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Investigation the ergodic distribution of a semi-Markovian inventory model of type (s,S) with intuitive approximation approach

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Abstract

This paper concerns a stochastic process $X(t)$ expressing (s, S) type inventory system with intuitive approximation approach. The ergodic distributions of the process $X(t)$ can be analyzed with the help of the renewal function. Obtaining an explicit formula for renewal function $U(x)$ is difficult from a practical standpoint. Mitov and Omey recently present some intuitive approximations in literature for renewal function which cover a large number of existing results. Using their approach we were able to establish asymptotic approximations for ergodic distribution of a stochastic process $X(t)$. Obtained results can be used in many situations where demand random variables have different distributions from different classes such as the $\Gamma(g)$ class.

1. Introduction

In areas like stock control, queuing, stochastic finance, and reliability, semi-Markov processes have been applied, along with renewal theory. Through the use of these processes, many important stock control problems are also expressed (see, Smith, 1959; Brown and Solomon, 1975; Gikhman and Skorohod, 1975; Borovkov, 1984; Chen and Zheng, 1997; Csenki, 2000; Asmussen, 2000). For instance, it is typical to assume that the demand sequences creates a renewal process when analyzing inventory processes. Also, common inventory systems use renewal sequences to express stock replenishment times. The stochastic control model of type (*s, S*) is one of the inventory models where in the formula of the ergodic distribution the renewal function is encountered. There has been a substantial amount of research in the literature on the stationary characteristics of (*s, S*) type inventory models with different modifications. These modifications are mostly given with some kinds of discrete interference of chances or by using different distributions for inter-arrival times (see, Khaniyev et. al., 2011; Khaniyev et. al., 2013; Kesemen et. al., 2016; Hanalioglu and Khaniyev, 2019). In recent years, these studies have moved towards examining (*s, S*) type inventory models with heavy-tailed distributions and the literature on this subject has been enriched considerably (see, Aliyev, 2017; Bektaş et. al., 2018; Bektaş et. al., 2019). But some important distributions for example the gamma distribution and the exponential distribution are excluded from the heavy-tailed situation. The $\Gamma(g)$ class is encountered in many applications, especially in extreme value theory. For more detail and representation theorems for the class of the $\Gamma(g)$ we refer to Geluk and de Haan (1981). Recently Mitov and Omey (2014) studied asymptotic behavior of renewal function generated by the $\Gamma(g)$ distributed random variables and provided intuitive approximations. We will provide a summary in preliminaries.

The stochastic control model of type (s, S) was previously investigated in the study by Bektaş et. al. (2020) with the $\Gamma(g)$ class of distributions. In the study by Bektaş et. al. (2020) demand random variables were assumed to have logistic distribution from the $\Gamma(g)$ class. But the findings of their study can be only used when demand random variables have a logistic distribution specifically. However, in order to analyze the stationary characteristics of such systems, it is more appropriate to use general formulas that cover as many distributions as possible. Differently from previous literature, in this research it is assumed that random variables expressing the demands could have any distribution from the $\Gamma(g)$ class and the process was examined under this assumption. Growth models and a particular kind of regression known as the logistic regression have both employed the logistic distribution. This distribution has numerous other applications, including in inventory management, geological sciences, survival analysis, and population modeling. The logistic distribution is primarily employed across a variety of disciplines since it closely resembles the normal distribution and has a straightforward cumulative distribution formula. The normal distribution can be approximated by a wide variety of functions, however these functions frequently have quite difficult mathematical formulations. Comparatively, the cumulative distribution formula for the logistic distribution is substantially more straightforward.

In this study previous results were expanded here to include all random variables from the $\Gamma(g)$ class. First we provided approximations for ergodic distribution of the model. As a result the formulas obtained for the general case are applied to the special cases such as generalized extreme value distribution and generalized gamma distribution. This approximation can be used in practice with ease, especially for large values of *t.*

2. Preliminaries

Before examining the primary problem, we offer the key notations and mathematical description of the considered model.

Definition 2.1: A positive and measurable function h belongs to the class of $\Gamma(g)$ with auxiliary function g if and only if every fixed $y \in \mathbb{R}$

$$\lim_{x\to\infty} h(x + yg(x))/h(x) = e^{-y}. \tag{1}$$

Here the function g in (1) ought to satisfy $g(x) = o(x),\ x \to \infty$, where $f(x) = o\big(g(x)\big)$ means:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

For more detailed information about the $\Gamma(g)$ class and its asymptotic properties, see Mitov and Omey (2014). In this study, we will consider the renewal function produced by the $\Gamma(g)$ class of distributions. Consider the renewal process $N(t);\ t \geq 0$. The renewal function is defined as

$$E[N(t)] = U(t) = \sum_{n=0}^{\infty} F^{*n}(t)\,;\ \ t \geq 0.$$

where $F^{*n}(t)$ is n-fold convolution of $F(.)$ with $F^{*1} = F(t)$. The intuitive approximation for renewal function $U(x)$ is denoted by $\widehat{U}(x)$. Mitov and Omey (2014) suggested the following intuitive approximation for the renewal function where $\bar{F}(x) = \mathbb{P}\{\eta_1 > x\} \in \Gamma(g)$:

$$\hat{U}(x) = \frac{x}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2} g^2(x)\bar{F}(x)\,. \tag{2}$$

Here $\mu_1 = E(\eta_1)$. When $\mu_1 < \infty$, $\mu_e = \int_0^\infty \big(1 - F_e(x)\big)dx$. The definition of equilibrium distribution F_e is given as follows:

$$F_e(x) = \frac{1}{\mu_1}\int_0^x \bar{F}(y)dy\,.$$

For $\mu_2 < \infty$, then $\mu_e < \infty$ and the statement (2) can also be expressed as:

$$\hat{U}(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{1}{\mu_1^2} g^2(x)\bar{F}(x)\,. \tag{3}$$

2.1. Model Formulation

Consider a stochastic process $X(t)$ with an initial stock level S. Suppose the demands with random amount arrive on the system at random times $T_1, T_2, \dots, T_n$. The system must respond to demands promptly. Therefore, we do not want the system's stock level to drop below a certain control level. On the other hand overstocking might be costly so we also do not want the stock amount in the system to exceed a present stock level S. The stock level decreases continuously in time starting from maximum stock level until it falls below a certain control level. Demand random variables are independent and identically distributed (i.i.d.). Time intervals between two successive demands are likewise considered to be i.i.d.. When the stock level falls below a predetermined control level, the system is intervened and refilled with stock up to its maximum capacity. Here the system completes its first period, starts its second period with an initial stock level S and proceeds similarly to the first period. Hence stock level varies between S and s.

Notations:

$X(t)$	: Stock level in the system at a random time t.
S	: Maximum stock level.
$\{\eta_n\}_{n\geq 1}$	: Demand random variables.
$\{T_n\}_{n\geq 1}$	: Random times when demands arrive in the system.
$F(x)$	: The distribution function corresponding to η_1.
$\{\xi_n\}_{n\geq 1}$	: Random variables that express time between two consecutive demands.
$\phi(t)$	: The distribution function corresponding to ξ_1.
s	: Stock control level.
τ_n	: The time when the stock level falls below s for the n^{th} time.
$U(t)$	: Renewal function of demand random variable .
μ_n	: First three raw moments of η_1, for n=1,2,3.

By assuming that $\{\xi_n, \eta_n\}, n \geq 1$ be a vector of i.i.d. random variables defined on $(\Omega, F, \mathbb{P})$ their distribution functions are defined as follows:

$$\Phi(t) = \mathbb{P}\{\xi_1 \leq t\},\;\; F(x) = \mathbb{P}\{\eta_1 \leq x\},\;\; t \geq 0,\;\; x \geq 0.$$

Renewal sequences T_n, S_n defined as follows:

$$T_n = \sum_{i=1}^{n} \xi_i\,,\;\; Y_n = \sum_{i=1}^{n} \eta_i. \tag{4}$$

Since $X(t)$ decreases by $\eta_1, \eta_2, \ldots, \eta_n$ the variation in stock level should be expressed as follows:

$$X(T_1) \equiv X_1 = S - \eta_1, \ X(T_2) \equiv X_2 = S - (\eta_1 + \eta_2), \ldots, X(T_n) \equiv X_n = S - \sum_{i=1}^{n} \eta_i.$$

Defining a sequence of integer valued random variables $\{N_n\}$, $n \geq 0$ as:

$$N_0 = 0, N_{n+1} = inf\{k \geq N_n + 1 : \ S - (Y_k - Y_{N_n}) < s\}, \ n \geq 0,$$

and letting

$$\tau_0 = 0, \tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i, n \geq 1, \ \nu(t) = max\{n \geq 0 : T_n \leq t\}, \ t \geq 0,$$

then for $t \in [\tau_n, \tau_{n+1})$ the construction of the stochastic process $X(t)$ will be as follows:

$$X(t) = S - (\eta_{N_n+1} + \cdots + \eta_{\nu(t)}) = S - (Y_{\nu(t)} - Y_{N_n}), \ n = 0,1,2 \ldots. \quad (5)$$

A sample trajectory of the process $X(t)$ is given with Figure 1.

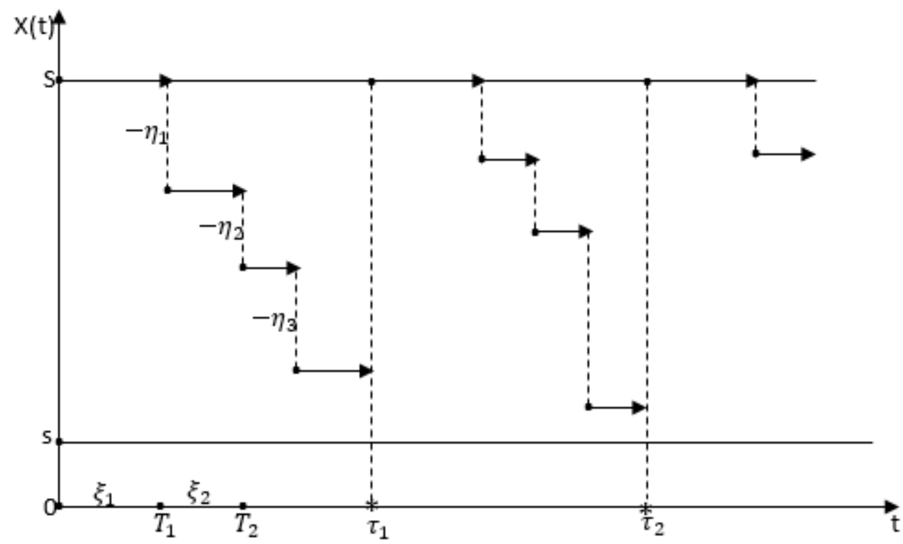

Figure 1. A sample trajectory of the process $X(t)$

The primary goal of this study is to provide general approximation results for the ergodic distribution of the stochastic process whose mathematical model was given above. We presume that $\bar{F}(x) = \mathbb{P}(\eta_1 > x) \in \Gamma(g)$. General definition of the $\Gamma(g)$ class is given with Definition 2.1. and this class covers many of the distribution classes used in practice.

2.2. Ergodicity of the Process $X(t)$ and Exact Formulas for Ergodic Distribution

Ergodicity of the process $X(t)$ and exact formula for ergodic distribution is given in publication Nasirova et. al. (1998) with the following proposition.

Proposition 2.1: Let the initial sequences $\{\xi_n\}$ and $\{\eta_n\}, n \geq 1$ satisfy the following conditions:

1) $E(\xi_1) < \infty$, 2) $0 < E(\eta_1^2) < \infty$, 3) η_1 is non-arithmetic random variable,

then, the process $X(t)$ is ergodic and the ergodic distribution function of the process $X(t)$ has the following explicit form:

$$Q_X(x) \equiv \lim_{t\to\infty} \mathbb{P}\{X(t) \leq x\} = 1 - \frac{U(S-x)}{U(S-s)}, \quad s \leq x \leq S, \tag{6}$$

where $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$ and $F(x) = \mathbb{P}(\eta_1 \leq x)$. ∎

Let define the standardized process $W_\beta(x) \equiv \frac{1}{\beta}(X(t) - s)$. Here $\beta = S - s$. Then the ergodic distribution function of the process $W_\beta(x)$ will be obtained as follows by using (6).

$$Q_{W_\beta}(x) = 1 - \frac{U\big(\beta(1-x)\big)}{U(\beta)}, \quad x \in (0,1). \tag{7}$$

As can be clearly seen by (7), an approximation of $U(x)$ is essential to obtain an approximate expression for the ergodic distribution of the process $X(t)$. The asymptotic behavior of $U(x)$ from various forms are the subject of numerous research in the literature (we refer the reader to, Asmussen, 2000; Brown and Solomon, 1975; Chen and Zheng, 1997; Csenki, 2000; Embrechts et. al., 1997; Feller, 1971; Geluk, 1997 and Levy and Taqqu, 2000). In paper by Mitov and Omey (2014) they proposed a simple and intuitive approach to approximate $U(x)$ which is given with (2). They first discussed the case of regularly varying case in their work. But as they indicated classes of distributions like the exponential or gamma distribution are not included in the regularly varying situation. For this reason, they proposed an approximation method for the renewal function produced by the distributions from the $\Gamma(g)$ class, which is a larger class and is frequently encountered in extreme value theory. Using the approximation method proposed by Mitov and Omey (2014), we were able to approximate $Q_{W_\beta}(x)$. By using (2) firstly asymptotic expansions for $Q_{W_\beta}(x)$ will be provided while demand random variables have $\Gamma(g)$ distribution in the general case. We then consider the process $X(t)$ with generalized extreme value distributed and generalized gamma distributed (which are two important distributions from the $\Gamma(g)$ class) demand random variables respectively.

3. Approximate Expressions for the Ergodic Distribution of the Standardized Process

To get approximate expressions for $Q_{W_\beta}(x)$, we will employ the approximate expression for $U(x)$ given with (3). First of all, we will need the Lemma 3.1 whose statement and proof are given below.

Lemma 3.1: Let the conditions of Proposition 2.1 be satisfied. Furthermore, let $\{\eta_n\}, n \geq 1$; be a sequence of i.i.d. random variables and $\bar{F}(x) = \mathbb{P}(\eta_1 > x) \in \Gamma(g)$. Here $g^2(x)$ is a constant or non-increasing function. In this case for $E(\eta_1^n) = \mu_n$, $n = 1,2$ and each $x \in (0,1)$ the following approximation is true, as $\beta \equiv S - s \to \infty$:

$$\hat{U}\big(\beta(1-x)\big) = \frac{1}{\mu_1}\beta(1-x) + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2} g^2\big(\beta(1-x)\big)\bar{F}\big(\beta(1-x)\big), \quad x \in [0,1), \tag{8}$$

$$\left(\hat{U}(\beta)\right)^{-1} = \frac{\mu_1}{\beta + \mu_e}\left(1 + \frac{1}{\beta + \mu_e}\frac{1}{\mu_1} g^2(\beta)\bar{F}(\beta)\right). \tag{9}$$

Proof : By substituting β and $\beta(1-x)$ for each $x \in [0,1)$ in expression (3), expressions (8) and (9) are easily obtained.
Using Proposition 2.1 and Lemma 3.1, the following main result was reached.

Theorem 3.1: Let the conditions of Lemma 3.1. be satisfied. In this case for each $x \in (0,1)$ we propose following approximation for $Q_{W_\beta}(x)$ as $\beta \equiv S-s \to \infty$:

$$\hat{Q}_{W_\beta}(x) = \frac{\beta x}{\beta+\mu_e} + \frac{1}{\mu_1(\beta+\mu_e)} g^2\big(\beta(1-x)\big)\, \bar{F}\big(\beta(1-x)\big) - \frac{\beta(1-x)}{\mu_1(\beta+\mu_e)^2} g^2(\beta)\bar{F}(\beta)\,. \tag{10}$$

Here $\hat{Q}_{W_\beta}(x)$ presents the approximation for ergodic distribution of the standardized process $W_\beta(t)$, as $\beta \equiv S-s \to \infty$.

Proof: For each $x \in (0,1)$ we have

$$\begin{aligned}
\frac{\widehat{U}\big(\beta(1-x)\big)}{\widehat{U}(\beta)} &= \left[\frac{\beta(1-x)}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{1}{\mu_1^2} g^2\big(\beta(1-x)\big)\, \bar{F}\big(\beta(1-x)\big)\right] \\
&\times \left\{\left(\frac{\mu_1}{\beta+\mu_e}\right)\left[1 + \frac{1}{\mu_1(\beta\mu_e)} + o\left(\frac{1}{\beta} g^2(\beta)\bar{F}(\beta)\right)\right]\right\} \\
&= \left[\frac{\beta(1-x)}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2} g^2\big(\beta(1-x)\big)\, \bar{F}\big(\beta(1-x)\big)\right] \\
&\times \left[\frac{\mu_1}{\beta+\mu_e} + \frac{1}{(\beta+\mu_e)^2} g^2(\beta)\bar{F}(\beta) + o\left(\frac{1}{\beta^2} g^2(\beta)\bar{F}(\beta)\right)\right].
\end{aligned} \tag{11}$$

Taking into account that

$$Q_{W_\beta}(x) = 1 - \frac{U\big(\beta(1-x)\big)}{U(\beta)}$$

for each $x \in (0,1)$ we obtain following asymptotic expansion for $Q_{W_\beta}(x)$, as $\beta \equiv S-s \to \infty$:

$$\begin{aligned}
Q_{W_\beta}(x) &= 1 - \frac{U\big(\beta(1-x)\big)}{U(\beta)} \\
&= \frac{\beta x}{\beta+\mu_e} + \frac{1}{\mu_1(\beta+\mu_e)} g^2\big(\beta(1-x)\big)\, \bar{F}\big(\beta(1-x)\big) \\
&- \frac{\beta(1-x)}{\mu_1(\beta+\mu_e)^2} g^2(\beta)\bar{F}(\beta) + o\left(\frac{1}{\beta^2} g^2\big(\beta(1-x)\big)\bar{F}\big(\beta(1-x)\big)\right).
\end{aligned} \tag{12}$$

On the other hand since $\beta > 0$ for $0 < x < 1$, $\beta(1-x) < \beta$. From the definition of the class of $\Gamma(g)$ we know that $g^2(x)$ is constant or nonincreasing function and any $\bar{F} \in \Gamma(g)$ is monotone decreasing function hence; $g^2\big(\beta(1-x)\big)\, \bar{F}\big(\beta(1-x)\big) > g^2(\beta)\bar{F}(\beta)$. Hence desired approximation given with (10) is obtained from (12).

Remark: Theorem 3.1 concludes that $W_\beta(t) = \frac{1}{\beta}(X(t) - s)$ converges weakly to uniform distribution defined in $[0,1]$, as $\beta \equiv S - s \to \infty$. This result provides consistency with previous studies on this subject.

3.1. Applications of Obtained Approximation for Special Distributions from $\Gamma(g)$ Class

In the previous section we examined the general form of an approximation for $Q_{W_\beta}(x)$, when the demand quantities have any distribution from the $\Gamma(g)$ class. In this section, we will consider at the scenario where two special distributions from the $\Gamma(g)$ class are present for demand random variables. Thus, we will exemplify the results we have obtained before.

3.1.1. Generalized Extreme Value Distribution of Type-I

In this section we suppose that η_1 have a generalized extreme value distribution from the $\Gamma(g)$ class. In this case we propose an approximation for $Q_{W_\beta}(x)$ with following corollary:

Corollary 3.1: Let the conditions of Theorem 3.1 be satisfied. Moreover suppose that η_1 have a generalized extreme value distribution of type-I from the $\Gamma(g)$ class. In this case as $\beta \to \infty$ and for each $x \in (0,1)$ we propose following approximation for ergodic distribution of the process $W_\beta(t)$:

$$\hat{Q}_{W_\beta}(x) = \frac{\beta x}{\beta + \mu_e} + \frac{1}{\mu_1(\beta + \mu_e)}\Big(1 - exp\big(- exp\big(-\beta(1-x)\big)\big)\Big) - \frac{\beta(1-x)}{\mu_1(\beta + \mu_e)^2}\big(1 - exp(- exp(-\beta))\big). \tag{13}$$

Proof: In the general case, the approximate expression of the renewal function produced by random variables from the $\Gamma(g)$ class is given in (3). The cumulative distribution functions of random variables from this class are given by $F(x) = exp(-e^{-x}), x \in \mathbb{R}$. Since $f(x) = e^{-x}exp(-e^{-x}), x \in \mathbb{R}$, then according to the definition of the $\Gamma(g)$ class $\bar{F}(x) \in \Gamma(1)$. In this case for $E(\eta_1^n) = \mu_n; n = 1,2$ we propose following approximation for $U(x)$:

$$\hat{U}(x) = \frac{x}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2}\big(1 - exp(- exp(-x))\big). \tag{14}$$

In this case for each $x \in (0,1)$ we have the following result as $\beta \to \infty$:

$$\hat{U}(\beta) = \frac{\beta}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2}\big(1 - exp(- exp(-\beta))\big) \tag{15}$$

and

$$\hat{U}\big(\beta(1-x)\big) = \frac{\beta(1-x)}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2}\Big(1 - exp\big(- exp\big(-\beta(1-x)\big)\big)\Big), \; x \in (0,1). \tag{16}$$

Using (15) and (16) in (7) we have the following asymptotic expansion:

$$Q_{W_\beta}(x) = 1 - \frac{U\big(\beta(1-x)\big)}{U(\beta)}$$
$$= \frac{\beta x}{\beta + \mu_e} + \frac{1}{\mu_1(\beta + \mu_e)}\big(1 - exp\big(- exp\big(-\beta(1-x)\big)\big)\big)$$

$$-\frac{\beta(1-x)}{\mu_1(\beta+\mu_e)^2}\left(1-exp\left(-e^{-\beta}\right)\right)+o\left(\frac{1}{\beta^2}\left(1-exp\left(-exp\left(-\beta(1-x)\right)\right)\right)\right). \tag{17}$$

From here desired result holds.

3.1.2. Gamma Distribution of Type-1

Similarly, now we will consider the demand random variable η_1 have gamma distribution of type 1 from the $\Gamma(g)$ class,

$$\mathbb{P}(\eta_1 \le x)=F(x)=\frac{\gamma(\alpha,x)}{\Gamma(\alpha)}, x\in[0,\infty).$$

It can easily be seen that this distribution belongs to the $\Gamma(1)$ class, by definition of the $\Gamma(g)$ class. In order to examine asymptotic behavior of $Q_{W_\beta}(x)$ first we need to give definitions of some well known functions.

Definition 3.1: The following functions are defined as classical gamma function, lower incomplete gamma function and upper incomplete gamma function respectively:

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\,, Re(x)>0,\quad \gamma(x)=\int_0^x t^{s-1}e^{-t}dt\,, Re(s)>0,\quad \bar{\Gamma}(s,x)=\int_x^\infty t^{s-1}e^{-t}dt\,, Re(s)>0.$$

Following corollary proposes an approximation for ergodic distribution of the process $W_\beta(t)$ under the condition that η_1 have a generalized gamma distribution from the $\Gamma(g)$ class using intuitive approximation approach.

Corollary 3.2: Let the conditions of Theorem 3.1. be satisfied. Moreover suppose the demand random variable η_1, have a generalized gamma distribution from the $\Gamma(g)$ class. In this case as $\beta\to\infty$ and $x\in(0,1)$ we obtain following approximation for $Q_{W_\beta}(x)$ as follows:

$$\hat{Q}_{W_\beta}(x)==\frac{\beta x}{\beta+\mu_e}+\frac{1}{\mu_1(\beta+\mu_e)}\left(\frac{\Gamma\left(\alpha,\beta(1-x)\right)}{\Gamma(\alpha)}\right)-\frac{\beta(1-x)}{\mu_1(\beta+\mu_e)^2}\frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)}. \tag{18}$$

Proof: For $F(x)=\gamma(\alpha,x)/\Gamma(\alpha)$, $\bar{F}(x)=\Gamma(\alpha,x)/\Gamma(\alpha)$. In this case for each $x\in(0,1)$ an approximation for the renewal function generated from generalized gamma distributions becomes:

$$\widehat{U}(x)=\frac{x}{\mu_1}+\frac{\mu_e}{\mu_1}-\frac{1}{\mu_1^2}\frac{\Gamma(\alpha,x)}{\Gamma(\alpha)}. \tag{19}$$

From here we get

$$\widehat{U}(\beta)=\frac{\beta}{\mu_1}+\frac{\mu_e}{\mu_1}-\frac{1}{\mu_1^2}\frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)}, \tag{20}$$

$$\widehat{U}\left(\beta(1-x)\right)=\frac{\beta(1-x)}{\mu_1}+\frac{\mu_e}{\mu_1}-\frac{1}{\mu_1^2}\frac{\Gamma\left(\alpha,\beta(1-x)\right)}{\Gamma(\alpha)}. \tag{21}$$

Taking into account that

$$Q_{W_\beta}(x)\ =1-\frac{U\left(\beta(1-x)\right)}{U(\beta)}$$

we propose asymptotic expansion for $Q_{W_\beta}(x)$ using intuitive approximation approach as follows:

$$Q_{W_\beta}(x) = \frac{\beta x}{\beta+\mu_e} + \frac{1}{\mu_1(\beta+\mu_e)}\left(\frac{\Gamma\big(\alpha,\beta(1-x)\big)}{\Gamma(\alpha)}\right) - \frac{\beta(1-x)}{\mu_1(\beta+\mu_e)^2}\frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)}$$

$$+o\left(\frac{\Gamma\big(\alpha,\beta(1-x)\big)}{\beta\,\Gamma(\alpha)}\right). \qquad (22)$$

Desired result holds form (22).

4. Conclusion

In this study, intuitive approximations are suggested for ergodic distribution of a specific stochastic process which characterize a stochastic control model of type (s,S). For demands belong to a broad class of distributions satisfying the condition $\mu_e < \infty$ asymptotic results are obtained for ergodic distribution of the process $X(t)$. Then the approximate formula for the ergodic distribution of the process is obtained when the demand random variables have a generalized extreme value distribution and gamma distribution of type-1, respectively. Many real-world problems have lately been captured using heavy tailed distributions and the class of $\Gamma(g)$. Renewal functions produced by these distributions and the use of these functions in different variations of the inventory models have become a popular field of study (see for example, Geluk, 1997; Gaffeo et. al, 2008; Foss et. al., 2011; Embrechts et. al., 1975). In literature the process $X(t)$ has been specifically examined with logistic distributed demands from the $\Gamma(g)$ class. In this study, however, the results obtained previously using a specific distribution were generalized to cover all distributions from the $\Gamma(g)$ class. Therefore, the results obtained in this study are more general results.

We believe that our approximation results from using these distributions in inventory models will serve as a useful guide for further research. Similar outcomes can be attained in the scenario where $\mu_e = \infty$. Moreover, the semi-Markovian random walk process can be studied in the future using similar approximation techniques. In addition, examining similar processes using only moment-based approaches can be another suggestion for future studies.

Contribution of Authors

Authors contributed equally to each part.

Conflicts of Interest

The authors declared that there is no conflict of interest.

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