by very definition of

A_η, we get $(u,v) \in A_\eta$.
(iii) It is trivial. □

Proposition 2.2. *Consider the space (X, F, δ). If $F_{uv}(a+b) \geq \delta(F_{uw}(a), F_{wv}(b))$ for all a, b positive numbers and $u, v, w \in X$, then the function ρ satisfies the triangle inequality.*

Proof. Let $\epsilon > 0$. Since $\rho(u,w) = inf\{1 - F_{uw}(\eta) + \eta : \eta > 0\}$ and $\rho(w,v) = inf\{1 - F_{wv}(\eta) + \eta : \eta > 0\}$, there exist $\lambda, \mu > 0$ such that

$$1 - F_{uw}(\lambda) + \lambda < \rho(u,w) + \frac{\epsilon}{2}$$

and

$$1 - F_{wv}(\mu) + \mu < \rho(w,v) + \frac{\epsilon}{2}.$$

From these inequalities, by taking summation from both sides, we get

$$1 - (F_{uw}(\lambda) + F_{wv}(\mu) - 1) + \lambda + \mu < \rho(u,w) + \rho(w,v) + \epsilon. \tag{2.1}$$

On the other hand, from the hypothesis, as $\delta \geq \delta_0$, we have

$$F_{uv}(\lambda + \mu) \geq \delta(F_{uw}(\lambda), F_{wv}(\mu)) \geq F_{uw}(\lambda) + F_{wv}(\mu) - 1.$$

Hence,

$$1 - F_{uv}(\lambda + \mu) + \lambda + \mu \leq 1 - \delta(F_{uw}(\lambda), F_{wv}(\mu)) + \lambda + \mu \leq 1 - (F_{uw}(\lambda) + F_{wv}(\mu) - 1) + \lambda + \mu$$

From here, by taking into account the inequality (2.1) and the definition of ρ, one can easily get that

$$\rho(u,v) \leq 1 - F_{uv}(\lambda + \mu) + \lambda + \mu < \rho(u,w) + \rho(w,v) + \epsilon$$

for all $\epsilon > 0$. Hence, $\rho(u,v) \leq \rho(u,w) + \rho(w,v)$. □

Theorem 2.3. *Consider the space (X, F, δ). Suppose that $F_{uu} = \varrho_0$ and $F_{uw}(a+b) \geq \delta(F_{uv}(a), F_{vw}(b))$ for all $a, b > 0$ and $u, v, w \in X$. In this case, X is a quasi-metrizable space.*

Proof. Since $F_{uu} = \epsilon_0$, $\rho(u,u) = inf\{\eta : \eta > 0\} = 0$. Thus, by Proposition 2.2, the function ρ is a quasi-metric on X. Now we will prove that the collection $S(u)$ is a local basis at each $u \in X$. At this point, we remark that the sub-collection $\mathfrak{E} = \{A_\eta : \eta > 0\}$ satisfies the conditions of Theorem 1.2. Indeed, as $F_{uu} = \varrho_0$, $D \subseteq A_\eta$ for each $\eta > 0$. On the other hand, by Proposition 2.1 (iii), we get $A_{min\{\eta_1,\eta_2\}} \subseteq A_{\eta_1} \cap A_{\eta_2}$. Thus, $\mathfrak{E}$ is a filter basis whose each element containing the diagonal. Moreover, let $\eta > 0$ and $(u,v) \in A_{\frac{\eta}{2}} \circ A_{\frac{\eta}{2}}$. Then, there exists $w \in X$ satisfying $(u,w), (w,v) \in A_{\frac{\eta}{2}}$. It follows from the hypothesis that

$$F_{uv}(\eta) \geq \delta(F_{uw}(\frac{\eta}{2}), F_{wv}(\frac{\eta}{2})) \geq \delta_0(F_{uw}(\frac{\eta}{2}), F_{wv}(\frac{\eta}{2})) \geq F_{uw}(\frac{\eta}{2}) + F_{wv}(\frac{\eta}{2}) - 1 > 1 - \eta.$$

Thus, $(u,v) \in A_\eta$. We conclude from here that $\mathfrak{E}$ is a basis for a quasi-uniformity $\mathfrak{I}$ on X.
Let $\tau_{\mathfrak{I}}$ be the topology generated by $\mathfrak{I}$. It's well-known that the family $(A_\eta(u))_{\eta>0}$ is a local basis at each $u \in X$, here, we recall that $A_\eta(u) = \{v \in X : (u,v) \in A_\eta\}$. In addition, Proposition 2.1 (i) and (ii) imply that $A_{\frac{\eta}{2}}(u) \subseteq B_\eta(u)$ and $B_\eta(u) \subseteq A_\eta(u)$ respectively. Thus, the collection $S(u) = \{B_\eta(u) : \eta > 0\}$ is a local basis at each $u \in X$. Hence, X is quasi-metrizable. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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