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Type 2-Positional Adapted Frame and Its Application to Tzitzeica and Smarandache Curves

2. Tip Konumsal Uyarlanmış Çatı ve Onun Tzitzeica ve Smarandache Eğrilerine Uygulaması

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Abstract

The topic of the orthonormal moving frames is one of the basic topics in differential geometry and it has an important place. On the other hand, the topic of special curves is among the popular topics of interest in this field. In this study, these two important concepts are taken into consideration. Firstly, a new version of the Positional Adapted Frame introduced by Özen and Tosun in 2021 is defined and this new frame is called Type 2-Positional Adapted Frame. In addition, some characterizations are obtained according to this frame for Tzitzeica curves in 3-dimensional Euclidean space. Lastly, the trajectories generated by special Smarandache curves according to the Type 2-Positional Adapted Frame are discussed and the Frenet elements of these trajectories are calculated. The aim of this study is to introduce the Type 2-Positional Adapted Frame, which we foresee it will be useful in some specific applications of differential geometry and particle kinematics, to researchers who may be interested in this topic.

Özet

Ortonormal hareketli çatılar konusu diferansiyel geometrinin temel konularından bir tanesidir ve önemli bir yere sahiptir. Diğer yandan, özel eğriler konusu da bu alandaki popüler ilgi konuları arasında yer almaktadır. Bu çalışmada bu iki önemli kavram göz önüne alınmıştır. İlk olarak, Özen ve Tosun tarafından 2021 yılında tanıtılan Konumsal Uyarlanmış Çatı'nın yeni bir versiyonu tanımlanmış ve bu yeni çatıya 2. Tip Konumsal Uyarlanmış Çatı adı verilmiştir. Ayrıca, 3-boyutlu Öklid uzayında Tzitzeica eğrileri için bu çatıya göre bazı karakterizasyonlar elde edilmiştir. Son olarak, 2. Tip Konumsal Uyarlanmış Çatıya göre özel Smarandache eğrilerinin ürettiği yörüngeler ele alınmış ve bu yörüngelerin Frenet elemanları hesaplanmıştır. Bu çalışmanın amacı diferansiyel geometri ve parçacık kinematiğinin bazı özel uygulamalarında faydalı olacağını öngördüğümüz 2. Tip Konumsal Uyarlanmış Çatıyı bu konuya ilgi duyabilecek araştırmacılara tanıtmaktır.

1 INTRODUCTION

The concept of special curves is one of the most popular basic concepts in the theory of curves. Involute-evolute curves, Mannheim curves, Bertrand curves, Tzitzeica curves and Smarandache curves can be given as some of examples to the special curves. The geometric properties of these kinds of special curves are mostly investigated by means of the moving frames in the literature. Hence the moving frames are useful tools to study the local theory of special curves. For the topic of moving frames, the investigation of the Frenet frame in 19th century was a turning point. After that discov-

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ery, researchers have been obtained many other moving frames such as Darboux frame (Darboux 1896), Bishop frame (Bishop 1975), Sabban frame (Koenderink 1990), Type 2-Bishop frame (Yılmaz and Turgut 2010), q-frame (Dede et al. 2015) etc. In the study (Özen and Tosun 2021a), Özen and Tosun introduced one of the newest of these kinds of moving frames in 2021. The name of this frame is Positional Adapted Frame (PAF) and it is well defined when the motion of the moving particle is a motion which have non-zero angular momentum.

The Positional Adapted Frame (PAF) has so far found application areas in the topics such as Smarandache curves, Tzitzeica curves, Mannheim curves, Bertrand curves, electric field, Lorentz force, Fermi-Walker derivative, spinors and has been used as an important tool in the studies (Özen and Tosun 2021a; Özen and Tosun 2021b; Solouma 2021; Özen et al. 2022; Gürbüz 2022; İşbilir et al. 2022a; İşbilir et al. 2022b; İşbilir et al. 2022c). In this paper, we will define a new version of PAF and will call it Type 2-Positional Adapted Frame. This frame will provide the opportunity to work together both the differential geometry of the orbit and particle kinematics of the moving point particle, just like the first version. In this respect, we think that this article can benefit researchers working on particle kinematics and differential geometry.

Let us consider the Euclidean 3-space and standard scalar product $\langle \delta, \gamma \rangle = \delta_1\gamma_1 + \delta_2\gamma_2 + \delta_3\gamma_3$ where $\delta = (\delta_1, \delta_2, \delta_3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ are any vectors in space. The equation $\|\delta\| = \sqrt{\langle \delta, \delta \rangle}$ gives us the norm of the vector δ. A differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$, satisfying $\left\| \frac{d\alpha}{ds} \right\| = 1$ for every $s \in I$, is said to be a unit speed curve. Then s is called arc-length parameter of α. A differentiable curve, whose derivative does not equal to zero for all values s, is said to be a regular curve. As is well known, every regular curve may be reparameterized by the arc-length of itself (Shifrin 2008). Note that the differentiation with respect to s will be shown with a prime.

In E^3, assume that a particle travels along a unit speed curve $\alpha = \alpha(s)$. Let the vector system $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ show the Frenet frame of $\alpha = \alpha(s)$. The unit tangent vector $\mathbf{T}(s)$ is calculated by $\mathbf{T}(s) = \alpha'(s)$, the unit principal normal vector $\mathbf{N}(s)$ is calculated by $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and the unit binormal vector $\mathbf{B}(s)$ is calculated by $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$. Also

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}. \tag{1}$$

Here $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle$ is the torsion function (Shifrin 2008). We must emphasize that we suppose everywhere $\kappa \not\equiv 0$ throughout the paper.

After giving the Type 2-PAF in Section 2, we investigate Tzitzeica curves and some special Smarandache trajectories using Type 2-PAF in Section 3. Tzitzeica curves which are a member of a class of curves were presented firstly in the study (Tzitzeica 1911) in the early 1900s. Tzitzeica curves are spatial curves in 3-dimensional Euclidean space E^3 and they have non-zero torsion and positive curvature. For a Tzitzeica curve the equation

$$\frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} = c \tag{2}$$

holds where c is a non-zero constant. Here the expression $\langle \alpha, \mathbf{B} \rangle^2$ corresponds to the square of the distance from the origin to the instantaneous osculating plane. The topic of Tzitzeica curves has been studied widely in the literature (Crâşmăreanu 2002; Karacan and Bükcü 2009; Agnew et al. 2010; Aydın and Ergut 2014; Bayram et al. 2018; Eren and Ersoy 2021). On the other hand, special Smarandache curves in 3-

dimensional Euclidean space were defined in (Ali 2010) by A. T. Ali. He obtained these curves by using the Serret-Frenet frame of them. After this study, many studies performed in different spaces by using various moving frames. (Turgut and Yılmaz 2008; Bektaş and Yüce 2013; Çetin et al. 2014; Taşköprü and Tosun 2014; Şenyurt and Eren 2020; Yazıcı et al. 2022) can be given some examples to these studies.

The aim of why we include the application in two separate topics we mentioned above is to give idea to researchers who will deal with the other areas of use of Type 2-Positional Adapted Frame. We expect that this frame will arouse the interest of researchers and we think it will be useful.

2 TYPE 2-PAF

Let us consider any fixed origin O in 3-dimensional Euclidean space. Tangent, velocity and linear momentum vectors of the aforementioned particle at $\alpha(s)$ (corresponding to time t) are expressed as

$$\begin{aligned} \mathbf{T}(s) &= \frac{d\mathbf{x}}{ds} \\ \mathbf{v}(t) &= \frac{d\mathbf{x}}{dt} = \left(\frac{ds}{dt}\right)\mathbf{T}(s) \\ \mathbf{p}(t) &= m\mathbf{v}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s) \end{aligned} \tag{3}$$

where $\mathbf{x}$ is the position vector according to O. (Casey 2011). One can easily write the following:

$$\mathbf{x} = \langle \mathbf{T}(s),\ \alpha(s)\rangle\,\mathbf{T}(s) + \langle \mathbf{N}(s),\ \alpha(s)\rangle\,\mathbf{N}(s) + \langle \mathbf{B}(s),\ \alpha(s)\rangle\,\mathbf{B}(s) \tag{4}$$

As it is well known the angular momentum about the origin O is obtained as in the following:

$$\mathbf{H}^O = m\,\langle \mathbf{B}(s),\ \alpha(s)\rangle \left(\frac{ds}{dt}\right)\mathbf{N}(s) - m\,\langle \mathbf{N}(s),\ \alpha(s)\rangle \left(\frac{ds}{dt}\right)\mathbf{B}(s) \tag{5}$$

by vector product $\mathbf{x} \wedge \mathbf{p}(t)$.

Suppose that the normal component of the angular momentum vector never vanishes. By this assumption, we ensure that $\langle \alpha(s), \mathbf{B}(s)\rangle$ is never zero. Thus we ensure that the line, which is generated by the normal vector, never passes through the origin along $\alpha = \alpha(s)$. Let us focus on the opposite of the position vector:

$$-\mathbf{x} = \langle \mathbf{T}(s), -\alpha(s)\rangle\,\mathbf{T}(s) + \langle \mathbf{N}(s), -\alpha(s)\rangle\,\mathbf{N}(s) + \langle \mathbf{B}(s), -\alpha(s)\rangle\,\mathbf{B}(s). \tag{6}$$

The projections of it on the instantaneous normal and osculating planes $\pi_1(s)$ and $\pi_2(s)$ yield two vectors which are important to build our orthonormal moving frame. These cases will be stated below:

Draw a perpendicular from origin to $\pi_1(s)$ and consider the foot of this perpendicular. Then construct a vector which is oriented from $\alpha(s)$ to the aforementioned foot. This vector may be written in the form:

$$\mathbf{w}(s) = \langle \mathbf{N}(s),\ -\alpha(s)\rangle\,\mathbf{N}(s) + \langle \mathbf{B}(s),\ -\alpha(s)\rangle\,\mathbf{B}(s) \tag{7}$$

and it matches up with the aforementioned projection on $\pi_1(s)$. Similarly, draw a perpendicular from origin to $\pi_2(s)$ and consider the foot of this perpendicular. Then construct a vector which is oriented from $\alpha(s)$ to this foot. This vector may be written in the form:

$$\mathbf{r}(s) = \langle \mathbf{T}(s),\ -\alpha(s)\rangle\,\mathbf{T}(s) + \langle \mathbf{N}(s),\ -\alpha(s)\rangle\,\mathbf{N}(s) \tag{8}$$

and it matches up with the aforementioned projection on $\pi_2(s)$. With the aid of the equations (7) and (8), the vector

$$\mathbf{w}(s) - \mathbf{r}(s) = \langle \mathbf{T}(s),\ \alpha(s)\rangle\,\mathbf{T}(s) + \langle \mathbf{B}(s),\ -\alpha(s)\rangle\,\mathbf{B}(s) \tag{9}$$

is found. As can be seen easily, this vector lies in the the instantaneous rectifying plane $\pi_3(s)$ and its initial point is $\alpha(s)$. It must be noted that $\mathbf{w}(s) - \mathbf{r}(s)$ is in the same equivalence class with the vector directed from the aforesaid foot on $\pi_2(s)$ to the aforesaid foot on $\pi_1(s)$.

Let us deal with the question "When the length of the vector $\mathbf{w}(s) - \mathbf{r}(s)$ is non-zero". The answer of this question is important from the view point of that the unit vector in direction of this vector is definable or not. If the origin does not lie on both the planes $\pi_1(s)$ and $\pi_2(s)$, the aforementioned foots are different from each other and they do not coincide with the origin. Hence, two different foots generate the vector $\mathbf{w}(s) - \mathbf{r}(s)$ as a non-zero vector. Then, the unit vector in direction $\mathbf{w}(s) - \mathbf{r}(s)$ is easily obtained. When only one of $\pi_1(s)$ and $\pi_2(s)$ passes from O, the length of the vector $\mathbf{w}(s) - \mathbf{r}(s)$ is non-zero similarly. The case both $\pi_1(s)$ and $\pi_2(s)$ include O at the same time gives rise not to be obtained of the necessary unit vector because the aforesaid foots coincide with O. That situation occurs only when the line, which is generated by the normal vector, passes through O. But, our assumption about the normal component of angular momentum prevents this possibility. Let the aforesaid necessary unit vector be symbolized by $\mathbf{K}(s)$. It means that

$$\mathbf{K}(s) = \frac{\mathbf{w}(s) - \mathbf{r}(s)}{\|\mathbf{w}(s) - \mathbf{r}(s)\|} = \frac{\langle \mathbf{T}(s), \alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}}\mathbf{T}(s) + \frac{\langle \mathbf{B}(s), -\alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}}\mathbf{B}(s). \tag{10}$$

We can have another basis vector by vector product as in the following:

$$\mathbf{R}(s) = \mathbf{K}(s) \wedge \mathbf{N}(s) = \frac{\langle \mathbf{B}(s), \alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}}\mathbf{T}(s) + \frac{\langle \mathbf{T}(s), \alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}}\mathbf{B}(s). \tag{11}$$

Consequently, $\{\mathbf{K}(s), \mathbf{N}(s), \mathbf{R}(s)\}$ is a moving frame which is orthonormal. Because $\mathbf{T}(s)$, $\mathbf{B}(s)$, $\mathbf{K}(s)$ and $\mathbf{R}(s)$ lie in $\pi_3(s)$, a relationship exists between them as in the following:

$$\begin{pmatrix}\mathbf{K}(s)\\ \mathbf{N}(s)\\ \mathbf{R}(s)\end{pmatrix} = \begin{pmatrix}\cos\psi(s) & 0 & -\sin\psi(s)\\ 0 & 1 & 0\\ \sin\psi(s) & 0 & \cos\psi(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} \tag{12}$$

where $\psi(s)$ is the angle between $\mathbf{B}(s)$ and $\mathbf{R}(s)$.
By utilizing (1) and (12),

$$\begin{aligned}\mathbf{K}'(s) &= (\cos\psi(s)\mathbf{T}(s) - \sin\psi(s)\mathbf{B}(s))'\\ &= -\psi'(s)\sin\psi(s)\mathbf{T}(s) + \cos\psi(s)\kappa(s)\mathbf{N}(s)\\ &\quad -\psi'(s)\cos\psi(s)\mathbf{B}(s) + \tau(s)\sin\psi(s)\mathbf{N}(s)\\ &= (\kappa(s)\cos\psi(s) + \tau(s)\sin\psi(s))\,\mathbf{N}(s)\\ &\quad -\psi'(s)\,(\sin\psi(s)\mathbf{T}(s) + \cos\psi(s)\mathbf{B}(s))\\ &= (\kappa(s)\cos\psi(s) + \tau(s)\sin\psi(s))\,\mathbf{N}(s) - \psi'(s)\mathbf{R}(s)\end{aligned}$$

and

$$\begin{aligned}\mathbf{R}'(s) &= (\sin\psi(s)\mathbf{T}(s) + \cos\psi(s)\mathbf{B}(s))'\\ &= \psi'(s)\cos\psi(s)\mathbf{T}(s) + \kappa(s)\sin\psi(s)\mathbf{N}(s)\\ &\quad -\psi'(s)\sin\psi(s)\mathbf{B}(s) - \tau(s)\cos\psi(s)\mathbf{N}(s)\\ &= \psi'(s)\,(\cos\psi(s)\mathbf{T}(s) - \sin\psi(s)\mathbf{B}(s))\\ &\quad + (\kappa(s)\sin\psi(s) - \tau(s)\cos\psi(s))\,\mathbf{N}(s)\\ &= \psi'(s)\mathbf{K}(s) + (\kappa(s)\sin\psi(s) - \tau(s)\cos\psi(s))\,\mathbf{N}(s)\end{aligned}$$

can be given. Also, differentiating $\mathbf{R}(s) \wedge \mathbf{K}(s)$ yields

$$\begin{aligned}\mathbf{N}'(s) &= (\mathbf{R}(s) \wedge \mathbf{K}(s))'\\ &= \mathbf{R}'(s) \wedge \mathbf{K}(s) + \mathbf{R}(s) \wedge \mathbf{K}'(s)\\ &= \left[\psi'(s)\mathbf{K}(s) + (\kappa(s)\sin\psi(s) - \tau(s)\cos\psi(s))\,\mathbf{N}(s)\right] \wedge \mathbf{K}(s)\\ &\quad + \mathbf{R}(s) \wedge \left[(\kappa(s)\cos\psi(s) + \tau(s)\sin\psi(s))\,\mathbf{N}(s) - \psi'(s)\mathbf{R}(s)\right]\\ &= -(\kappa(s)\cos\psi(s) + \tau(s)\sin\psi(s))\,\mathbf{K}(s)\\ &\quad + (-\kappa(s)\sin\psi(s) + \tau(s)\cos\psi(s))\,\mathbf{R}(s).\end{aligned}$$

Summarizing above

$$\begin{pmatrix}\mathbf{K}'(s)\\ \mathbf{N}'(s)\\ \mathbf{R}'(s)\end{pmatrix} = \begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{K}(s)\\ \mathbf{N}(s)\\ \mathbf{R}(s)\end{pmatrix} \tag{13}$$

can be written where

$$\begin{aligned}k_1(s) &= \kappa(s)\cos\psi(s) + \tau(s)\sin\psi(s)\\ k_2(s) &= -\psi'(s)\\ k_3(s) &= -\kappa(s)\sin\psi(s) + \tau(s)\cos\psi(s).\end{aligned} \tag{14}$$

We call this frame as Type 2-Positional Adapted Frame (or shortly Type 2-PAF). Also, $\mathbf{K}(s), \mathbf{N}(s), \mathbf{R}(s), k_1(s), k_2(s), k_3(s)$ are said to be Type 2-PAF apparatuses of $\alpha = \alpha(s)$. By comparing (12) with (10) and (11),

$$\sin\psi(s) = \frac{\langle \mathbf{B}(s), \alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}} \tag{15}$$

$$\cos\psi(s) = \frac{\langle \mathbf{T}(s), \alpha(s)\rangle}{\sqrt{\langle \mathbf{T}(s), \alpha(s)\rangle^2 + \langle \mathbf{B}(s), \alpha(s)\rangle^2}} \tag{16}$$

are obtained immediately. In this case, we find

$$\tan\psi(s) = \frac{\langle \mathbf{B}(s),\ \alpha(s)\rangle}{\langle \mathbf{T}(s),\ \alpha(s)\rangle}. \tag{17}$$

If we consider (15), (16) and (17), $\psi(s)$ is specified as in the following:

$$\psi(s) = \begin{cases} \arctan\left(\frac{\langle \mathbf{B}(s),\alpha(s)\rangle}{\langle \mathbf{T}(s),\alpha(s)\rangle}\right) & \text{if } \langle \mathbf{T}(s),\ \alpha(s)\rangle > 0 \\ \arctan\left(\frac{\langle \mathbf{B}(s),\alpha(s)\rangle}{\langle \mathbf{T}(s),\alpha(s)\rangle}\right) + \pi & \text{if } \langle \mathbf{T}(s),\ \alpha(s)\rangle < 0 \\ \frac{\pi}{2} & \text{if } \langle \mathbf{T}(s),\ \alpha(s)\rangle = 0\ ,\ \langle \mathbf{B}(s),\ \alpha(s)\rangle > 0 \\ -\frac{\pi}{2} & \text{if } \langle \mathbf{T}(s),\ \alpha(s)\rangle = 0\ ,\ \langle \mathbf{B}(s),\ \alpha(s)\rangle < 0. \end{cases} \tag{18}$$

When $\langle \mathbf{T}(s), \alpha(s)\rangle = 0$ and $\langle \mathbf{B}(s), \alpha(s)\rangle > 0$, Type 2-PAF apparatuses $\{\mathbf{K}(s), \mathbf{N}(s), \mathbf{R}(s), k_1(s), k_2(s), k_3(s)\}$ coincide with the apparatuses $\{-\mathbf{B}(s), \mathbf{N}(s), \mathbf{T}(s), \tau(s), 0, -\kappa(s)\}$. Similarly, if $\langle \mathbf{T}(s), \alpha(s)\rangle = 0, \langle \mathbf{B}(s), \alpha(s)\rangle < 0$, apparatuses $\{\mathbf{K}(s),\ \mathbf{N}(s),\ \mathbf{R}(s),\ k_1(s),\ k_2(s),\ k_3(s)\}$ coincide with $\{\mathbf{B}(s),\ \mathbf{N}(s),\ -\mathbf{T}(s),\ -\tau(s),\ 0,\ \kappa(s)\}$.

Of course, considering (14), one can use the relation $\psi(s) = \int -k_2(s)\,ds$ to generate a rotation angle. But we want to obtain $\psi(s)$ uniquely in accordance with the explanations indicated above. Hence we prefer the equation (18) to determine $\psi(s)$.

Now, the angular velocity vector $\omega(s)$ will be obtained for Type 2-PAF. $\omega(s)$ must satisfy the following relations:

$$\begin{aligned} \mathbf{K}'(s) &= \omega(s)\wedge \mathbf{K}(s) \\ \mathbf{N}'(s) &= \omega(s)\wedge \mathbf{N}(s) \\ \mathbf{R}'(s) &= \omega(s)\wedge \mathbf{R}(s). \end{aligned} \tag{19}$$

Assume that $\omega(s)$ is written in terms of Type 2-PAF basis vectors as:

$$\omega(s) = a(s)\mathbf{K}(s) + b(s)\mathbf{N}(s) + c(s)\mathbf{R}(s)$$

In this case, (19) takes the following form

$$\begin{aligned} \mathbf{K}'(s) &= -b(s)\mathbf{R}(s) + c(s)\mathbf{N}(s) \\ \mathbf{N}'(s) &= a(s)\mathbf{R}(s) - c(s)\mathbf{K}(s) \\ \mathbf{R}'(s) &= -a(s)\mathbf{N}(s) + b(s)\mathbf{K}(s). \end{aligned} \tag{20}$$

Comparing (13) with (20) gives us

$$\begin{aligned} a &= k_3 \\ b &= -k_2 \\ c &= k_1. \end{aligned}$$

Hence, $\omega(s)$ is expressed as

$$\omega(s) = k_3(s)\mathbf{K}(s) - k_2(s)\mathbf{N}(s) + k_1(s)\mathbf{R}(s) \tag{21}$$

for Type 2-PAF.

3 APPLICATIONS

In this section, we continue to take into account of any moving point particle satisfying the aforementioned supposition (related to the normal component of angular momentum) and symbolize the unit speed form of its trajectory with $\alpha = \alpha(s)$. At first, the conditions for $\alpha = \alpha(s)$ to be a Tzitzeica curve are obtained according to Type 2-PAF and some results are presented for spherical Tzitzeica curves. Secondly, the special trajectories generated by Smarandache curves are discussed according to Type 2-PAF in E^3.

3.1 Characterization of Tzitzeica curves According to Type 2-PAF

If the trajectory $\alpha = \alpha(s)$ of the moving point particle is a Tzitzeica curve, $\langle \mathbf{B}(s),\ \alpha(s)\rangle \neq 0$ for all the values s, and so the normal component of $\mathbf{H}^0$(about origin) is non-zero along the trajectory. As a result of this case, Type 2-PAF is well defined for $\alpha = \alpha(s)$ when $\alpha = \alpha(s)$ is a Tzitzeica curve.

Theorem 3.1. *Let the trajectory* $\alpha = \alpha(s)$ *be given. Then, the equation*

$$\frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} = \frac{k_1 \sin\psi + k_3 \cos\psi}{\langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle^2} \tag{22}$$

holds.

Proof. Using the basic knowledge

$$\tau = \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2}, \quad \mathbf{B} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}$$

of the curve theory, we can easily write

$$\begin{aligned} \frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} &= \frac{\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2}}{\left\langle \alpha, \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|} \right\rangle^2} \\ &= \frac{\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2}}{\frac{1}{\|\alpha' \wedge \alpha''\|^2} \langle \alpha, \alpha' \wedge \alpha'' \rangle^2} \\ &= \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2}. \end{aligned}$$

Also, from (12) it is not difficult to see the following relation:

$$\mathbf{T} = \cos\psi \mathbf{K} + \sin\psi \mathbf{R}.$$

Then, we find

$$\begin{aligned} \alpha' &= \cos\psi \mathbf{K} + \sin\psi \mathbf{R} \\ \alpha'' &= (\cos\psi \mathbf{K} + \sin\psi \mathbf{R})' \\ &= -(k_2 + \psi') \sin\psi \mathbf{K} + (k_1 \cos\psi - k_3 \sin\psi) \mathbf{N} \\ &\quad + (k_2 + \psi') \cos\psi \mathbf{R} \\ &= (k_1 \cos\psi - k_3 \sin\psi) \mathbf{N} \\ \alpha''' &= ((k_1 \cos\psi - k_3 \sin\psi) \mathbf{N})' \\ &= -k_1 (k_1 \cos\psi - k_3 \sin\psi) \mathbf{K} \\ &\quad + \left(\left(k_1' - k_3 \psi' \right) \cos\psi - \left(k_3' + k_1 \psi' \right) \sin\psi \right) \mathbf{N} \\ &\quad + k_3 (k_1 \cos\psi - k_3 \sin\psi) \mathbf{R}. \end{aligned}$$

These equations yield the followings:

$$\begin{aligned} \alpha' \wedge \alpha'' &= \begin{vmatrix} \mathbf{K} & \mathbf{N} & \mathbf{R} \\ \cos\psi & 0 & \sin\psi \\ 0 & k_1 \cos\psi - k_3 \sin\psi & 0 \end{vmatrix} \\ &= (k_3 \sin\psi - k_1 \cos\psi) \sin\psi \mathbf{K} \\ &\quad + (k_1 \cos\psi - k_3 \sin\psi) \cos\psi \mathbf{R} \\ \left\langle \alpha, \alpha' \wedge \alpha'' \right\rangle^2 &= ((k_1 \cos\psi - k_3 \sin\psi) \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle)^2 \\ &= (k_1 \cos\psi - k_3 \sin\psi)^2 \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle^2 \\ \left\langle \alpha' \wedge \alpha'', \alpha''' \right\rangle &= (k_1 \cos\psi - k_3 \sin\psi)^2 (k_1 \sin\psi + k_3 \cos\psi). \end{aligned}$$

In the light of these equations, we obtain

$$\begin{aligned} \frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} &= \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2} \\ &= \frac{k_1 \sin\psi + k_3 \cos\psi}{\langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle^2} \end{aligned}$$

and finish the proof. □

Considering the condition of being a Tzitzeica curve and the last theorem, we can easily give a characterization for Tzitzeica curves as follows.

Corollary 3.2. *The trajectory* $\alpha = \alpha(s)$ *is a Tzitzeica curve iff the function*

$$\frac{k_1 \sin\psi + k_3 \cos\psi}{\langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle^2}$$

is a non-zero constant function.

Remark 3.3. *The derivative formula for the principal normal vector can be rewritten in the following form:*

$$\begin{aligned} \mathbf{N}' &= -k_1 \mathbf{K} + k_3 \mathbf{R} \\ &= -k_1 \left(\sin^2\psi + \cos^2\psi \right) \mathbf{K} + k_3 \left(\sin^2\psi + \cos^2\psi \right) \mathbf{R} \\ &= -k_1 \cos^2\psi \mathbf{K} + k_3 \sin^2\psi \mathbf{R} - k_1 \sin^2\psi \mathbf{K} + k_3 \cos^2\psi \mathbf{R} \\ &= -k_1 \cos^2\psi \mathbf{K} - k_1 \cos\psi \sin\psi \mathbf{R} + k_3 \cos\psi \sin\psi \mathbf{K} + k_3 \sin^2\psi \mathbf{R} \\ &\quad - k_1 \sin^2\psi \mathbf{K} + k_1 \cos\psi \sin\psi \mathbf{R} - k_3 \cos\psi \sin\psi \mathbf{K} + k_3 \cos^2\psi \mathbf{R} \\ &= (-k_1 \cos\psi + k_3 \sin\psi)(\cos\psi \mathbf{K} + \sin\psi \mathbf{R}) \\ &\quad + (k_1 \sin\psi + k_3 \cos\psi)(-\sin\psi \mathbf{K} + \cos\psi \mathbf{R}). \end{aligned}$$

On the other hand, one can similarly show

$$\begin{aligned} (-\sin\psi \mathbf{K} + \cos\psi \mathbf{R})' &= -(k_1 \sin\psi + k_3 \cos\psi) \mathbf{N} \\ (\cos\psi \mathbf{K} + \sin\psi \mathbf{R})' &= (k_1 \cos\psi - k_3 \sin\psi) \mathbf{N}. \end{aligned}$$

We must emphasize that these formulas will be used in *the proof of the next theorem.*

Theorem 3.4. *If $\alpha = \alpha(s)$ is a unit speed spherical curve on 2-sphere of radius r centered at the origin (on $S^2_{0,r}$),*

$$\frac{k_1 \sin\psi + k_3 \cos\psi}{k_1 \cos\psi - k_3 \sin\psi} = \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \sin\psi + k_3 \cos\psi)\,(k_1 \cos\psi - k_3 \sin\psi)^2}\right)' \tag{23}$$

is satisfied.

Proof. Let $\alpha = \alpha(s)$ be a unit speed spherical curve on 2-sphere of radius r centered at the origin. In that case, we get

$$\langle \alpha, \alpha \rangle = r^2.$$

Differentiating this equation with respect to arc-length parameter s, we find

$$\langle \alpha, \alpha' \rangle = 0$$

and so

$$\langle \alpha, \cos\psi \mathbf{K} + \sin\psi \mathbf{R} \rangle = 0. \tag{24}$$

The equation (24) yields the following

$$\begin{aligned} 0 &= \langle \alpha', \cos\psi \mathbf{K} + \sin\psi \mathbf{R} \rangle + \left\langle \alpha, (\cos\psi \mathbf{K} + \sin\psi \mathbf{R})' \right\rangle \\ &= \langle \cos\psi \mathbf{K} + \sin\psi \mathbf{R}, \cos\psi \mathbf{K} + \sin\psi \mathbf{R} \rangle \\ &\quad + \langle \alpha, (k_1 \cos\psi - k_3 \sin\psi)\,\mathbf{N} \rangle \\ &= \left(\cos^2\psi + \sin^2\psi\right) + \langle \alpha, (k_1 \cos\psi - k_3 \sin\psi)\,\mathbf{N} \rangle \\ &= 1 + \langle \alpha, (k_1 \cos\psi - k_3 \sin\psi)\,\mathbf{N} \rangle . \end{aligned}$$

Thus we get

$$\langle \alpha, (k_1 \cos\psi - k_3 \sin\psi)\,\mathbf{N} \rangle = -1. \tag{25}$$

That is

$$\langle \alpha, \mathbf{N} \rangle = -\frac{1}{(k_1 \cos\psi - k_3 \sin\psi)}. \tag{26}$$

Differentiating (25) gives us

$$\begin{aligned} 0 &= \left\langle \alpha, (k_1 \cos\psi - k_3 \sin\psi)'\mathbf{N} + (k_1 \cos\psi - k_3 \sin\psi)\,\mathbf{N}' \right\rangle \\ &= (k_1 \cos\psi - k_3 \sin\psi)' \langle \alpha, \mathbf{N} \rangle \\ &\quad - (k_1 \cos\psi - k_3 \sin\psi)^2 \langle \alpha, \cos\psi \mathbf{K} + \sin\psi \mathbf{R} \rangle \\ &\quad + (k_1 \cos\psi - k_3 \sin\psi)\,(k_1 \sin\psi + k_3 \cos\psi) \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle \\ &= (k_1 \cos\psi - k_3 \sin\psi)' \langle \alpha, \mathbf{N} \rangle \\ &\quad + (k_1 \cos\psi - k_3 \sin\psi)\,(k_1 \sin\psi + k_3 \cos\psi) \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle . \end{aligned}$$

If we write the equation (26) in the last equation, we obtain

$$\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2} = (k_1 \sin\psi + k_3 \cos\psi) \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle . \tag{27}$$

The equation (27) yields the following:

$$\begin{aligned} \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2}\right)' &= (k_1 \sin\psi + k_3 \cos\psi)' \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle \\ &\quad + (k_1 \sin\psi + k_3 \cos\psi) \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle' \\ &= (k_1 \sin\psi + k_3 \cos\psi)' \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle \\ &\quad + (k_1 \sin\psi + k_3 \cos\psi) \langle \alpha, -(k_1 \sin\psi + k_3 \cos\psi)\,\mathbf{N} \rangle . \end{aligned}$$

Thus we have

$$\begin{aligned} \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2}\right)' &= (k_1 \sin\psi + k_3 \cos\psi)' \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle \\ &\quad - (k_1 \sin\psi + k_3 \cos\psi)^2 \langle \alpha, \mathbf{N} \rangle . \end{aligned}$$

Taking into consideration the equation (27),

$$\langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle = \frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2\,(k_1 \sin\psi + k_3 \cos\psi)} \tag{28}$$

can be easily written. Using (26) and (28), we find

$$\begin{aligned} \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2}\right)' &= \frac{(k_1 \sin\psi + k_3 \cos\psi)'(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2\,(k_1 \sin\psi + k_3 \cos\psi)} \\ &\quad + \frac{(k_1 \sin\psi + k_3 \cos\psi)^2}{(k_1 \cos\psi - k_3 \sin\psi)} . \end{aligned}$$

This equation gives us

$$\begin{aligned} \frac{(k_1 \sin\psi + k_3 \cos\psi)^2}{k_1 \cos\psi - k_3 \sin\psi} &= \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2}\right)' \\ &\quad - \frac{(k_1 \sin\psi + k_3 \cos\psi)'(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \cos\psi - k_3 \sin\psi)^2\,(k_1 \sin\psi + k_3 \cos\psi)} . \end{aligned}$$

Then

$$\frac{k_1 \sin\psi + k_3 \cos\psi}{k_1 \cos\psi - k_3 \sin\psi} = \left(\frac{(k_1 \cos\psi - k_3 \sin\psi)'}{(k_1 \sin\psi + k_3 \cos\psi)\,(k_1 \cos\psi - k_3 \sin\psi)^2}\right)'$$

is obtained. This finishes the proof. □

Proposition 3.5. *If $\alpha = \alpha(s)$ is a Tzitzeica curve, in that case the equation*

$$(k_1 \sin\psi + k_3 \cos\psi)' \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle = -2\,(k_1 \sin\psi + k_3 \cos\psi)^2 \langle \alpha, \mathbf{N} \rangle$$

holds.

Proof. If the trajectory $\alpha = \alpha(s)$ is a Tzitzeica curve, then

$$\frac{k_1 \sin\psi + k_3 \cos\psi}{\langle \alpha, \ -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle^2} = c$$

can be easily written where c is a non-zero real constant. Differentiating the last equation gives us

$$\begin{aligned} 0 &= (k_1 \sin\psi + k_3 \cos\psi)' \langle \alpha, -\sin\psi \mathbf{K} + \cos\psi \mathbf{R} \rangle \\ &+2\,(k_1 \sin\psi + k_3 \cos\psi)^2 \langle \alpha, \mathbf{N} \rangle \end{aligned}$$

and so the desired equation is satisfied. □

Theorem 3.6. *Assume that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical curve on $S^2_{0,r}$. If it is a Tzitzeica curve, then*

$$\frac{(k_1 \sin\psi + k_3 \cos\psi)'}{2(k_1 \sin\psi + k_3 \cos\psi)^3} = \frac{k_1 \cos\psi - k_3 \sin\psi}{(k_1 \cos\psi - k_3 \sin\psi)'} \qquad (29)$$

holds.

PROOF. Suppose that $\alpha = \alpha(s)$ is a unit speed spherical curve on $S^2_{0,r}$ and also it is a Tzitzeica curve. Then, we have the equation given in Proposition 3.5. Hence the desired result is obtained by using the equations (26) and (28) in the aforesaid equation. □

Corollary 3.7. *Assume that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical Tzitzeica curve on $S^2_{0,r}$. In that case*

$$k_1 \sin\psi + k_3 \cos\psi = \sqrt{\frac{(k_1 \cos\psi - k_3 \sin\psi)'' \,(k_1 \cos\psi - k_3 \sin\psi) - 2((k_1 \cos\psi - k_3 \sin\psi)')^2}{3(k_1 \cos\psi - k_3 \sin\psi)^2}}$$

holds.

Corollary 3.8. *Assume that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical Tzitzeica curve on $S^2_{0,r}$. Let the functions $k_1 \sin\psi + k_3 \cos\psi$ and $k_1 \cos\psi - k_3 \sin\psi$ be constant function and non-constant function, respectively. In that case, the equation*

$$k_1 \cos\psi - k_3 \sin\psi = \lambda_2 \sec\left(\sqrt{3}\,(k_1 \sin\psi + k_3 \cos\psi)\,(\lambda_1 + s)\right)$$

is satisfied. Here λ_1 and λ_2 are any real constants.

PROOF. Assume that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical Tzitzeica curve on $S^2_{0,r}$. Let the function $k_1 \sin\psi + k_3 \cos\psi$ be a constant function and let the function $k_1 \cos\psi - k_3 \sin\psi$ be a non-constant function. Then we have the equation indicated in Corollary 3.7 which gives us the differential equation

$$\begin{aligned} 0 &= (k_1 \cos\psi - k_3 \sin\psi)\,(k_1 \cos\psi - k_3 \sin\psi)'' \\ &-2\Big((k_1 \cos\psi - k_3 \sin\psi)'\Big)^2 \\ &-3(k_1 \cos\psi - k_3 \sin\psi)^2 (k_1 \sin\psi + k_3 \cos\psi)^2. \end{aligned}$$

The solution of this differential equation corresponds to the desired result. □

3.2 Some Special Trajectories Generated by Smarandache Curves According to Type 2-PAF

In (Ali 2010), the special Smarandache curves in 3-dimensional Euclidean space were defined by A. T. Ali. A unit speed regular curve $\gamma = \gamma(s)$ was taken into consideration in this study and **TN**, **NB**, **TNB**− Smarandache curves were stated as

$$\begin{aligned} \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}) \\ \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{B}) \\ \beta(s^*) &= \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{N} + \mathbf{B}), \end{aligned}$$

respectively where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ symbolize the Frenet frame of $\gamma = \gamma(s)$.

In this subsection, we will discuss the special trajectories generated by Smarandache curves with respect to Type 2-PAF in E^3. Note that Type 2-PAF apparatus of $\alpha = \alpha(s)$ will be denoted by $\{\mathbf{K}_\alpha(s), \mathbf{N}_\alpha(s), \mathbf{R}_\alpha(s), k_1(s), k_2(s), k_3(s)\}$ in the rest of the study. Lastly, we note that here we will use similar steps used in (Özen and Tosun (2021b)).

Definition 3.9. *The special trajectories generated by $\mathbf{N}_\alpha\mathbf{R}_\alpha-$ Smarandache curves are defined as in the fol-*

lowing:

$$\sigma_1(s^*) = \frac{1}{\sqrt{2}}(\mathbf{N}_\alpha + \mathbf{R}_\alpha). \quad (30)$$

For convenience, we call them as $\mathbf{N}_\alpha\mathbf{R}_\alpha-$*Smarandache trajectories.*

Now, we try to find the Frenet apparatuses of $\mathbf{N}_\alpha\mathbf{R}_\alpha-$ Smarandache trajectories. If we differentiate (30) with respect to s, we get

$$\sigma_1' = \frac{d\sigma_1}{ds^*}\frac{ds^*}{ds} = \mathbf{T}_{\sigma_1}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\,\mathbf{K}_\alpha - k_3\mathbf{N}_\alpha + k_3\mathbf{R}_\alpha\right). \quad (31)$$

Taking into account of the equation (31),

$$\frac{ds^*}{ds} = \sqrt{{k_3}^2 + \frac{(k_1+k_2)^2}{2}} \quad (32)$$

can be obtained. Hence, the equation (31) can be given by

$$\mathbf{T}_\sigma\sqrt{{k_3}^2 + \frac{(k_1+k_2)^2}{2}} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\,\mathbf{K}_\alpha - k_3\mathbf{N}_\alpha + k_3\mathbf{R}_\alpha\right). \quad (33)$$

The equation (33) gives us

$$\mathbf{T}_{\sigma_1} = \frac{1}{\sqrt{2{k_3}^2 + (k_1+k_2)^2}}\left((-k_1-k_2)\,\mathbf{K}_\alpha - k_3\mathbf{N}_\alpha + k_3\mathbf{R}_\alpha\right).$$

Differentiating this equation with respect to s, we find

$$\frac{d\mathbf{T}_{\sigma_1}}{ds^*}\frac{ds^*}{ds} = \left(2{k_3}^2 + (k_1+k_2)^2\right)^{-3/2}\left(\upsilon_1\mathbf{K}_\alpha + \upsilon_2\mathbf{N}_\alpha + \upsilon_3\mathbf{R}_\alpha\right) \quad (34)$$

where

$$\begin{aligned}
\upsilon_1 &= k_3(k_1+k_2)\left[2k_3' + {k_1}^2 - {k_2}^2\right] + \\
&\quad 2{k_3}^2\left[k_1k_3 - k_2k_3 - k_1' - k_2'\right] \\
\upsilon_2 &= (k_1+k_2)k_3(k_1'+k_2') - 2k_1{k_3}^2 - k_3'(k_1+k_2) \\
&\quad -{k_3}^2(k_1+k_2) - k_1(k_1+k_2)^2 - 2k_3^4 \\
\upsilon_3 &= (k_1+k_2)\Big[-k_3(k_1'+k_2') - 2k_2{k_3}^2 + k_3'(k_1+k_2) \\
&\quad {k_3}^2(k_1+k_2) - k_2(k_1+k_2)^2\Big] - 2k_3^4.
\end{aligned}$$

Using the equation (32) in the equation (34), we get

$$\frac{d\mathbf{T}_{\sigma_1}}{ds^*} = \sqrt{2}\left(2{k_3}^2 + (k_1+k_2)^2\right)^{-2}\left(\upsilon_1\mathbf{K}_\alpha + \upsilon_2\mathbf{N}_\alpha + \upsilon_3\mathbf{R}_\alpha\right).$$

In the light of the last equation, the curvature and principal normal vector of σ_1 are found as in the following:

$$\begin{aligned}
\kappa_{\sigma_1} &= \frac{\sqrt{2\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}{\left(2{k_3}^2 + (k_1+k_2)^2\right)^2} \\
\mathbf{N}_{\sigma_1} &= \frac{1}{\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}}\left(\upsilon_1\mathbf{K}_\alpha + \upsilon_2\mathbf{N}_\alpha + \upsilon_3\mathbf{R}_\alpha\right).
\end{aligned}$$

By vector product of $\mathbf{T}_{\sigma_1}$ and $\mathbf{N}_{\sigma_1}$, we may immediately have the binormal vector of σ_1 as

$$\mathbf{B}_{\sigma_1} = \frac{(-k_3\upsilon_3 - k_3\upsilon_2)\mathbf{K}_\alpha + (k_3\upsilon_1 + k_2\upsilon_3 + k_1\upsilon_3)\mathbf{N}_\alpha + (-k_1\upsilon_2 - k_2\upsilon_2 + k_3\upsilon_1)\mathbf{R}_\alpha}{\sqrt{\left(2{k_3}^2 + (k_1+k_2)^2\right)\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}.$$

The torsion of σ_1 is obtained similarly. This is left to the readers.

Definition 3.10. *The special trajectories generated by* $\mathbf{K}_\alpha\mathbf{N}_\alpha-$ *Smarandache curves are defined as in the following:*

$$\sigma_2(s^*) = \frac{1}{\sqrt{2}}(\mathbf{K}_\alpha + \mathbf{N}_\alpha).$$

For convenience, we call them as $\mathbf{K}_\alpha\mathbf{N}_\alpha-$ *Smarandache trajectories.*

Definition 3.11. *The special trajectories generated by* $\mathbf{K}_\alpha\mathbf{R}_\alpha-$ *Smarandache curves are defined as in the following:*

$$\sigma_3(s^*) = \frac{1}{\sqrt{2}}(\mathbf{K}_\alpha + \mathbf{R}_\alpha).$$

For convenience, we call them as $\mathbf{K}_\alpha\mathbf{R}_\alpha-$*Smarandache trajectories.*

For the trajectories σ_2 and σ_3, by following steps similar to those we have followed until now, one can easily find .

$$\mathbf{T}_{\sigma_2} = \frac{1}{\sqrt{2{k_1}^2 + (k_2+k_3)^2}} \left(-k_1\mathbf{K}_\alpha + k_1\mathbf{N}_\alpha + (k_2+k_3)\,\mathbf{R}_\alpha\right)$$

$$\kappa_{\sigma_2} = \frac{\sqrt{2\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}{\left(2{k_1}^2+(k_2+k_3)^2\right)^2}$$

$$\mathbf{N}_{\sigma_2} = \frac{1}{\sqrt{{\xi_1}^2+{\xi_2}^2+{\xi_3}^2}}\left(\xi_1\mathbf{K}_\alpha + \xi_2\mathbf{N}_\alpha + \xi_3\mathbf{R}_\alpha\right)$$

$$\mathbf{B}_{\sigma_2} = \frac{(k_1\xi_3 - k_2\xi_2 - k_3\xi_2)\mathbf{K}_\alpha + (k_2\xi_1 + k_3\xi_1 + k_1\xi_3)\mathbf{N}_\alpha - (k_1\xi_2 + k_1\xi_1)\mathbf{R}_\alpha}{\sqrt{\left(2{k_1}^2+(k_2+k_3)^2\right)\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}$$

$$\mathbf{T}_{\sigma_3} = \frac{1}{\sqrt{2{k_2}^2+(k_1-k_3)^2}}\left(-k_2\mathbf{K}_\alpha + (k_1-k_3)\,\mathbf{N}_\alpha + k_2\mathbf{R}_\alpha\right)$$

$$\kappa_{\sigma_3} = \frac{\sqrt{2\left({\mu_1}^2+{\mu_2}^2+{\mu_3}^2\right)}}{\left(2{k_2}^2+(k_1-k_3)^2\right)^2}$$

$$\mathbf{N}_{\sigma_3} = \frac{1}{\sqrt{{\mu_1}^2+{\mu_2}^2+{\mu_3}^2}}\left(\mu_1\mathbf{K}_\alpha + \mu_2\mathbf{N}_\alpha + \mu_3\mathbf{R}_\alpha\right)$$

$$\mathbf{B}_{\sigma_3} = \frac{(k_1\mu_3 - k_3\mu_3 - k_2\mu_2)\mathbf{K}_\alpha + (k_2\mu_1 + k_2\mu_3)\mathbf{N}_\alpha - (k_2\mu_2 - k_3\mu_1 + k_1\mu_1)\mathbf{R}_\alpha}{\sqrt{\left(2{k_2}^2+(k_1-k_3)^2\right)\left({\mu_1}^2+{\mu_2}^2+{\mu_3}^2\right)}}$$

where

$$\xi_1 = -2{k_1}^4 + \left[k_1k'_2 + k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 - k'_1(k_2+k_3) - k_2\left(2{k_1}^2+(k_2+k_3)^2\right)\right](k_2+k_3)$$

$$\xi_2 = -2{k_1}^4 - \left[k_1k'_2 + k_1k'_3 + {k_1}^2k_2 + {k_1}^2k_3 - k'_1(k_2+k_3) + k_3\left(2{k_1}^2+(k_2+k_3)^2\right)\right](k_2+k_3)$$

$$\xi_3 = 2{k_1}^2\left[k'_2 + k'_3 + k_1k_3 - k_1k_2\right] - \left[2k'_1 + {k_2}^2 - {k_3}^2\right]k_1(k_2+k_3)$$

$$\mu_1 = -2{k_2}^4 - \left[k_2(k'_1 - k'_3) - 2k_1{k_2}^2 + k'_2(k_3-k_1) + {k_2}^2(k_3-k_1) - k_1(k_3-k_1)^2\right](k_3-k_1)$$

$$\mu_2 = 2{k_2}^2\left[k'_1 - k'_3 - k_1k_2 - k_2k_3\right] - \left[2k'_2 + {k_1}^2 - {k_3}^2\right]k_2(k_1-k_3)$$

$$\mu_3 = -2{k_2}^4 + \left[-k_2(k'_1 - k'_3) + 2k_3{k_2}^2 - k'_2(k_3-k_1) + {k_2}^2(k_3-k_1) + k_3(k_3-k_1)^2\right](k_1-k_3).$$

Finally, we must emphasize that the results given in this subsection are in accordance with the results given in (Özen and Tosun (2021b)). This case arises from the similarity between Type 2-PAF and PAF derivative formulas. We suggest the readers to keep the differences between the Type 2-PAF and PAF apparatuses in mind to avoid misunderstanding.

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