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A Modelling on the Exponential Curves as $Cubic$, 5^{th} and 7^{th} Bézier Curve in Plane

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Abstract
In this study, it has been researched the exponential curve as a 3^{rd}, 5^{th} and 7^{th} order Bézier curve in $\mathbf{E}^2$. Also, the numerical matrix representations of these curves have been calculated using the Maclaurin series in the plane via the control points.

Keywords: Bézier curves, Exponential curve, Maclaurin series, 5^{th} and 7^{th} $order$ Bézier curve.
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1. Introduction and Preliminaries

Bezier curves have special mathematical representations and are obtained with the help of polynomial functions. Since these curves are used in computer aided geometric design and modelling [1], they have an important place in applied fields. The Bezier curve has a control polygon that contains it, and only the start and end points are on the curve, so it provides an advantage in terms of use in modelling. Thus, it provides the opportunity to make the desired changes over the control polygon. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For three dimension animation Bézier curves are often used to define 3D paths as well as two dimension curves for keyframe interpolation. Apart from the Bézier-curves' frequent use in applied sciences, the theory has been studied by many researchers in mathematical points of view. The matrix form was first coined in [2]. The derivatives of the Bezier curves in matrix notation was studied in [3]. Particularly, the 5^{th} order Bezier curve and its derivatives were studied by matrices in [4]. Besides, it has been investigated approximation methods in matrix form for Helix, sin waves and cosin curves by different order Bézier curves in [5–7]. The curve is also subjected to the differential geometry. For example: In [8], A dual unit spherical Bézier-like curve corresponds to a ruled surface by using Study's transference principle and closed ruled surfaces are determined via control points and also, integral invariants of these surfaces are investigated. In [9], Bezier-curves with curvature and torsion continuity has been examined. In [10–12], Bezier curves and surfaces has been given and Bezier curves are designed for Computer-Aided Geometric [13]. Recently equivalence conditions of control points and application to planar Bézier curves have been examined. In [14], Frenet apparatus of the cubic Bezier curves has been examined in E^3. In here, first 5^{th} order Bezier curve and its first, second and third derivatives have been examined based on the control points of 5^{th}order Bezier Curve in E^3. Subsequently, in [15, 16] involutes of cubic Bezier curves, in [17] and [18] the Bertrand and the Mannheim mate of a cubic Bézier curve by using matrix representation have been researched in E^3. In [19], it has been researched the answer of the question "How to find a n^{th} order Bezier curve if we know the first, second and third derivatives?".

Generally Bézier curves can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}(t)[P_i]. \tag{1.1}$$

In this study, it will be researched the exponential curve as a 3^{rd}, 5^{th} and 7^{th} order Bézier curve in E^2. Also, the numerical matrix representations of these curves will be calculated via the control points. For more detail, see respectively [20,21].

It is well known that Taylor series of a function $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$ is an infinite sum of the functions derivatives at a single point a, also a Maclaurin series $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$ is a taylor series where $a = 0$.

2. The Curve e^x as a *Cubic* Bézier Curve

We will examine the curve e^x as a *cubic* or 3^{rd} *order* Bézier curve.

Theorem 2.1. *The numerical matrix representation of the curve $f(x) = e^x$ as a cubic Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{11}{6} \\ 1 & \frac{8}{3} \end{bmatrix}$$

where the control points P_0, P_1, P_2, and P_3 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{11}{6} \\ 1 & \frac{8}{3} \end{bmatrix}.$$

Proof. For e^x function cubic Maclaurin series expansion is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

It can be written as in parametric form and a 5^{th} degree polynomial function

$$\left(t, e^t\right) = \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) = \left(t, a_3 t^3 + a_2 t^2 + a_1 t + a_0\right).$$

Also this can be written as a *cubic* Bézier curve in matrix representation with the coefficients

$$\begin{aligned} a_3 &= \tfrac{1}{3!}, \\ a_2 &= \tfrac{1}{2!}, \\ a_1 &= 1, \\ a_0 &= 1. \end{aligned}$$

Hence we get the following equation

$$\begin{aligned} \left(t, e^t\right) &= \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) \\ &= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

where the coefficients matrix of any cubic Bézier curve and inverse matrix are respectively

$$\left[B^3\right]=\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix},\quad \left[B^3\right]^{-1}=\begin{bmatrix}0&0&0&1\\0&0&\frac{1}{3}&1\\0&\frac{1}{3}&\frac{2}{3}&1\\1&1&1&1\end{bmatrix}.$$

For more detail see in [18]. □

3. The Curve e^{ax+b} as a *Cubic* Bézier Curve

Theorem 3.1. *The numerical matrix representation of the curve $f(x)=e^{ax+b}$ as a cubic Bézier curve is*

$$\left(t,e^{at+b}\right)=\begin{bmatrix}t^3\\t^2\\t\\1\end{bmatrix}^T\left[B^3\right]\begin{bmatrix}P_0\\P_1\\P_2\\P_3\end{bmatrix}$$

where the control points P_0, P_1, P_2, and P_3 are

$$\begin{bmatrix}P_0\\P_1\\P_2\\P_3\end{bmatrix}=\begin{bmatrix}0&e^b\\\frac{1}{3}&\frac{1}{3}e^b(a+3)\\\frac{2}{3}&\frac{1}{6}e^b\left(a^2+4a+6\right)\\1&\frac{1}{6}e^b\left(a^3+3a^2+6a+6\right)\end{bmatrix}.$$

Proof. Taylor series of a function is an infinite sum of terms of the functions derivatives at a single point a , also a Maclaurin series is a taylor series where $a=0$. 5th degree Maclaurin series expansion for the function e^{ax+b} is

$$\begin{aligned}f(x)=e^{ax+b}&=\sum_{n=0}^{3}f^{(n)}(0)\frac{x^n}{n!}\\&=e^b+ae^bx+a^2e^b\frac{x^2}{2!}+a^3e^b\frac{x^3}{3!}.\end{aligned}$$

It can be written as in parametric form and a *cubic* polynomial function

$$\begin{aligned}\left(t,e^{at+b}\right)&=\left(t,\frac{a^3e^b}{3!}t^3+\frac{a^2e^b}{2!}t^2+ae^bt+e^b\right)\\&=\left(t,a_3t^3+a_2t^2+a_1t+a_0\right).\end{aligned}$$

Also this can be written as a *cubic* Bézier curve in matrix representation with the coefficients

$$\begin{aligned}a_3&=\tfrac{a^3e^b}{3!},\\a_2&=\tfrac{a^2e^b}{2!},\\a_1&=ae^b,\\a_0&=e^b.\end{aligned}$$

Hence we get the following equation

$$\begin{bmatrix}t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}0&\frac{a^3e^b}{3!}\\0&\frac{a^2e^b}{2!}\\1&ae^b\\0&e^b\end{bmatrix}=\begin{bmatrix}t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}P_0\\P_1\\P_2\\P_3\end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^b \\ \frac{1}{3} & \frac{1}{3}e^b(a+3) \\ \frac{2}{3} & \frac{1}{6}e^b\left(a^2+4a+6\right) \\ 1 & \frac{1}{6}e^b\left(a^3+3a^2+6a+6\right) \end{bmatrix}.$$

□

4. The Curve e^x as a 5^{th} *Order* Bézier Curve

Now, we will examine the curve e^x as a 5^{th} *order* Bézier curve. We have already known that the matrix representation of $\alpha(t) = (t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + at_1 + a_0)$ is

$$\alpha(t) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^5\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

where the coefficient matrix and inverse matrix of 5^{th} order Bézier curve are

$$\left[B^5\right] = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \left[B^5\right]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 4.1. *The numerical matrix representation of the curve $f(x) = e^x$ as a 5^{th} order Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{7}{12} \\ \frac{4}{5} & \frac{11}{15} \\ 1 & \frac{101}{120} \end{bmatrix}$$

where the control points $P_0, P_1, P_2, P_3, P_4,$ and P_5 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{7}{12} \\ \frac{4}{5} & \frac{11}{15} \\ 1 & \frac{101}{120} \end{bmatrix}.$$

Proof. 5th degree Maclaurin series expansion for the function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

It can be written as in parametric form and a 5^{th} degree polynomial function

$$\left(t,e^{t}\right)=\left(t,1+t+\frac{t^{2}}{2!}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+\frac{t^{5}}{5!}\right)=\left(t,a_{5}t^{5}+a_{4}t^{4}+a_{3}t^{3}+a_{2}t^{2}+a_{1}t+a_{0}\right).$$

Also this can be written as a 5^{th} $order$ Bézier curve in matrix representation with the coefficients

$$\left[\begin{array}{cccccc} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{array}\right]=\left[\begin{array}{cccccc} \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 \end{array}\right].$$

Hence we get the following equation

$$\begin{aligned}
\left(t,e^{t}\right)&=\left(t,1+t+\frac{t^{2}}{2!}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+\frac{t^{5}}{5!}\right)\\
&=\left[\begin{array}{c} t^5\\ t^4\\ t^3\\ t^2\\ t\\ 1\end{array}\right]^T\left[\begin{array}{cc} 0 & \frac{1}{5!}\\ 0 & \frac{1}{4!}\\ 0 & \frac{1}{3!}\\ 0 & \frac{1}{2!}\\ 1 & 1\\ 0 & 1\end{array}\right]=\left[\begin{array}{c} t^5\\ t^4\\ t^3\\ t^2\\ t\\ 1\end{array}\right]^T\left[\begin{array}{cccccc} -1 & 5 & -10 & 10 & -5 & 1\\ 5 & -20 & 30 & -20 & 5 & 0\\ -10 & 30 & -30 & 10 & 0 & 0\\ 10 & -20 & 10 & 0 & 0 & 0\\ -5 & 5 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{array}\right]\left[\begin{array}{c} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{array}\right],
\end{aligned}$$

$$\left[\begin{array}{c} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{array}\right]=\left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1\\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1\\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1\\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1\\ 1 & 1 & 1 & 1 & 1 & 1\end{array}\right]\left[\begin{array}{cc} 0 & \frac{1}{5!}\\ 0 & \frac{1}{4!}\\ 0 & \frac{1}{3!}\\ 0 & \frac{1}{2!}\\ 1 & 1\\ 0 & 1\end{array}\right],$$

solving these equation we obtained the control numbers

$$\left[\begin{array}{c} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{array}\right]=\left[\begin{array}{cc} 0 & 1\\ \frac{1}{5} & \frac{6}{5}\\ \frac{2}{5} & \frac{29}{20}\\ \frac{3}{5} & \frac{53}{30}\\ \frac{4}{5} & \frac{87}{40}\\ 1 & \frac{163}{60}\end{array}\right].$$

□

5. The Curve e^{ax+b} as a 5^{th} $Order$ Bézier Curve

In this section we have investigated the curve e^{ax+b} as a 5^{th} $order$ Bézier curve.

$$\begin{aligned}
f(x)&=\sum_{n=0}^{\infty}f^{(n)}(0)\frac{x^n}{n!},\\
f(x)&=e^{ax+b},\\
f'(x)&=ae^{ax+b},\\
f''(x)&=a^2e^{ax+b},\\
f'''(x)&=a^3e^{ax+b},\\
f^{(4)}(x)&=a^4e^{ax+b},\\
f^{(5)}(x)&=a^5e^{ax+b},\\
f^{(6)}(x)&=a^6e^{ax+b},\\
f^{(7)}(x)&=a^7e^{ax+b}.
\end{aligned}$$

Theorem 5.1. *The numerical matrix representation of the curve $f(x) = e^{ax+b}$ as a 5^{th} order Bézier curve is*

$$\left(t, e^{at+b}\right) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

where the control points P_0, P_1, P_2, P_3, P_4 and P_5 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{5} & \frac{1}{5}e^b(a+5) \\ \frac{2}{5} & \frac{1}{20}e^b\left(a^2+8a+20\right) \\ \frac{3}{5} & \frac{1}{60}e^b\left(a^3+9a^2+36a+60\right) \\ \frac{4}{5} & \frac{1}{120}e^b\left(a^4+8a^3+36a^2+96a+120\right) \\ 1 & \frac{1}{120}e^b\left(a^5+5a^4+20a^3+60a^2+120a+120\right) \end{bmatrix}.$$

Proof. Taylor series of a function is an infinite sum of terms of the functions derivatives at a single point a , also a Maclaurin series is a taylor series where $a = 0$. 5th degree Maclaurin series expansion for the function e^{ax+b} is

$$\begin{aligned} f(x) = e^{ax+b} &= \sum_{n=0}^{5} f^{(n)}(0)\frac{x^n}{n!} \\ &= e^b + ae^bx + a^2e^b\frac{x^2}{2!} + a^3e^b\frac{x^3}{3!} + a^4e^b\frac{x^4}{4!} + a^5e^b\frac{x^5}{5!} \end{aligned}$$

and it can be written as in parametric form and a 5^{th} degree polynomial function

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, \frac{a^5e^b}{5!}t^5 + \frac{a^4e^b}{4!}t^4 + \frac{a^3e^b}{3!}t^3 + \frac{a^2e^b}{2!}t^2 + ae^bt + e^b\right) \\ &= \left(t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0\right). \end{aligned}$$

Also this can be written as a 5^{th} *order* Bézier curve in matrix representation with the coefficients

$$\begin{bmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} = \begin{bmatrix} \frac{a^5e^b}{5!} & \frac{a^4e^b}{4!} & \frac{a^3e^b}{3!} & \frac{a^2e^b}{2!} & ae^b & e^b \end{bmatrix}.$$

Hence we get the following equation

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, \frac{a^5e^b}{5!}t^5 + \frac{a^4e^b}{4!}t^4 + \frac{a^3e^b}{3!}t^3 + \frac{a^2e^b}{2!}t^2 + ae^bt + e^b\right) \\ &= \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^5e^b}{5!} \\ 0 & \frac{a^4e^b}{4!} \\ 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix} = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^5e^b}{5!} \\ 0 & \frac{a^4e^b}{4!} \\ 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{5} & e^b + \frac{1}{5}ae^b \\ \frac{2}{5} & \frac{1}{20}e^b a^2 + \frac{2}{5}e^b a + e^b \\ \frac{3}{5} & \frac{1}{60}e^b a^3 + \frac{3}{20}e^b a^2 + \frac{3}{5}e^b a + e^b \\ \frac{4}{5} & \frac{1}{120}e^b a^4 + \frac{1}{15}e^b a^3 + \frac{3}{10}e^b a^2 + \frac{4}{5}e^b a + e^b \\ 1 & \frac{1}{120}e^b a^5 + \frac{1}{24}e^b a^4 + \frac{1}{6}e^b a^3 + \frac{1}{2}e^b a^2 + e^b a + e^b \end{bmatrix},$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!},$$
$$f(x) = e^{ax+b}\frac{x^n}{0!} + ae^{ax+b}\frac{x^n}{1!} + a^2e^{ax+b}\frac{x^n}{2!} + a^3e^{ax+b}\frac{x^n}{3!} + a^4e^{ax+b}\frac{x^n}{4!} + a^5e^{ax+b}\frac{x^n}{5!} + a^6e^{ax+b}\frac{x^n}{6!} + a^7e^{ax+b}\frac{x^n}{7!}.$$

□

6. The Curve e^x as a 7^{th} $Order$ Bézier Curve

Theorem 6.1. *The matrix of any 7^{th} order Bézier curve is*

$$[B^7] = \begin{bmatrix} -\binom{7}{0}\binom{7}{7} & \binom{7}{1}\binom{7-1}{7-1} & -\binom{7}{2}\binom{7-2}{7-2} & \binom{7}{3}\binom{7-3}{7-3} & -\binom{7}{4}\binom{7-4}{7-4} & \binom{7}{5}\binom{7-5}{7-5} & -\binom{7}{6}\binom{7-6}{7-6} & \binom{7}{7}\binom{0}{0} \\ \binom{7}{0}\binom{7}{7-1} & -\binom{7}{1}\binom{7-1}{7-2} & \binom{7}{2}\binom{7-2}{7-3} & -\binom{7}{3}\binom{7-3}{7-4} & \binom{7}{4}\binom{7-4}{7-5} & -\binom{7}{5}\binom{7-5}{7-6} & \binom{7}{6}\binom{7-6}{7-7} & 0 \\ -\binom{7}{0}\binom{7}{7-2} & \binom{7}{1}\binom{7-1}{7-3} & -\binom{7}{2}\binom{7-2}{7-4} & \binom{7}{3}\binom{7-3}{7-5} & -\binom{7}{4}\binom{7-4}{7-6} & \binom{7}{5}\binom{7-5}{7-7} & 0 & 0 \\ \binom{7}{0}\binom{7}{7-3} & -\binom{7}{1}\binom{7-1}{7-4} & \binom{7}{2}\binom{7-2}{7-5} & -\binom{7}{3}\binom{7-3}{7-6} & \binom{7}{4}\binom{7-4}{7-7} & 0 & 0 & 0 \\ -\binom{7}{0}\binom{7}{7-4} & \binom{7}{1}\binom{7-1}{7-5} & -\binom{7}{2}\binom{7-2}{7-6} & \binom{7}{3}\binom{7-3}{7-7} & 0 & 0 & 0 & 0 \\ \binom{7}{0}\binom{7}{7-5} & -\binom{7}{1}\binom{7-1}{7-6} & \binom{7}{2}\binom{7-2}{7-7} & 0 & 0 & 0 & 0 & 0 \\ -\binom{7}{0}\binom{7}{7-6} & \binom{7}{1}\binom{7-1}{7-7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \binom{7}{0}\binom{7}{7-7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also the inverse matrix of 7^{th} order Bézier curves in $\mathbf{E}^2$ is

$$[B^7]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now, we will examine the e^x curve as a 7^{th} *order* Bézier curve.

Theorem 6.2. *The numerical matrix representation of the curve $f(x) = e^x$ as a 7^{th} order Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}$$

where the control points $P_0, P_1, P_2, \ldots, P_7$ are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{7} & \frac{8}{7} \\ \frac{2}{7} & \frac{55}{42} \\ \frac{3}{7} & \frac{158}{105} \\ \frac{4}{7} & \frac{1457}{840} \\ \frac{5}{7} & \frac{632}{315} \\ \frac{6}{7} & \frac{11743}{5040} \\ 1 & \frac{685}{252} \end{bmatrix}.$$

Proof. 7^{th} degree Maclaurin series expansion for the function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$$

and it can be written as in parametric form and a 7^{th} degree polynomial function

$$\left(t, e^t\right) = \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right) = \left(t, a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0\right).$$

Also this can be written as a 7^{th} *order* Bézier curve in matrix representation with the coefficients. Hence we get the following equation

$$\begin{aligned} \left(t, e^t\right) &= \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right) \\ &= \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{7!} \\ 0 & \frac{1}{6!} \\ 0 & \frac{1}{5!} \\ 0 & \frac{1}{4!} \\ 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{7!} \\ 0 & \frac{1}{6!} \\ 0 & \frac{1}{5!} \\ 0 & \frac{1}{4!} \\ 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

$\square$

7. The Curve e^{ax+b} as a 7^{th} $Order$ Bézier Curve

In this section, we will research the curve e^{ax+b} as a 7^{th} order Bézier curve.

$$\begin{aligned} f(x) = e^{ax+b} &= \sum_{n=0}^{7} f^{(n)}(0)\frac{x^n}{n!} \\ &= e^{ax+b} + ae^{ax+b}x + a^2e^{ax+b}\frac{x^2}{2!} + a^3e^{ax+b}\frac{x^3}{3!} + a^4e^{ax+b}\frac{x^4}{4!} + a^5e^{ax+b}\frac{x^5}{5!} + a^6e^b\frac{x^6}{6!} + a^7e^b\frac{x^7}{7!}. \end{aligned}$$

Theorem 7.1. *The numerical matrix representation of the curve $f(x) = e^{ax+b}$ as a 7^{th} order Bézier curve is*

$$\left(t, e^{at+b}\right) = \alpha(t) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}$$

where the control points $P_0, P_1, P_2, \ldots, P_7$ are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{7} & e^b + \frac{1}{7}ae^b \\ \frac{2}{7} & \frac{1}{42}e^ba^2 + \frac{2}{7}e^ba + e^b \\ \frac{3}{7} & \frac{1}{210}e^ba^3 + \frac{1}{14}e^ba^2 + \frac{3}{7}e^ba + e^b \\ \frac{4}{7} & \frac{1}{840}e^ba^4 + \frac{2}{105}e^ba^3 + \frac{1}{7}e^ba^2 + \frac{4}{7}e^ba + e^b \\ \frac{5}{7} & \frac{1}{2520}e^ba^5 + \frac{1}{168}e^ba^4 + \frac{1}{21}e^ba^3 + \frac{5}{21}e^ba^2 + \frac{5}{7}e^ba + e^b \\ \frac{6}{7} & \frac{1}{5040}e^ba^6 + \frac{1}{420}e^ba^5 + \frac{1}{56}e^ba^4 + \frac{2}{21}e^ba^3 + \frac{5}{14}e^ba^2 + \frac{6}{7}e^ba + e^b \\ 1 & \frac{1}{5040}e^ba^7 + \frac{1}{720}e^ba^6 + \frac{1}{120}e^ba^5 + \frac{1}{24}e^ba^4 + \frac{1}{6}e^ba^3 + \frac{1}{2}e^ba^2 + e^ba + e^b \end{bmatrix}.$$

Proof. 7^{th} degree Maclaurin series expansion for the function e^{ax+b} is

$$f(x) = e^{ax+b} = \sum_{n=0}^{7} f^{(n)}(0)\frac{x^n}{n!},$$

$$f(x) = e^{ax+b} + ae^{ax+b}x + a^2e^{ax+b}\frac{x^2}{2!} + a^3e^{ax+b}\frac{x^3}{3!} + a^4e^{ax+b}\frac{x^4}{4!} + a^5e^{ax+b}\frac{x^5}{5!} + a^6e^b\frac{x^6}{6!} + a^7e^b\frac{x^7}{7!},$$

and it can be written as in parametric form and a 5^{th} degree polynomial function

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, e^{at+b} + ae^{at+b}t + \frac{a^2e^{ax+b}}{2!}t^2 + \frac{a^3e^{ax+b}}{3!}t^3 + \frac{a^4e^{ax+b}}{4!}t^4 + \frac{a^5e^{ax+b}}{5!}t^5\right) \\ &= \left(t, a^7e^b\frac{t^7}{7!} + a^6e^b\frac{t^6}{6!} + \frac{a^5e^{ax+b}}{5!}t^5 + \frac{a^4e^{ax+b}}{4!}t^4 + \frac{a^3e^{ax+b}}{3!}t^3 + \frac{a^2e^{ax+b}}{2!}t^2 + ae^{ax+b}t + e^{ax+b}\right) \\ &= \left(t, a_7t^7 + a_6t^6 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0\right). \end{aligned}$$

Also, this can be written as a 7^{th} *order* Bézier curve in matrix representation with the coefficients. Hence we get the following equation

$$\left(t, e^{at+b}\right) = \left(t, \frac{a^5 e^{ax+b}}{5!} t^5 + \frac{a^4 e^{ax+b}}{4!} t^4 + \frac{a^3 e^{ax+b}}{3!} t^3 + \frac{a^2 e^{ax+b}}{2!} t^2 + a e^{ax+b} t + e^{ax+b}\right),$$

$$\left(t, e^{at+b}\right) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T,$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{7} & \frac{1}{7} e^b (a+7) \\ \frac{2}{7} & \frac{1}{42} e^b \left(a^2 + 12a + 42\right) \\ \frac{3}{7} & \frac{1}{210} e^b \left(a^3 + 15a^2 + 90a + 210\right) \\ \frac{4}{7} & \frac{1}{840} e^b \left(a^4 + 16a^3 + 120a^2 + 480a + 840\right) \\ \frac{5}{7} & \frac{1}{2520} e^b \left(a^5 + 15a^4 + 120a^3 + 600a^2 + 1800a + 2520\right) \\ \frac{6}{7} & \frac{1}{5040} e^b \left(a^6 + 12a^5 + 90a^4 + 480a^3 + 1800a^2 + 4320a + 5040\right) \\ 1 & \frac{1}{5040} e^b \left(a^7 + 7a^6 + 42a^5 + 210a^4 + 840a^3 + 2520a^2 + 5040a + 5040\right) \end{bmatrix},$$

and so, the result give us the proof. □

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