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Research Article

New algorithms for solving pseudo-monotone variational inequalities in Banach spaces

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Abstract

In this paper, we introduce new algorithms for finding a solution of a variational inequality problem involving pseudo-monotone operator which is also a fixed point of a Bregman relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces that are more general than Hilbert spaces. We prove weak and strong convergence theorems for proposed algorithms. Finally, we give some numerical experiments for supporting our main results.

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Keywords. variational inequality, pseudo-monotone operator, Bregman distance, Bregman projection

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and its norm $\|.\|$, C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be an operator. The classical variational inequality problem (VI) for F on C is to find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote by $Sol(C, F)$ the solution set of problem (1.1). This problem was first introduced in [17, 45] for modeling problems arising from mechanics. Variational inequality problem plays an important role in many fields such as in transportation, engineering mechanics, economics and others [2, 15, 28, 30]. There are several papers available in the literature which are devoted to this subject, most of which deal with conditions for the existence of a solution (cf. [8,16,26,46,47,53]). Many numerical iterative methods have been constructed for solving variational inequalities and their related optimization problems (see [9,10,20] and the references therein). The simplest one is the following projection method, which can be considered an extension of the projected gradient method for optimization problems:

$$x_{n+1} = P_C(x_n - \lambda F(x_n)), \quad n \geq 1, \tag{1.2}$$

where P_C denotes the metric projection from H onto C [46]. Convergence results for this method require some monotonicity properties of F. This method converges under quite

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strong hypotheses. If F is Lipschitz continuous with Lipschitz constant L and α-strongly monotone, then the sequence generated by (1.2) converges to an element of $Sol(C,F)$ for $\lambda \in (0, \frac{2\alpha}{L^2})$. In order to find an element of $Sol(C,F)$ under weaker hypotheses, Korpelevich [31] introduced the following double projection method in Euclidean space:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda F(y_n)), \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$ and $F : \mathbb{R}^m \to \mathbb{R}^m$ is monotone and L-Lipschitz continuous. The sequence $\{x_n\}$ generated by (1.3) converges to an element of $Sol(C,F)$ provided that $Sol(C,F)$ is nonempty. In recent years, the extragradient method has been extended to infinite dimensional spaces in various ways; see, for example, [8–10, 14, 35, 49] and the references cited therein.

We may observe that, when F is not Lipschitz continuous or the constant L is very difficult to compute, Korpelevich's method is not so practical because we cannot determine the step size λ. To overcome this difficulty, Iusem [23] proposed in the Euclidean space $\mathbb{R}^n$ the following iterative algorithm for solving $Sol(C,F)$:

$$y_n = P_C(x_n - \gamma_n F x_n), \quad x_{n+1} = P_C(x_n - \lambda_n F y_n) \tag{1.4}$$

where $\gamma_n > 0$ is computed through an Armijo-type search and $\lambda_n = \frac{\langle F y_n, x_n - y_n \rangle}{\|F y_n\|^2}$. This modification has allowed the authors to establish convergence without assuming Lipschitz continuity of the operator F.

In (1.4), we require an Armijo-like line search procedure to compute the step size γ_n with a new projection needed for each trial, which leads to expensive computation. To overcomes this difficulty Iusem and Svaiter [26] proposed a modified extragradient method for solving monotone variational inequalities which only requires one projection onto C at each iteration. A few years later, this method was improved by Solodov and Svaiter [47]. They introduced an algorithm for solving (1.1) in finite dimensional spaces. As a matter of fact, their method applies to a more general case, where F is merely continuous and satisfies the following condition:

$$\langle Fx, x - x^* \rangle \geq 0, \quad \forall x \in C \text{ and } x^* \in Sol(C,F). \tag{1.5}$$

Property (1.5) holds if F is monotone or, more generally, pseudo-monotone on C in the sense of Karamardian [27]. Vuong and Shehu [51] have recently modified the result of Solodov and Svaiter [47] in the spirit of Halpern [18], and obtained strong convergence in infinite-dimensional real Hilbert spaces. Recently, Reich et al. [43] introduced new algorithms for solving variational inequalities with uniformly continuous pseudo-monotone operators. In particular, they used a different Armijo-type line search in order to obtain a hyperplane which strictly separates the current iterate from the solutions of the variational inequality under consideration. Their algorithm is of the following form:

Algorithm 1.1. Initialization: Choose $\mu>0, \lambda\in(0, \frac{1}{\mu}), l\in(0,1)$. Let $x_1\in C$ be arbitrary.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute

$$z_n = P_C(x_n - \lambda F x_n),$$

and $r_\lambda(x_n) :=x_n - z_n$. If $r_\lambda(x_n)=0$, then stop; x_n belongs to $Sol(C,F)$. Otherwise,
Step 2. Compute

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := l^{j_n}$ and j_n is the smallest non-negative integer j satisfying:

$$\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \frac{\mu}{2}\|r_\lambda(x_n)\|^2,$$

Step 3. Compute

$$x_{n+1} = P_{C_n}(x_n),$$

where

$$C_n := \{x \in C : h_n(x) \leq 0\},$$

and

$$h_n(x) = \langle Fy_n, x - x_n\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(x_n)\|^2.$$

Set $n := n+1$ and go to Step 1.
Reich et al. proved that if the operator $F : C \to H$ is pseudo-monotone and uniformly continuous on C and satisfies:

$$\text{whenever} \quad \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|F(z)\| \leq \liminf_{n\to\infty} \|Fx_n\|,$$

then any sequence $\{x_n\}$ generated by Algorithm 1.1 converges weakly to an element of $Sol(C,F)$. In addition, they introduced another algorithm and proved strong convergence theorem for the sequences generated by this new method.
Many iterative methods have been proposed for finding a common element of fixed point set $Fix(T)$ and the solution set $Sol(C,F)$ of variational inequality problem (1.1) in Hilbert space H, see, e.g.,[6, 7, 36, 37, 49] and the references therein. The motivation for studying this problem is that many mathematical models such as signal processing, image recovery and network resource allocation can be expressed as fixed point problems and variational inequality problems, see [21, 22, 33] and the references therein. Takahashi and Toyoda [48] introduced the following iterative algorithm for finding a common element of solution set $Sol(C,F)$ and $Fix(T)$:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n TP_C(x_n - \lambda_n Fx_n).$$

Precisely, they proved that the sequence $\{x_n\}$ generated by the above algorithm converges weakly to some element of $Sol(C,F) \cap Fix(T)$.
Throughout this paper, let E be a p-uniformly convex and uniformly smooth Banach space and C be a nonempty, closed and convex subset of E. We shall denote the dual space of E by E^*. The norm and the duality pairing between E and E^* are respectively denoted by $\|.\|$ and $\langle ., .\rangle$.

Motivated and inspired by [43] and by the ongoing research in these directions, we introduce new algorithms for finding a solution of a variational inequality problem involving pseudo-monotone operator which is also a fixed point of a Bregman relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces that are more general than Hilbert spaces. We prove weak and strong convergence theorems for the proposed algorithms. Finally, we give some numerical experiments which support our main results.
The paper is organized as follows: In section 2, we recall some definitions and preliminary results for further use. Section 3 deals with our algorithms and the relevant convergence analysis. Finally, in section 4, we present some numerical experiments which illustrate the performance of the algorithms.

2. Preliminaries

In this section, we recall some definitions and preliminaries. Let C be a nonempty, closed and convex subset of Banach space E. Let $rB = \{z \in E : \|z\| \leq r\}$ for all $r > 0$.

A function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded sets if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by:

$$\rho_r(t) = \inf_{x,y\in rB, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$ [54]. The function ρ_r is called the gauge of uniform convexity of f. It is known that ρ_r is a nondecreasing function. Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_E : [0, 2] \to [0, 1]$ is defined by:

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}.$$

E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, p-uniformly convex if there is a $c_p > 0$ so that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness $\rho_E : [0, \infty) \to [0, \infty)$ is defined by:

$$\rho_E(\tau) = \sup\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called uniformly smooth if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$, q-uniformly smooth if there is a $C_q > 0$ so that $\rho_E(\tau) \leq C_q\tau^q$ for any $\tau > 0$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth [32].

For any convex mapping $f : E \to \mathbb{R}$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at $x \in E$ in the direction y, that is

$$f^\circ(x, y) := \lim_{t\downarrow 0} \frac{f(x+ty) - f(x)}{t}. \tag{2.1}$$

If the limit as $t \to 0$ in (2.1) exists for each y, then the function f is said to be Gâteaux differentiable at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^\circ(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in E$. When the limit as $t \to 0$ in (2.1) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is Fréchet differentiable at x. Finally, f is said to be uniformly Fréchet differentiable on a subset K of E if the limit is attained uniformly for $x \in K$ and $\|y\| = 1$.

A Banach space E is called smooth if its norm is Gâteaux differentiable. Let $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by:

$$J_E^p(x) = \{f \in E^*, \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in E$. We know that E is smooth if and only if J_E^p is single-valued mapping of E into E^*. We also know that E is reflexive if and only if J_E^p is surjective, and E is strictly convex if and only if J_E^p is one-to-one. Therefore, if E is smooth, strictly convex and reflexive Banach space, then J_E^p is a single-valued bijection and in this case, $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the duality mapping of E^*. Furthermore, we known that E is uniformly smooth if and only if the mapping $f_p(x) = \frac{1}{p}\|x\|^p$ is uniformly Fréchet differentiable on bounded sets if and only if J_E^p is single-valued and uniformly continuous on bounded sets. We also known that E is uniformly convex if and only if the mapping f_p is uniformly convex. (see [1, 12, 41]).

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined as:

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \quad x, y \in E.$$

It should be noted that the Bregman distance is not a distance in the usual sense of the term. Clearly $D_f(x, x) = 0$ but $D_f(x, y) = 0$ may not imply $x = y$. In general, D_f is not symmetric and does not satisfy the triangle inequality. However, D_f satisfies the three point identity

$$D_f(x,y)+D_f(y,z)-D_f(x,z)=\langle \nabla f(z)-\nabla f(y), x-y\rangle.$$

More information regarding Bregman functions and distances can be found in [40].
It is worth noting that the duality mapping J_E^p on the smooth Banach space E is the derivative of the function f_p. Then the Bregman distance with respect to f_p is given by:

$$\begin{aligned} D_{f_p}(x,y) =& \frac{1}{p}(\|x\|^p-\|y\|^p)+\langle J_E^p(y), y-x\rangle \\ =& \frac{1}{p}\|x\|^p+\frac{1}{q}\|y\|^p-\langle J_E^p(y),x\rangle \\ =& \frac{1}{q}(\|y\|^p-\|x\|^p)-\langle J_E^p(y)-J_E^p(x),x\rangle. \end{aligned}$$

For the p-uniformly convex space, the metric and Bregman distance have the following relation :

$$\tau\|x-y\|^p \leq D_{f_p}(x,y)\leq \langle J_E^p(x)-J_E^p(y), x-y\rangle, \tag{2.2}$$

where $\tau>0$ is some fixed number [44].
Let C be a nonempty closed and convex subset of reflexive, smooth and strictly convex Banach space E. Bregman projections are defined as minimizers of Bregman distances. The Bregman projection of $x\in E$ onto C with respect to the function f_p is the unique element $\Pi_C x\in C$ such that

$$D_{f_p}(\Pi_C x, x)=\min_{y\in C} D_{f_p}(y,x).$$

In Hilbert spaces the Bregman projection with respect to the function f_2 coincides with the metric projection, but in general they differ from each other. Using ([4, Corollary 4.4]) and [5, Theorem 2.1]), in uniformly convex Banach spaces Bregman projections can be characterized by the variational inequality:

$$\langle J_E^p(x)-J_E^p(\Pi_C x), \Pi_C x-y\rangle \geq 0, \quad \forall y\in C. \tag{2.3}$$

Moreover this variational inequality is equivalent to the descent property

$$D_{f_p}(y,\Pi_C x)+D_{f_p}(\Pi_C x,x)\leq D_{f_p}(y,x), \quad \forall y\in C. \tag{2.4}$$

For $p=2$, the duality mapping J_E^p, is called the normalized duality and is denoted by J. The function $\phi: E^2\to\mathbb{R}$ is defined by:

$$\phi(y,x)=\| y\|^2-2\langle Jx,y\rangle+\| x\|^2, \quad \forall x,y\in E,$$

and

$$\Pi_C(x)=argmin_{y\in C}\phi(y,x), \quad \forall x\in E.$$

Following [1, 11], we make use of the function $V_{f_p}: E\times E^*\to[0,+\infty)$ associated with f_p which is defined by:

$$V_{f_p}(x,x^*)=\frac{1}{p}\|x\|^p-\langle x^*,x\rangle+\frac{1}{q}\|x^*\|^q, \ \ \forall x\in E, x^*\in E^*. \tag{2.5}$$

So $V_{f_p}(x,x^*)=D_{f_p}(x,J_{E^*}^q(x^*))$ for all $x\in E$ and $x^*\in E^*$. Moreover, by the subdifferential inequality, we have

$$V_{f_p}(x,x^*)+\langle y^*, J_{E^*}^q(x^*)-x\rangle \leq V_{f_p}(x,x^*+y^*), \tag{2.6}$$

for all $x \in X$ and $x^*, y^* \in E^*$ [29]. Furthermore V_{f_p} is convex in the second variable. Thus for all $z \in X$, we have

$$D_{f_p}\left(z, J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_{f_p}(z, x_i), \tag{2.7}$$

where $\{x_i\}_{i=1}^N \subset X$ and $\{t_i\}_{i=1}^N \subset [0,1]$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.1. *[5] Let E be a uniformly convex Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences in E such that the first one is bounded. If $\lim_{n\to\infty} D_{f_p}(y_n, x_n) = 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Let $T : C \to C$ be a mapping. We denote the set of fixed points of T by $F(T)$, that is $F(T) = \{x \in C : x = Tx\}$. A point $x \in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C that converges weakly to x such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ for the set of asymptotic fixed points of T. The concept of an asymptotic fixed point was introduced by Reich in [42].
A mapping $T : C \to C$ is called Bregman relatively nonexpansive with respect to f_p, if $F(T) = \widehat{F}(T) \neq \emptyset$ and $D_{f_p}(u, Tx) \leq D_{f_p}(u, x)$, for all $x \in C$ and $u \in F(T)$.

Definition 2.2. Let C be a nonempty subset of E, the mapping $F{:}C{\to}E^*$ is said to be
(i) monotone on C if for any $x, y \in C$,

$$\langle F(x) - F(y), x - y\rangle \geq 0,$$

(ii) pseudo-monotone on C if for any $x, y \in C$ the following implication holds:

$$\langle F(x), x - y\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0,$$

(iii) L-Lipschitz continuous on C if there exists a scalar $L > 0$ satisfying

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in C,$$

$(iiii)$ weakly sequentially continuous if for each sequence $\{x_n\}$we have, $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $F(x)$.

We can establish the following lemmas.

Lemma 2.3. *[38] Let E be a Banach space, $r > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f(\sum_{k=0}^n \alpha_k x_k) \leq \sum_{k=0}^n \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in rB$, $\alpha_k \in (0, 1)$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.4. *[24, 25] Let E_1 and E_2 be two Banach spaces. Suppose $F : E_1 \to E_2$ is uniformly continuous on bounded subsets of E_1 and M is a bounded subset of E_1. Then $F(M)$ is bounded.*

Lemma 2.5. *[13] Let C be a nonempty, closed and convex subset of a real Banach space E, and let $F : C \to E^*$ be pseudo-monotone and continuous. Then x^* belongs to $Sol(C, F)$ if and only if*

$$\langle F(x), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.6. *Let E be a smooth and reflexive Banach space such that the duality mapping J_E^p is weakly sequentially continuous. Let $\{x_n\}$ be a sequence in E and C be a nonempty subset of E. Suppose that for every $x \in C$, $\{D_{f_p}(x, x_n)\}$ converges and every weak cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges weakly to a point in C.*

Proof. It follows from (2.2) that $\{x_n\}$ is bounded. So there is at least one weak cluster point of $\{x_n\}$. Hence it is suffices to show that there is exactly one weak cluster point of $\{x_n\}$. Assume that x and y are two weak cluster point of $\{x_n\}$ in C, say $x_{k_n} \rightharpoonup x$ and $x_{l_n} \rightharpoonup y$. Since J_E^p is weakly sequentially continuous, we have $J_E^p(x_{k_n}) \rightharpoonup J_E^p x$ and $J_E^p(x_{l_n}) \rightharpoonup J_E^p y$. It follows from $x, y \in C$, that the sequences $\{D_{f_p}(x, x_n)\}$ and $\{D_{f_p}(y, x_n)\}$ converge. Since

$$D_{f_p}(x, y) + D_{f_p}(y, x_n) - D_{f_p}(x, x_n) = \langle J_E^p x_n - J_E^p y, x - y \rangle,$$

passing to the limit along x_{k_n} and along x_{l_n}, respectively, yields

$$\langle J_E^p x - J_E^p y, x - y \rangle = \langle J_E^p y - J_E^p y, x - y \rangle = 0.$$

Thus $D_{f_p}(x, y) + D_{f_p}(y, x) = 0$ and hence $x = y$. □

It is known that, for $1 < p < \infty$, the sequence space l_p has a sequentially weakly continuous duality map J^p [39].
The following lemma was given in $\mathbb{R}^n$ in [19]. The proof of the lemma is the same if given in Banach spaces. Hence, we state the lemma and omit the proof in Banach spaces.

Lemma 2.7. *Let C be a nonempty closed and convex subset of a Banach space X. Let h be a real-valued function on X and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$dist(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \quad \forall x \in C.$$

where $dist(x, K)$ denotes the distance function from x to K.

Lemma 2.8. *[52] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions:
(i) $\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, or $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.9. *[34] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main results

Now, we introduce our algorithms for finding a solution of a variational inequality problem involving pseudo-monotone and non Lipschitz continuous operator which is also a fixed point of a Bregman relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. In this section, we assume that the following conditions hold.
(B_1) The feasible set C is a nonempty, closed and convex subset of the Banach space E.
(B_2) The operator $F : C \to E^*$ associated with the problem (1.1) is pseudo-monotone and uniformly continuous on C.
(B_3) The operator $F : C \to E^*$ satisfies the following property:

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{one has } \|F(z)\| \leq \liminf_{n\to\infty} \|Fx_n\|.$$

(B_4) $T : C \to C$ is a Bregman relatively nonexpansive mapping.
(B_5) $\Omega := Sol(C, F) \cap F(T) \neq \emptyset$.

3.1. Weak convergence

Algorithm 3.1

Initialization: Let $x_1 \in C$ be arbitrary. Choose $\mu > 0$, $\lambda \in (0, \frac{1}{\mu})$, $l \in (0, 1)$, $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Iterative steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$z_n = \Pi_C(J^q_{E^*}(J^p_E x_n - \lambda F x_n)),$$

and $r_\lambda(x_n) := x_n - z_n$. If $r_\lambda(x_n) = 0$ and $T x_n = x_n$, then stop, x_n belongs to $Sol(C, F) \cap F(T)$. Otherwise,

Step2. Compute

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle F x_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \frac{\mu}{2} D_{f_p}(x_n, z_n). \tag{3.1}$$

Step3. Compute

$$t_n = J^q_{E^*}\Big(\alpha_n J^p_E(x_n) + (1 - \alpha_n) J^p_E(T x_n)\Big),$$

$$x_{n+1} = \Pi_{C_n \cap Q_n}(x_n),$$

where

$$Q_n := \{x \in C : D_{f_p}(x, t_n) \leq D_{f_p}(x, x_n)\},$$

$$C_n := \{x \in C : h_n(x) \leq 0\},$$

and

$$h_n(x) = \langle F y_n, x - x_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n). \tag{3.2}$$

Set $n := n + 1$ and go to Step 1.

The following lemmas are used in the sequel in the proofs of our main results.

Lemma 3.1. *Assume that the sequence $\{x_n\}$ is generated by Algorithm 3.1. Then, we have*

$$\langle F y_n, r_\lambda(x_n)\rangle \geq \frac{1}{\lambda} D_{f_p}(x_n, z_n).$$

Proof. By the definition of z_n and properties of Π_C, we have

$$\langle J^p_E(x_n) - \lambda F x_n - J^p_E(z_n), z - z_n\rangle \leq 0, \quad \forall z \in C.$$

Substituting $z = x_n$ into the last inequality and using the definition of Bregman distance, we have

$$D_{f_p}(x_n, z_n) \leq \langle J^p_E(x_n) - J^p_E(z_n), x_n - z_n\rangle \leq \lambda\langle F x_n, x_n - z_n\rangle,$$

hence, we get the desired result. □

Lemma 3.2. *The Armijo-type search rule* (3.1) *and the sequence $\{x_n\}$ generated by Algorithm 3.1 are well defined.*

Proof. For the proof of first part see [43]. It is easy to see that for every $n \in \mathbb{N}$, C_n and Q_n are closed and convex. We show that $\Omega \subset C_n \cap Q_n$. Let $x^* \in \Omega$. Using (2.7), we have

$$\begin{aligned} D_{f_p}(x^*, t_n) \leq& \alpha_n D_{f_p}(x^*, x_n) + (1 - \alpha_n) D_{f_p}(x^*, T x_n) \\ \leq& \alpha_n D_{f_p}(x^*, x_n) + (1 - \alpha_n) D_{f_p}(x^*, x_n) = D_{f_p}(x^*, x_n), \end{aligned}$$

hence $x^* \in Q_n$. On the other hand, applying Lemma 2.5, we have $\langle Fy_n, y_n - x^*\rangle \geq 0$. Hence

$$\begin{aligned} h_n(x^*) &= \langle Fy_n, x^* - x_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n) \\ &= \langle Fy_n, x^* - y_n\rangle + \langle Fy_n, y_n - x_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n) \\ &\leq -\tau_n \langle Fy_n, r_\lambda(x_n)\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n). \end{aligned} \tag{3.3}$$

Due to (3.1), we have

$$\langle Fx_n - Fy_n, r_\lambda(x_n)\rangle \leq \frac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n).$$

Using this and Lemma 3.1, we have

$$\begin{aligned} \langle Fy_n, r_\lambda(x_n)\rangle &\geq \langle Fx_n, r_\lambda(x_n)\rangle - \frac{\mu}{2} D_{f_p}(x_n, z_n) \\ &\geq \frac{1}{\lambda} D_{f_p}(x_n, z_n) - \frac{\mu}{2} D_{f_p}(x_n, z_n). \end{aligned}$$

This together with (3.3) implies that

$$h_n(x^*) \leq -\frac{\tau_n}{2}\left(\frac{1}{\lambda} - \mu\right) D_{f_p}(x_n, z_n) \leq 0,$$

and hence $\Omega \subset C_n \cap Q_n$. Hence the sequence $\{x_n\}$ is well define. □

Lemma 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$ and $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$, then $z \in Sol(C, F)$.*

Proof. Applying (2.3), we have

$$\langle J_E^p x_{n_k} - \lambda F x_{n_k} - J_E^p z_{n_k}, x - z_{n_k}\rangle \leq 0, \quad \forall x \in C,$$

therefore

$$\left\langle \frac{J_E^p x_{n_k} - J_E^p z_{n_k}}{\lambda}, x - z_{n_k}\right\rangle \leq \langle F x_{n_k}, x - z_{n_k}\rangle, \qquad \forall x \in C,$$

which implies

$$\left\langle \frac{J_E^p x_{n_k} - J_E^p z_{n_k}}{\lambda}, x - z_{n_k}\right\rangle + \langle F x_{n_k}, z_{n_k} - x_{n_k}\rangle \leq \langle F x_{n_k}, x - x_{n_k}\rangle \quad \forall x \in C. \tag{3.4}$$

Using the boundedness of $\{Fx_{n_k}\}$ and uniform continuity of J_E^p on bounded subsets of E, we get

$$\liminf_{k\to\infty} \langle F x_{n_k}, x - x_{n_k}\rangle \geq 0. \tag{3.5}$$

Now, we choose a decreasing sequence $\{\epsilon_k\}$ of positive numbers which $\epsilon_k \to 0$ as $k \to \infty$. For each k, we denote by N_k the smallest positive integer such that

$$\langle F x_{n_j}, x - x_{n_j}\rangle + \epsilon_k \geq 0, \qquad \forall j \geq N_k, \tag{3.6}$$

where the existence of N_k follows from (3.5). Since the sequence $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Let $A := \{k \in \mathbb{N} : F(x_{N_k}) = 0\}$. If A is infinite, then the assertion is obvious. Otherwise there exists a subsequence $\{N_{k_m}\}$ of $\{N_k\}$ such that $F(x_{N_{k_m}}) \neq 0$. Without loss of generality, we can assume that $F(x_{N_k}) \neq 0$. Setting

$$\nu_{N_k} := \frac{F x_{N_k}}{\|F x_{N_k}\|^{\frac{q}{q-1}}},$$

we have $\langle Fx_{N_k}, J^q_{E^*}\nu_{N_k}\rangle = 1$ for each k. Indeed, using Proposition 4.7 of [12], we get

$$\langle Fx_{N_k}, J^q_{E^*}\nu_{N_k}\rangle = \frac{\|Fx_{N_k}\|^{q-1}}{(\|Fx_{N_k}\|^{\frac{q}{q-1}})^{q-1}} \times \frac{1}{\|Fx_{N_k}\|^{q-1}}\langle Fx_{N_k}, J^q_{E^*}(Fx_{N_k})\rangle = 1,$$

for each k. It follows from (3.6) that

$$\langle Fx_{N_k}, x - x_{N_k}\rangle + \epsilon_k\langle Fx_{N_k}, J^q_{E^*}\nu_{N_k}\rangle \geq 0,$$

therefore

$$\langle Fx_{N_k}, x + \varepsilon_k J^q_{E^*}\nu_{N_k} - x_{N_k}\rangle \geq 0.$$

By the pseudo-monotonicity of F, we obtain

$$\langle F(x + \epsilon_k J^q_{E^*}\nu_{N_k}), x + \epsilon_k J^q_{E^*}\nu_{N_k} - x_{N_k}\rangle \geq 0. \tag{3.7}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k J^q_{E^*}\nu_{N_k}=0$. Since, $x_{n_k} \rightharpoonup z$, using the condition B_3, we obtain

$$0 \leq \|Fz\| \leq \liminf_{k\to\infty} \|Fx_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and ϵ_k tends to zero, we get

$$0 \leq \limsup \|\epsilon_k J^q_{E^*}\nu_{N_k}\| = \limsup \frac{\epsilon_k}{\|Fx_{N_k}\|} \leq \frac{\limsup_{k\to\infty}\epsilon_k}{\liminf_{k\to\infty}\|Fx_{N_k}\|} = 0,$$

this implies that $\lim_{k\to\infty} \epsilon_k J^q_{E^*}\nu_{N_k} = 0$. Hence, taking the limit as $k \to \infty$ in (3.7) and using condition B_2, we get $\langle Fx, x - z\rangle \geq 0$. Using Lemma 2.5, we get $z \in Sol(C, F)$.

□

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. If $\lim_{n\to\infty} \tau_n D_{f_p}(x_n, z_n) = 0$, then $\lim_{n\to\infty} D_{f_p}(x_n, z_n) = 0$.*

Proof. For the proof we consider two possible cases:
Case 1. In this case, we suppose that $\liminf_{n\to\infty} \tau_n > 0$. Therefore, there is a constant $\tau > 0$ such that $\tau_n \geq \tau > 0$ for all $n \in \mathbb{N}$. We obtain that

$$D_{f_p}(x_n, z_n) = \frac{1}{\tau_n}\tau_n D_{f_p}(x_n, z_n) \leq \frac{1}{\tau}\tau_n D_{f_p}(x_n, z_n). \tag{3.8}$$

Considering the limit in the last inequality as $n \to \infty$ and using the assumptions, we have

$$\lim_{n\to\infty} D_{f_p}(x_n, z_n) = 0.$$

Case 2. We suppose that $\liminf_{n\to\infty} \tau_n = 0$. Taking a subsequence if necessary, we may assume without loss of generality that $\lim_{n\to\infty} \tau_n=0$. Define $y_n=\frac{1}{l}\tau_n z_n+(1-\frac{1}{l}\tau_n)x_n$. Applying (2.2) and noting that $\lim_{n\to\infty} \tau_n D_{f_p}(x_n, z_n)=0$, we have $\lim_{n\to\infty} \tau_n\|x_n - z_n\|^p=0$ and hence

$$\lim_{n\to\infty} \|y_n - x_n\|^p = \lim_{n\to\infty} (\frac{\tau_n^{p-1}}{l^p})\tau_n\|z_n - x_n\|^p = 0. \tag{3.9}$$

Since F is uniformly continuous on bounded subsets of C, we obtain

$$\lim_{n\to\infty} \|Fx_n - Fy_n\| = 0. \tag{3.10}$$

Using (3.1) and the definition of y_n, we have

$$\langle Fx_n - Fy_n, x_n - z_n\rangle > \frac{\mu}{2}D_{f_p}(x_n, z_n). \tag{3.11}$$

Now, letting $n\to\infty$ and from (3.10) we have $\lim_{n\to\infty} D_{f_p}(x_n, z_n)=0$ and hence $\lim_{n\to\infty} \|x_n-z_n\|=0$.

□

Now, we are ready to prove the weak convergence theorem.

Theorem 3.5. *Let E be a p-uniformly convex and uniformly smooth Banach space such that the duality mapping J_E^p is weakly sequentially continuous. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to an element of Ω.*

Proof. Let $w \in \Omega$. From (2.2) and (2.4) we have

$$\begin{aligned} D_{f_p}(w, x_{n+1}) \leq & D_{f_p}(w, x_n) - D_{f_p}(x_{n+1}, x_n) \\ = & D_{f_p}(w, x_n) - D_{f_p}(\Pi_{C_n \cap Q_n} x_n, x_n) \\ \leq & D_{f_p}(w, x_n) - D_{f_p}(\Pi_{C_n} x_n, x_n) \\ \leq & D_{f_p}(w, x_n) - \tau \|x_n - \Pi_{C_n} x_n\|^p \\ \leq & D_{f_p}(w, x_n) - \tau \|x_n - P_{C_n} x_n\|^p \\ = & D_{f_p}(w, x_n) - \tau dist^p(C_n, x_n). \end{aligned} \tag{3.12}$$

This implies that $\lim_{n\to\infty} D_{f_p}(w, x_n)$ exists. Hence the sequence $\{x_n\}$ is bounded. consequently, we conclude that $\{Fx_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{t_n\}$ are also bounded. Since $\{x_n\}$ is bounded and X is reflexive, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$. We show that $z \in \Omega$. Since $x_{n+1} \in Q_n$, from the definition of Q_n and (3.12), we have

$$\begin{aligned} D_{f_p}(x_{n+1}, t_n) &\leq D_{f_p}(x_{n+1}, x_n) \\ &\leq D_{f_p}(w, x_n) - D_{f_p}(w, x_{n+1}). \end{aligned}$$

This implies that $\lim_{n\to\infty} D_{f_p}(x_{n+1}, x_n) = \lim_{n\to\infty} D_{f_p}(x_{n+1}, t_n) = 0$, and hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty} \|x_{n+1} - t_n\|.$$

Hence,

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.13}$$

On the other hand using Lemma 2.3, we get

$$\begin{aligned} D_{f_p}(w, t_n) = & V_{f_p}(w, \alpha_n J_E^p x_n + (1-\alpha_n) J_E^p T x_n) \\ \leq & \frac{1}{p}\|w\|^p - \alpha_n \langle J_E^p x_n, w\rangle - (1-\alpha_n)\langle J_E^p T x_n, w\rangle + \frac{\alpha_n}{q}\|J_E^p x_n\|^q \\ & + \frac{(1-\alpha_n)}{q}\|J_E^p T x_n\|^q - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p x_n - J_E^p T x_n\| \\ = & \frac{1}{p}\|w\|^p - \alpha_n \langle J_E^p x_n, w\rangle - (1-\alpha_n)\langle J_E^p T x_n, w\rangle + \frac{\alpha_n}{q}\|x_n\|^p \\ & + \frac{(1-\alpha_n)}{q}\|T x_n\|^p - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p x_n - J_E^p T w_n\| \\ = & \alpha_n D_{f_p}(w, x_n) + (1-\alpha_n) D_{f_p}(w, T x_n) - \alpha_n(1-\alpha_n)\rho_{b^*}\|J_E^p x_n - J_E^p T x_n\| \\ \leq & D_{f_p}(w, x_n) - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p x_n - J_E^p T x_n\|. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p x_n - J_E^p T x_n\| &\leq D_{f_p}(w, x_n) - D_{f_p}(w, t_n) \\ &\leq D_{f_p}(w, x_n) - D_{f_p}(w, t_n) + D_{f_p}(x_n, t_n) \\ &= \langle J_E^p t_n - J_E^p x_n, w - x_n\rangle. \end{aligned}$$

Taking the limit in the above inequality as $n \to \infty$ and using uniform continuity of J_E^p on bounded subsets of E, (3.13) and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we get $\lim_{n\to\infty} \rho_b^*\|J_E^p x_n - J_E^p T x_n\| = 0$ and so $\lim_{n\to\infty} \|J_E^p x_n - J_E^p T x_n\| = 0$. This together with uniform continuity of $J_{E^*}^q$ on bounded subset of E^* implies that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. Therefore $z \in \widehat{F}(T) = F(T)$. Now we show that $z \in Sol(C, F)$. Since $\{Fy_n\}$ is bounded there exists $L > 0$ such that $\|Fy_n\| < L$. Let $x, y \in C$, then

$$|h_n x - h_n y| = |\langle Fy_n, x - y\rangle| \leq \|Fy_n\|\|x - y\| \leq L\|x - y\|,$$

which show that $h_n(x)$ is L-Lipschitz continuous on C. Using Lemma 2.7, we get

$$dist(C_n, x_n) \geq \frac{1}{L} h_n(x_n) = \frac{\tau_n}{2\lambda L} D_{f_p}(x_n, z_n). \tag{3.14}$$

Using (3.12) and (3.14), we obtain

$$\Big(\frac{\tau\tau_n}{2\lambda L} D_{f_p}(x_n, z_n)\Big)^p \leq D_{f_p}(w, x_n) - D_{f_p}(w, x_{n+1}). \tag{3.15}$$

Hence $\lim\limits_{n\to\infty} \tau_n D_{f_p}(x_n, z_n) = 0$. By Lemma 3.4, $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$. This together with Lemma 3.3 imply that $z \in Sol(C, F)$. Now, applying Lemma 2.6, we conclude that $x_n \rightharpoonup z$. □

3.2. Strong convergence

In this subsection, we prove a strong convergence theorem for approximating a solution of a variational inequality which is also a fixed point of a Bregman relatively nonexpansive mapping.

Algorithm 3.2

Initialization. Choose $x_1 \in C$, $\mu > 0$, $l \in (0,1)$ and $\lambda \in (0, \frac{1}{\mu})$. $\beta_n \in (0,1)$ and $\liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$. Also, $\alpha_n \in (0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

Iterative steps: Assume that $x_n \in C$, calculate x_{n+1} as follows:

Step 1. Compute

$$z_n = \Pi_C(J_{E^*}^q(J_E^p x_n - \lambda F x_n)),$$

and $r_\lambda(x_n) := x_n - z_n$. If $r_\lambda(x_n) = 0$ and $Tx_n = x_n$, then stop. Otherwise,

Step 2. Compute

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \tfrac{\mu}{2} D_{f_p}(x_n, z_n).$$

Step 3. Compute

$$t_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n) J_E^p(T\Pi_{C_n} x_n)),$$

$$x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p t_n)),$$

where

$$C_n := \{x \in C : h_n(x) \leq 0\},$$

and

$$h_n(x) = \langle Fy_n, x - x_n\rangle + \tfrac{\tau_n}{2\lambda} D_{f_p}(x_n, z_n).$$

Set n:=n+1 and go to Step 1.

Theorem 3.6. *Suppose that Conditions $B_1 - B_5$ hold. Then, the sequence $\{x_n\}$ generated by Algorithm* 3.2 *converges strongly to* $\Pi_\Omega u$.

Proof. We divide our proof into four steps.

step 1. The sequence $\{x_n\}$ is bounded.

Set $w_n = \Pi_{C_n} x_n$ and $\hat{u} = \Pi_\Omega u$. Using (2.4) and (2.7), we have

$$\begin{aligned} D_{f_p}(\hat{u}, x_{n+1}) \leq & D_{f_p}(\hat{u}, J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E t_n)) \\ \leq & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, t_n) \\ \leq & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, x_n) + (1-\beta_n) D_{f_p}(\hat{u}, w_n)] \\ \leq & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, x_n) + (1-\beta_n) D_{f_p}(\hat{u}, x_n)] \\ \leq & \max\{D_{f_p}(\hat{u}, u), D_{f_p}(\hat{u}, x_n)\} \\ & \vdots \\ \leq & \max\{D_{f_p}(\hat{u}, u), D_{f_p}(\hat{u}, x_1)\}. \end{aligned}$$

This together with (2.2), implies that $\{x_n\}$ is bounded. Consequently, we conclude that $\{Fx_n\}$, $\{z_n\}$, $\{y_n\}$, $\{w_n\}$, $\{Tw_n\}$ and $\{t_n\}$ are also bounded.

Step 2. In this step, we show that

$$(1-\beta_n) D_{f_p}(w_n, x_n) \leq D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle.$$

Set $b = \sup\{\|x_n\|^{p-1}, \|Tw_n\|^{p-1}\}$. Using Lemma 2.3, we have

$$\begin{aligned} D_{f_p}(\hat{u}, t_n) = & V_{f_p}(\hat{u}, \beta_n J^P_E x_n + (1-\beta_n) J^P_E T w_n) \\ \leq & \frac{1}{p}\|\hat{u}\|^p - \beta_n \langle J^p_E x_n, \hat{u} \rangle - (1-\beta_n)\langle J^p_E T w_n, \hat{u} \rangle + \frac{\beta_n}{q}\|J^p_E x_n\|^q \\ & + \frac{(1-\beta_n)}{q}\|J^p_E T w_n\|^q - \beta_n(1-\beta_n)\rho^*_b \|J^p_E x_n - J^p_E T w_n\| \\ = & \frac{1}{p}\|\hat{u}\|^p - \beta_n \langle J^p_E x_n, \hat{u} \rangle - (1-\beta_n)\langle J^p_E T w_n, \hat{u} \rangle + \frac{\beta_n}{q}\|x_n\|^p \\ & + \frac{(1-\beta_n)}{q}\|T w_n\|^p - \beta_n(1-\beta_n)\rho^*_b \|J^p_E x_n - J^p_E T w_n\| \\ = & \beta_n D_{f_p}(\hat{u}, x_n) + (1-\beta_n) D_{f_p}(\hat{u}, T w_n) - \beta_n(1-\beta_n)\rho_{b^*} \|J^p_E x_n - J^p_E T w_n\| \\ \leq & D_{f_p}(\hat{u}, x_n) - \beta_n(1-\beta_n)\rho^*_b \|J^p_E x_n - J^p_E T w_n\|. \end{aligned}$$

Set $s_n = J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E t_n)$. Using (2.6), we have

$$\begin{aligned} D_{f_p}(\hat{u}, x_{n+1}) \leq & D_{f_p}(\hat{u}, J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E t_n)) \\ = & V_{f_p}(\hat{u}, \alpha_n J^p_E u + (1-\alpha_n) J^p_E t_n) \\ \leq & V_{f_p}(\hat{u}, \alpha_n J^p_E u + (1-\alpha_n) J^p_E t_n - \alpha_n (J^p_E u - J^p_E \hat{u})) \\ & + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle \\ \leq & \alpha_n D_{f_p}(\hat{u}, \hat{u}) + (1-\alpha_n) D_{f_p}(\hat{u}, t_n) + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle \\ \leq & (1-\alpha_n) D_{f_p}(\hat{u}, t_n) + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle \qquad (3.16) \\ \leq & (1-\alpha_n) D_{f_p}(\hat{u}, x_n) - \beta_n(1-\beta_n)\rho^*_b \|J^p_E x_n - J^p_E T w_n\| \\ & + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle \qquad (3.17) \\ \leq & (1-\alpha_n) D_{f_p}(\hat{u}, x_n) + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle. \qquad (3.18) \end{aligned}$$

On the other hand

$$\begin{aligned} D_{f_p}(\hat{u}, t_n) \leq & \beta_n D_{f_p}(\hat{u}, x_n) + (1-\beta_n) D_{f_p}(\hat{u}, w_n) \\ \leq & \beta_n D_{f_p}(\hat{u}, x_n) + (1-\beta_n)[(\hat{u}, x_n) - D_{f_p}(w_n, x_n)] \\ = & D_{f_p}(\hat{u}, x_n) - (1-\beta_n) D_{f_p}(w_n, x_n). \end{aligned}$$

Substituting the above inequality into (3.16), we get

$$D_{f_p}(\hat{u}, x_{n+1}) \leq D_{f_p}(\hat{u}, x_n) - (1-\beta_n) D_{f_p}(w_n, x_n) + \alpha_n \langle J^p_E u - J^p_E \hat{u}, s_n - \hat{u} \rangle.$$

This implies that

$$(1-\beta_n)D_{f_p}(w_n,x_n)\leq D_{f_p}(\hat{u},x_n)-D_{f_p}(\hat{u},x_{n+1})+\alpha_n\langle J_E^p u-J_E^p\hat{u},s_n-\hat{u}\rangle. \tag{3.19}$$

Step 3. In this step, we prove that

$$(1-\alpha_n)(1-\beta_n)\Big(\frac{\tau_n}{2\lambda L}D_{f_p}(x_n,z_n)\Big)^p\leq\alpha_n D_{f_p}(\hat{u},u)+D_{f_p}(\hat{u},x_n)-D_{f_p}(\hat{u},x_{n+1}).$$

Employing a similar argument to the one used in the proof of (3.15), we get

$$D_{f_p}(\hat{u},w_n)\leq D_{f_p}(\hat{u},x_n)-\Big(\frac{\tau_n}{2\lambda L}D_{f_p}(x_n,z_n)\Big)^p. \tag{3.20}$$

Applying (3.20), we have

$$\begin{aligned}
D_{f_p}(\hat{u},x_{n+1})\leq& D_{f_p}(\hat{u},J_{E^*}^q(\alpha_n J_E^p u+(1-\alpha_n)J_E^p t_n))\\
\leq&\alpha_n D_{f_p}(\hat{u},u)+(1-\alpha_n)D_{f_p}(\hat{u},t_n)\\
\leq&\alpha_n D_{f_p}(\hat{u},u)+(1-\alpha_n)\Big[\beta_n D_{f_p}(\hat{u},x_n)+(1-\beta_n)D_{f_p}(\hat{u},w_n)\Big]\\
\leq&\alpha_n D_{f_p}(\hat{u},u)+(1-\alpha_n)\beta_n D_{f_p}(\hat{u},x_n)\\
&+(1-\alpha_n)(1-\beta_n)\Big[D_{f_p}(\hat{u},x_n)-\Big(\frac{\tau_n}{2\lambda L}D_{f_p}(x_n,z_n)\Big)^p\Big]\\
\leq&\alpha_n D_{f_p}(\hat{u},u)+D_{f_p}(\hat{u},x_n)-(1-\alpha_n)(1-\beta_n)\Big(\frac{\tau_n}{2\lambda L}D_{f_p}(x_n,z_n)\Big)^p.
\end{aligned} \tag{3.21}$$

Step 4. In this step, we prove that $x_n\to\hat{u}$ as $n\to\infty$.
Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup z$, as $n\to\infty$. In order to prove that $x_n\to\hat{u}$ as $n\to\infty$, we consider two possible cases.
Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{D_{f_p}(\hat{u},x_n)\}_{n=n_0}^\infty$ is nonincreasing. In this situation, $\{D_{f_p}(\hat{u},x_n)\}$ is convergent. Since $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ and the sequence $\{s_n\}$ is bounded, from (3.19), we have $\lim_{n\to\infty}D_{f_p}(w_n,x_n)=0$ and hence,

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.22}$$

By similar argument from inequality (3.17), we obtain that

$$\lim_{n\to\infty}\rho_b^*\|J_E^p x_n-J_E^p Tw_n\|=0,$$

and hence

$$\lim_{n\to\infty}\|J_E^p x_n-J_E^p Tw_n\|=0. \tag{3.23}$$

From the uniform continity of $J_{E^*}^q$ on bounded subset of X^* and (3.23), we have

$$\lim_{n\to\infty}\|x_n-Tw_n\|=0. \tag{3.24}$$

This together with (3.22) implies that

$$\lim_{n\to\infty}\|w_n-Tw_n\|=0. \tag{3.25}$$

Since $x_{n_k}\rightharpoonup z$, from (3.22), we get $w_{n_k}\rightharpoonup z$. Hence, using (3.25) we get $z\in\widehat{F}(T)=F(T)$. Next we show that $z\in Sol(C,F)$. Applying (3.21), we have, $\lim_{n\to\infty}\frac{\tau_n}{2\lambda L}D_{f_p}(x_n,z_n)=0$, and therefore

$$\lim_{n\to\infty}\tau_n D_{f_p}(x_n,z_n)=0. \tag{3.26}$$

Using Lemma 3.4, we infer that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.27}$$

Applying Lemma 3.3 and (3.27), we conclude that $z \in Sol(C, F)$. Next, we show that $\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \leq 0$. We can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \rightharpoonup v \in \Omega$, as $k \to \infty$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup v$ as $j \to \infty$. Using (2.3), we deduce

$$\begin{aligned}\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle &= \lim_{j\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle \\ &= \langle J_E^p u - J_E^p \hat{u}, v - \hat{u}\rangle \leq 0. \end{aligned} \tag{3.28}$$

On the other hand

$$\begin{aligned} D_{f_p}(x_n, s_n) &\leq \alpha_n D_{f_p}(x_n, u) + (1-\alpha_n) D_{f_p}(x_n, t_n) \\ &\leq \alpha_n D_{f_p}(x_n, u) + (1-\alpha_n)[\beta_n D_{f_p}(x_n, x_n) + (1-\beta_n) D_{f_p}(x_n, Tw_n)]. \end{aligned} \tag{3.29}$$

Hence, taking the limit as $n\to\infty$ in (3.29) and using (2.2) and (3.24), we get that $\lim_{n\to\infty} D_{f_p}(x_n, s_n) = 0$, and hence

$$\lim_{n\to\infty}\|x_n - s_n\| = 0. \tag{3.30}$$

This together with (3.28) implies that

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \leq 0. \tag{3.31}$$

Using (3.18), (3.31) and Lemma 2.8, we get that $x_n \to \hat{u}$, as $n \to \infty$.
Case 2. There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $D_{f_p}(\hat{u}, x_{n_j}) < D_{f_p}(\hat{u}, x_{n_j+1})$ for all $j \in \mathbb{N}$. Then by Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_{f_p}(\hat{u}, x_{m_k}) \leq D_{f_p}(\hat{u}, x_{m_k+1}) \quad \textit{and} \quad D_{f_p}(\hat{u}, x_k) \leq D_{f_p}(\hat{u}, x_{m_k+1}), \tag{3.32}$$

for all $k \in \mathbb{N}$. Using (3.32) and employing a similar argument to the one used in case 1, we deduce

$$\lim_{k\to\infty}\|w_{m_k} - Tw_{m_k}\| = \lim_{k\to\infty}\|z_{m_k} - x_{m_k}\| = \lim_{k\to\infty}\|x_{m_k} - s_{m_k}\| = 0, \tag{3.33}$$

and

$$\limsup_{k\to\infty}\langle J_E^p u - J_E^p \hat{u}, s_{m_k} - \hat{u}\rangle \leq 0. \tag{3.34}$$

It follows from (3.18) that

$$D_{f_p}(\hat{u}, x_{m_k+1}) \leq (1-\alpha_{m_k}) D_{f_p}(\hat{u}, x_{m_k}) + \alpha_{m_k}\langle J_E^p u - J_E^p \hat{u}, s_{m_k} - \hat{u}\rangle. \tag{3.35}$$

This together with (3.32) implies that

$$\alpha_{m_k} D_{f_p}(\hat{u}, x_{m_k}) \leq \alpha_{m_k}\langle J_E^p u - J_E^p \hat{u}, s_{m_k} - \hat{u}\rangle.$$

Since $\alpha_{m_k} > 0$ and $\limsup_{k\to\infty}\langle J_E^p u - J_E^p \hat{u}, s_{m_k} - \hat{u}\rangle \leq 0$, we deduce

$$\lim_{k\to\infty} D_{f_p}(\hat{u}, x_{m_k}) = 0. \tag{3.36}$$

From (3.34), (3.35) and (3.36), we get that

$$\lim_{k\to\infty} D_{f_p}(\hat{u}, x_{m_k+1}) = 0. \tag{3.37}$$

Now from (3.32), we have $\lim_{k\to\infty} D_{f_p}(\hat{u}, x_k) = 0$. Hence $\lim_{k\to\infty}\|x_k - \hat{u}\| = 0$. This completes the proof. □

Setting $F=0$ in Algorithm 3.1, we immediately obtain the following result for the fixed pint problem.

Corollary 3.7. *Let E be a p-uniformly convex and uniformly smooth Banach space such that the duality mapping J_E^p is weakly sequantially continuous. Let $T:C\to C$ be a Bregman relatively nonexpansive mapping such that $F(T)\neq\emptyset$. For $x_1\in C$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} t_n = J_{E^*}^q\Big(\alpha_n J_E^p(x_n) + (1-\alpha_n)J_E^p(Tx_n)\Big),\\ Q_n = \{x\in C : D_{f_p}(x,t_n)\leq D_{f_p}(x,x_n)\},\\ x_{n+1} = \Pi_{Q_n}(x_n),\end{cases}$$

where $\alpha_n\in(0,1)$ and $\liminf\limits_{n\to\infty}\alpha_n(1-\alpha_n)>0$. Then the sequence $\{x_n\}$ converges weakly to an element of $F(T)$.

Putting $T=I$, the identity mapping, in Algorithm 3.2, we obtain the following corollary for the variational inequality problem.

Corollary 3.8. *Let $x_1\in C$ and $u\in E$ be arbitrary. Choose $\mu>0, \lambda\in(0,\dfrac{1}{\mu}), l\in(0,1), \beta_n\in(0,1)$ and $\liminf\limits_{n\to\infty}\beta_n(1-\beta_n)>0$. Also, $\alpha_n\in(0,1)$ such that $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$. Compute*

$$z_n = \Pi_C\Big(J_{E^*}^q(J_E^p(x_n)-\lambda Fx_n)\Big),$$

and $r_\lambda(x_n):=x_n-z_n$. If $r_\lambda(x_n)=0$, then stop. Otherwise,
Compute

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \frac{\mu}{2}D_{f_p}(x_n,z_n),$$

Compute

$$t_n = J_{E^*}^q\Big(\beta_n J_E^p(x_n) + (1-\beta_n)J_E^p(\Pi_{C_n}x_n)\Big),$$

where

$$C_n := \{x\in C : h_n(x)\leq 0\},$$

and

$$h_n(x) = \langle Fy_n, x-x_n\rangle + \frac{\tau_n}{2\lambda}D_{f_p}(x_n,z_n).$$

Compute

$$x_{n+1} = \Pi_C\left(J_{E^*}^q\left(\alpha_n J_E^p u + (1-\alpha_n)J_E^p t_n\right)\right).$$

Assume that Conditions B_1-B_5 hold. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{Sol(C,F)}u$.

4. Numerical experiments

In this section, we perform two numerical experiments to illustrate the behavior of Algorithm 3.1 and Algorithm 3.2. In the next remark, we provide closed form expressions for the projectors onto a half space.

Remark 4.1. [3] Let H be a Hilbert space, $u \in H$, $\eta \in \mathbb{R}$ and $C = \{x \in H : \langle x, u \rangle \leq \eta\}$. Then exactly one of the following holds:
(i) $u = 0$ and $\eta \geq 0$, in which case $C = H$ and $P_C = Id$.
(ii) $u = 0$ and $\eta < 0$, in which case $C = \emptyset$.
(iii) $u \neq 0$, in which case $C \neq \emptyset$ and for all x in H we have

$$P_C x = \begin{cases} x, & \langle x, u \rangle \leq \eta, \\ x + \dfrac{\eta - \langle x, u \rangle}{\|u\|^2} u, & \langle x, u \rangle > \eta. \end{cases}$$

Example 4.2. Let $E=\mathbb{R}$ and $C=[0,4]$. Define $F(x)=\sqrt{x}+\sin x$ and $Tx=\arctan x$. We consider $\mu=\frac{1}{2}$, $\lambda = 1$, $l = 0.9$ and $\alpha_n = \frac{1}{2} + \frac{1}{n+2}$. Note that F is not Lipschitz continuous and $F(T) \cap Sol(C, F) = \{0\}$. Using Algorithm 3.1 with the initial point $x_1 = 4$, we have the numerical results in Fig 1.

Example 4.3. Take $E = L^2[0,1]$ with inner product $\langle x, y \rangle := \int_0^1 x(t)y(t)dt$ and norm $\|x\|_2 := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$. Suppose $C = \{x \in E : \|x\|_2 \leq 2\}$. Define $T := C \to C$ by $Tx = \frac{1}{2}x$ and $F : L^2[0,1] \to L^2[0,1]$ by:

$$F(x)(t) := \exp^{-\|x\|_2} \int_0^t x(s)ds,$$

where $x \in L^2[0,1]$ and $t \in [0,1]$. It can also be shown that F is pseudo-monotone but not monotone on E [50]. Take $\mu = 0.3$, $\lambda = 3$ and $l = 0.9$. Also, we take $\beta_n = \frac{1}{2} + \frac{1}{n+2}$ and $\alpha_n = \frac{1}{100n+1}$. Using Remark 4.1 and Algorithm 3.2 with the initial point $x_1(t) = t^2$ we have the numerical results in Fig 2.

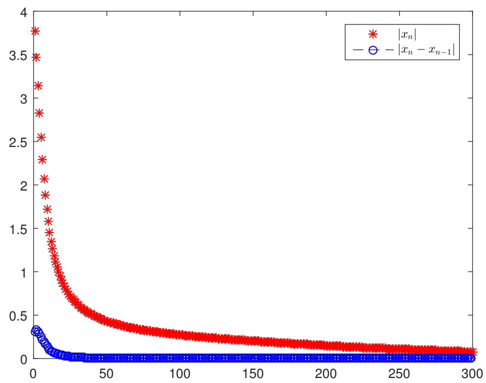

Figure 1. Plotting of $|x_n|$ and $|x_n - x_{n-1}|$ in Example 4.2

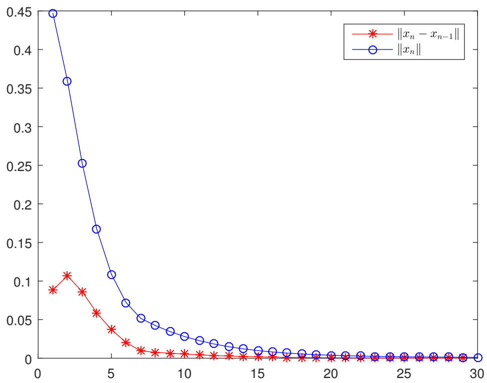

Figure 2. Plotting of $\|x_n\|_2$ and $\|x_n - x_{n-1}\|_2$ in Example 4.3

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