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μ-statistical Convergence of Multiple Sequences in Topological Vector Valued Cone Metric Spaces

Ahmet SAHINER[1], Nurullah YILMAZ[2]

[1]Department of Mathematics,
Suleyman Demirel University, Isparta, Türkiye
ahmetsahiner@sdu.edu.tr

[2]Department of Mathematics,
Suleyman Demirel University, Isparta, Türkiye
nurullahyilmaz@sdu.edu.tr

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Abstract — *The concept of μ-statistical convergence of double and multiple sequences in topological vector space (*tvs *for short) valued cone metric spaces is introduced in this paper. The relationships between μ-statistical convergence and convergence in μ-density is investigated.*

Keywords: Cone metric, Multiple sequence, Two valued measure, Statistical convergence.
Mathematics Subject Classification: 54E35, 40B05, 40A05, 40A35.

1 Introduction

A very important subject of functional analysis "metric space" was first introduced by Frechet [10, 11]. And there exists various generalizations of metric space in the literature [28, 15]. The cone metric spaces are the one of these generalizations and it was explored by Huang and Zhang in 2007 [15]. The generalization, metric space to cone metric space was made by replacing real numbers with ordered Banach spaces. Cone metric spaces were defined to generalize the fixed point theorems in metric spaces. Based on this study, different types of fixed point theorems have been investigated [25, 9, 16, 19, 32]. In 2010, the concept of cone metric space over topological vector space (tvs for short) was introduced by Du, for the first time [7]. It is shown that a kind of equivalence between metric and cone metric spaces can be established by using the nonlinear scalarization function [7]. After that, the studies on cone metric spaces continued as cone metric spaces over tvs [14, 13, 26, 1]. Recently, Proinov [21] develop the unified theory for cone metric spaces over a solid vector spaces. In recent years, various interesting generalizations of cone metric spaces have been introduced based on different ideas. For more details we

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refer to [13, 34]. On the other hand, some basic properties of cone metric spaces are introduced such as convergence and being Cauchy of sequences [15]. These results lead to new studies especially on convergence and summability.

As it stated in [4], if the comparison of the sets of ordinary convergent and not ordinary convergent sequences is done, it can be easily seen that the set of ordinary convergent sequences is very small. It is needed a new concept of convergence which includes ordinary convergence. The first studies on this direction was proposed in [8] and independently by [27] which introduce the notion of statistical convergence. At the following period this notion has been studied by Šalát [24], Fridy [12] and many others [5, 6]. In 1990, a new generalization of the concept of statistical convergence by using two valued $(\{0, 1\})$ complete measure μ defined on an algebra on $\mathbb{N}$ [2, 3]. These studies are pioneer for a new research areas [29, 30, 31].

For double sequences, the concept of statistical convergence was first introduced by Mursaleen and Edely [17] and Móricz [18] independently. Meanwhilee, Móricz investigate the statistical convergence for multiple sequences [18]. The concept of $\mu-$statistical convergence of double sequences in metric spaces is studied by Das [4, 5]. The convergence of double and multiple sequences is first studied in [23] and the concept of statistical convergence is first investigated in cone metric spaces by [33, 23].

In this paper, we aim to extend results about the statistical convergence to the $\mu-$statistical convergence, convergence in $\mu-$density and the relations between $\mu-$statistical convergence and convergence in $\mu-$density of double and multiple sequences in tvs valued cone metric spaces.

2 Preliminaries

Let E be a Hausdorff *tvs* with the zero vector θ. A subset P of E is called a (convex) cone if

i. P is closed, non-empty and $P \neq \{\theta\}$,

ii. $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

iii. $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stands $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. For $x, y \in E$ such that $x \preceq y$, the set

$$[x, y] = \{z \in E : x \preceq z \preceq y\}.$$

is called as *order-interval*.

A subset A of E is called as *order-convex* if $[x, y] \subset A$ whenever $x, y \in A$ and $x \preceq y$. Therefore, it is easy to see that *order-intervals* are convex.

Ordered topological vector space (E, P) is order-convex if it has a base of neighborhoods of θ consisting of *order-convex* subsets. In this case the cone P is said to be normal [14].

The cone P is called regular if $\{x_n\}$ is a sequence such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. It is easy to see that a regular cone is normal cone [15].

If V is an absolutely convex and absorbent subset of a *tvs* E, its Minkowski functional is defined from E to $\mathbb{R}$ by

$$x \mapsto q_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}.$$

Assume that (E, P) be an ordered *tvs* and $e \in \text{int}P$. Then order interval $[-e, e] = (P - e) \cap (e - P) = \{z \in E : -e \preceq z \preceq e\}$ is an absolutely convex neighborhood of θ; its Minkowski functional $q_{[-e,e]}$ is denoted by q_e. It can be easily seen that $\text{int}[-e, e] = (\text{int}P - e) \cap (e - \text{int}P)$. If P is solid and normal, q_e is the norm on E. Moreover, it is increasing function on P [14].

Definition 1. *[7, 15, 14] Assume that $X \neq \emptyset$ and the mapping $d : X \times X \to E$ satisfies the following conditions:*

i. $\theta \preceq d(y, z)$ *for all* $y, z \in X$ *and* $d(y, z) = \theta$ *if and only if* $x = y$*,*

ii. $d(y, z) = d(z, y)$ *for all* $x, y \in X$*,*

iii. $d(y, z) \preceq d(y, w) + d(w, z)$ *for all* $y, z, w \in X$*.*

Then d *is called as a cone metric on* X*, and* (X, d) *is called a cone metric space.*

Throughout the paper, we denote $d_q = q_e \circ d$ as a composition of corresponding Minkowski functional of the set $[-e, e]$ and cone metric d.

Example 1. *[15] Let* $E = \mathbb{R}^2$*,* $P = \{(y, z) \in E : y, z \geq 0\}$*,* $X = \mathbb{R}$ *and* $d(y, z) = (|y - z|, \alpha|y - z|)$*, where* $\alpha \geq 0$ *is a constant.* (X, d) *is a cone metric space.*

Now, we recall some important definitions related to convergence double sequences. The concept of convergence for double sequences was introduced by Pringsheim [20]. A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be convergent in Pringsheim sense if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ for every $i, j \geq n_0$ and L is called the Pringsheim limit of $\{x_{ij}\}_{i,j\in\mathbb{N}}$. A double natural density of a set $K \subset \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$ is defined as

$$\delta_2(E) = \lim_{n,m\to\infty} \frac{1}{mn} \left|\{i \leq m \text{ and } j \leq n : (n, m) \in K\}\right|,$$

where vertical bars denote the cardinality of enclosed set. Therefore, a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called as statistically convergent to ξ if for each $\varepsilon > 0$, the set

$$\{(i, j) : |x_{ij} - \xi| \geq \varepsilon \text{ for } i \leq m \text{ and } j \leq n\}$$

has double natural density zero [17, 18]. The concept of μ-statistical convergence need some extra knowledge except the notion of statistical convergence. The main element measure μ and the definitions of μ-statistical convergence and convergence in $\mu-$density are as follows: The measure μ is assumed a complete, finite additive, two $(\{0, 1\})$ valued measure defined on algebra $\Gamma \subset P(\mathbb{N}^2)$ that contains all the subset of $\mathbb{N}^2$ and $\mu(B) = 0$ if A is contained in the union of finite number of rows and columns of $\mathbb{N}^2$ as in the papers [4, 5].

Definition 2. *[4] Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be double sequence of real numbers*

i. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is called as $\mu-$statistically convergent to ξ if and only if for each $\varepsilon > 0$,*

$$\mu\left(\{(i,j),\ i \leq m \text{ and } j \leq n : |x_{ij} - \xi| \geq \varepsilon\}\right) = 0.$$

ii. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is called as convergent in $\mu-$density to ξ if there exists $B \in \Gamma$ such that $\{x_{ij}\}_{(i,j)\in B}$ is convergent to ξ.*

Definition 3. *[4] Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be double sequence of real numbers*

i. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is called as $\mu-$statistically Cauchy sequence if and only if for each $\varepsilon > 0$ there exists $B \subset \mathbb{N}^2$ with $\mu(B) = 0$ such that $(i,j),(k,l) \notin B$ implies that $|x_{ij} - x_{kl}| < \varepsilon$.*

ii. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is called as Cauchy sequence in $\mu-$density to ξ if there exists $B \subset \mathbb{N}^2$ with $\mu(B) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is a ordinary Cauchy double sequence.*

Ordinary and statistical convergence of multiple sequences in tvs valued cone metric spaces have been introduced in [23]. Now, we are ready to introduce the main results.

3 Main Results

In this section, we give the definitions of the μ-statistically convergence and convergence in μ-density of double sequences in tvs cone metric space. The relationship between μ-statistically convergence and convergence in μ-density is investigated.

Definition 4. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in a cone metric space (X,d). $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called as $\mu-$statistically convergent to $\xi \in X$ if for each pre-assigned $c \gg \theta$, $\mu(B(c)) = 0$ where $B(c) = \{(i,j) \in \mathbb{N}^2 : d(x_{ij},\xi) \succeq c\}$.*

If a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is μ-statistically convergent to ξ in a cone metric space (X,d) then we write

$$\mu - \lim_{i,j\to\infty} d(x_{ij},\xi) = \theta$$

where ξ is called as $\mu-$statistical limit of the sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$.

Definition 5. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in a cone metric space (X,d). $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called as convergent to $\xi \in X$ in μ density if there exists a set $B \in \Gamma$ with $\mu(B) = 1$ such that $\{x_{ij}\}_{(i,j)\in B}$ is convergent to ξ in (X,d).*

Definition 6. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in a cone metric space (X,d). $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called as $\mu-$statistically Cauchy sequence if and only if for every $c \in E$ with $c \gg \theta$ there exists an $B \subset \mathbb{N}^2$ with $\mu(B) = 0$ such that $(i,j),(i_1,j_1) \notin B$ implies $d\left(x_{ij}, x_{i_1j_1}\right) \prec c$.*

Definition 7. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in a cone metric space (X,d). $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called as Cauchy sequence in $\mu-$density if and only if there exists an $B \subset \mathbb{N}^2$ with $\mu(B) = 1$ such that $\{x_{ij}\}_{(i,j)\in B}$ is a ordinary Cauchy double sequence.*

We now give an example of a $\mu-$statistically convergent double sequence in tvs cone metric spaces.

Example 2. *Let d be a cone metric defined as $d : \mathbb{R}^3 \times \mathbb{R}^3 \to (\mathbb{R}^2, P)$, where P is the first of the quadrant of the* tvs $\mathbb{R}^2$. *Assume that μ is a two valued measure on $\mathbb{N}^2$ such that there exists $B \subseteq \mathbb{N}^2$ with $\mu(B) = 0$ which is not contained in any finite union of rows and columns of $\mathbb{N}^2$ and the double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is defined by*

$$x_{ij} = \begin{cases} (i^2 j, i, j^i) & \text{if } (i,j) \in B, \\ (0,0,0) & \text{otherwise}. \end{cases}$$

Let $\xi = 0$. Then for every $c \gg \theta$

$$\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, \xi) \gg c\right\} \subseteq B.$$

Hence

$$\mu\left(\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, \xi) \gg c\right\}\right) = 0.$$

Thus we obtain

$$\mu - \lim_{i,j\to\infty} d(x_{ij}, \xi) = 0.$$

but it is seen that the sequence $\{x_{ij}\}$ is not converge (ordinary) to ξ. It is clear that if B is contained in any finite union of rows and columns then $\mu(B) = 0$ and the convergence is also holds.

In a similar way, examples of double sequences which are convergent in $\mu-$density can be constructed in cone metric spaces. Now, introduce the following lemma which characterize the convergence in tvs cone metric spaces.

Lemma 1. *(X, d) be a tvs cone metric space, $e \in$ int P, q_e be the Minkowski functional of $[-e, e]$ and $d_q = q_e \circ d$. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in X. Then*

i. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is tvs cone converges to ξ if and only if $d_q(x_{ij}, \xi) \to 0$ as $i, j \to \infty$,

ii. $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is tvs cone Cauchy sequence if and only if $d_q(x_{ij}, x_{nm}) \to \theta$ as $(i, j, n, m \to \infty)$

Proof. i. Let $\varepsilon > 0$ choose $c \gg \theta$ such that $q_e(c) < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that $d(x_{ij}, \xi) \ll c$ for all $i, j \geq N$. Since the Minkowski functional is monotone $d_q(x_{ij}, \xi) = q_e(d(x_{ij}, \xi)) \leq q_e(c) < \varepsilon$.

Conversely, let $d_q(x_{ij}, \xi) = q_e(d(x_{ij}, \xi))$ converges to 0 and fix $c \in \text{int}P$. Then, there exists $\delta > 0$, such that $q_e(t) < \delta$ implies $c - t \in \text{int}P$. For this δ there is N, such that for all $i, j \geq N$, $q_e(d(x_{ij}, \xi)) < \delta$. Thus, $c - d(x_{ij}, \xi) \in \text{int}P$. This means $d(x_{ij}, \xi) \ll c$. Hence $\{x_{ij}\}$ converges to ξ.

ii. The proof is similar to i.

□

Now, we provide the following limit operation of double sequences in a cone metric space (X, d) with $\mu-$statistical sense.

Theorem 1. *(X,d) be a tvs cone metric space, $e \in$ int P and q_e be the Minkowski functional of $[-e,e]$. And let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a two double sequence in X and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is $\mu-$statistical convergent to ξ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is $\mu-$statistically convergent to η. Then $\{d(x_{ij},y_{ij})\}_{i,j\in\mathbb{N}}$ is $\mu-$statistically convergent to $d(\xi,\eta)$ as $i,j\to\infty$.*

Proof. For every ε, choose $c \in E$ with $\theta \ll c$ and $q_e(c)=\varepsilon$. From $x_{ij}\to\xi$ and $y_{ij}\to\eta$, there exists N such that for all $i,j>N$, $d(x_{ij},\xi)\ll c$ and $d(y_{ij},\eta)\ll c$. We have

$$d(x_{ij},y_{ij}) \preceq d(x_{ij},\xi)+d(\xi,\eta)+d(y_{ij},\eta)$$

Hence

$$d(x_{ij},y_{ij})-d(\xi,\eta)\preceq d(x_{ij},\xi)+d(x_{ij},\xi)$$

and

$$\left\{(i,j)\in\mathbb{N}^2: q_e\left(d(x_{ij},y_{ij})-d(\xi,\eta)\right)\geq\varepsilon\right\} \subset \left\{(i,j)\in\mathbb{N}^2: q_e\left(d(x_{ij},\xi)\right)\geq\frac{1}{2}\varepsilon\right\} \cup\left\{(i,j)\in\mathbb{N}^2: q_e\left(d(y_{ij},\eta)\right)\geq\frac{1}{2}\varepsilon\right\}$$

Thus $\mu\left(\left\{(i,j)\in\mathbb{N}^2: q_e\left(d(x_{ij},y_{ij})-d(\xi,\eta)\right)\geq\varepsilon\right\}\right)=0$ and the result follows. □

Theorem 2. *(X,d) be a tvs cone metric space, $e\in$ int P and q_e be the Minkowski functional of $[-e,e]$. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be convergent double sequence in μ-density then it is $\mu-$statistically convergent.*

Proof. Let a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a convergent in $\mu-$density. Then we have $B\subseteq\mathbb{N}^2$ with $\mu(B)=1$ such that $\{x_{ij}\}_{(i,j)\in B}$ is a convergent. Then for every $c\in E$ with $c\gg\theta$ there exists $k\in\mathbb{N}$ such that $d(x_{ij},\xi)\ll c$ for all $i,j\geq k$ and $(i,j)\in B$. Clearly $d\left(x_{ij},\xi\right)\ll c$ for all $i,j\geq k$ and $(i,j)\in B$. Thus we obtain $\left\{(i,j)\in\mathbb{N}^2: d\left(x_{ij},\xi\right)\gg c\right\}$ $\subseteq B^c\cup F$ where F is the union of the first k rows and first k columns of $\mathbb{N}^2$ and so $\mu\left(\left\{(i,j)\in\mathbb{N}^2: d\left(x_{ij},\xi\right)\gg c\right\}\right)=0$. Therefore we obtain the desired result. □

Theorem 3. *Let (X,d) be a tvs cone metric space and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $\mu-$statistically convergent double sequence in (X,d) then it is $\mu-$statistically Cauchy sequence in (X,d).*

Proof. Analogously the proof of Theorem 3 in [23], let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be μ-statistically convergent to ξ. Then for every $c\gg\theta$, such that $q_e(c)>\varepsilon$ the set

$$\mu\left(\{(i,j): d(x_{ij},\xi)\succeq c\}\right)=0$$

Choose m_ε and n_ε such that $d(x_{m_cn_c},\xi)\ll c$. Let

$$\begin{aligned} M_c &= \left\{(i,j): d\left(x_{ij},x_{m_cn_c}\right)\succeq c\right\}\\ N_c &= \left\{(i,j): d(x_{ij},\xi)\succeq c\right\}\\ P_c &= \left\{(i,j): d\left(x_{m_cn_c},\xi\right)\succeq c\right\}\end{aligned}$$

Then $M_c\subseteq N_c\cup P_c$ and therefore

$$\mu(M_c)\leq\mu(N_c)+\mu(P_c)=0.$$

Hence, the desired result is obtained. □

Remark 1. *The inverse of the Theorem 3 is not generally true. Let us consider* $X = \mathcal{P}[a,b]$ *and standard metric on* X*. Let us construct the following double sequence* $\{y_{ij}\}_{i,j\in\mathbb{N}}$ *such that* $y_{i1}(t) = 1, i \in \mathbb{N}$, $y_{i2}(t) = 1-\frac{t^2}{2}, i \in \mathbb{N}$, $y_{i,3}(t) = 1-\frac{t^2}{2!}+\frac{t^4}{4!}, i \in \mathbb{N}, \ldots, y_{in}(t) = 1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}, n \in \mathbb{N}$, *etc. It can be seen that* $\{y_{ij}\}_{i,j\in\mathbb{N}}$ *is a double Cauchy sequence and it is* $\mu-$*statistically Cauchy for any measure* μ *but it is not* $\mu-$*statistically convergent in* $\mathcal{P}[a,b]$*.*

Theorem 4. (X,d) *be a tvs cone metric space. If* $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is convergent in* $\mu-$*density in* (X,d) *then it also a double Cauchy sequence in* $\mu-$*density in* (X,d)*.*

Proof. The proof is similar proof of Theorem 3. □

Theorem 5. (X,d) *be a tvs cone metric space. Every Cauchy double sequence in* μ*-density is also* $\mu-$*statistically Cauchy.*

Proof. Assume that $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Cauchy double sequence in $\mu-$density. Then we have $B \subseteq \mathbb{N}^2$ with $\mu(B) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is a Cauchy double sequence. Moreover, for every $c \in E$ with $c \gg \theta$ there exists $m \in \mathbb{N}$ such that $d(x_{ij}, x_{kl}) \ll c$ for all $i,j,k,l \geq m$ and $(i,j),(k,l) \in B$. Choose $(k_0,l_0) \in B$ with $k_0, l_0 \geq m$. Clearly $d(x_{ij}, x_{k_0l_0}) \ll c$ for all $i,j \geq m$ and $(i,j) \in B$. Hence $\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{k_0l_0}) \succeq c\} \subseteq B^c \cup F$ where F is the union of the first m rows and first m columns of $\mathbb{N}^2$ and so we have $\mu(\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{k_0l_0}) \succeq c\}) = 0$. Therefore we obtain the desired result. □

Theorem 6. *Let* (X,d) *be a tvs cone metric space and* $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ *be a double sequence in* (X,d)*.* x *is* $\mu-$*statistically Cauchy if and only if for every* $c \gg \theta$ *there exists* $(k_0,k_0) \in \mathbb{N}\times\mathbb{N}$ *such that*

$$\mu\left(\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{k_0k_0}) \ll c\right\}\right) = 1.$$

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a μ-statistically Cauchy sequence in cone metric space (X,d). Then for every $c \gg \theta$ there exists an $B \subset \mathbb{N}^2$ with $\mu(B) = 0$ such that $(i,\ j)$, $(i_1,j_1) \notin B$ implies that $d(x_{ij}, x_{i_1j_1}) \ll c$. Then if $d(x_{ij}, x_{i_1j_1}) \succeq c$ for (i,j), $(i_1,j_1) \in \mathbb{N}^2$, then at least one of indices (i,j), (i_1,j_1) must be in A. Since $B^c \neq \emptyset$, choose $(m_0,n_0) \in B^c$. Then $d(x_{ij}, x_{m_0n_0}) \succeq c$ implies that $(i,\ j) \in B$. Hence we obtain $\{(i,\ j) \in \mathbb{N}^2 : d(x_{ij}, x_{m_0,n_0}) \succeq c\} \subseteq B$, which implies the following results

$$\mu\left(\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{m_0n_0}) \succeq c\right\}\right) = 0$$

and

$$\mu\left(\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij},\ x_{m_0n_0}) \ll c\right\}\right) = 1.$$

Suppose that for a given $c \gg \theta$ there exists $(k_0,k_0) \in \mathbb{N}^2$ such that

$$\mu\left(\left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{k_0l_0}) \ll c\right\}\right) = 1.$$

Then there exists $(k_0,l_0) \in \mathbb{N}^2$ such that $B = \left\{(i,j) \in \mathbb{N}^2 : d(x_{ij}, x_{k_0l_0}) \succeq \frac{1}{2}c\right\}$ with $\mu(B) = 0$. Let (i,j) , $(i_1,j_1) \notin B$. Then $d(x_{ij},\ x_{k_0l_0}) \ll \frac{1}{2}c$ and $d(x_{i_1j_1}, x_{k_0l_0}) \ll \frac{1}{2}c$ and consequently $d(x_{ij}, x_{i_1j_1}) \ll c$. Thus $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a $\mu-$ statistically Cauchy sequence. □

Remark 2. *The inverse of the Theorem 6 is not generally true. It can be seen by considering the example given in Remark 1.*

4 Generalization to Multiple Sequences

In this section, we give the definitions of $\mu-$ statistical convergence (convergence in $\mu-$density) and being $\mu-$statistical Cauchy (Cauchy in μ density) sequence for $n-$tuple sequences in tvs cone metric spaces analogously F. Moricz [18] and [23]. The definitions and obtained results of double sequences can be extended to $n-$tuple sequences. Let n is a fixed positive integer and $\mathbb{N}^n$ be the set of $n-$tuples $\mathbf{j} := (j_1, j_2, \ldots, j_n)$ with nonnegative integers for coordinates j_i. Two tuples $\mathbf{j}$ and $\mathbf{m} := (m_1, m_2, \ldots, m_n)$ are distinct if and only if $j_i \neq m_i$ for at least on i. $\mathbb{N}^n$ is partially ordered by agreeing that $\mathbf{j} \leq \mathbf{m}$ if and only if $j_i \leq m_i$ for each $i = 1, \ldots, n$. Before defining the concept of $\mu-$statistical convergence and being $\mu-$statistical Cauchy and convergence and being Cauchy in $\mu-$density for multiple sequences, we rearrange the definition of measure μ as follows: μ is assumed a complete, finite additive, two $(\{0,1\})$ valued measure defined on algebra $\Gamma \subset P(\mathbb{N}^n)$ that contains all the subset of $\mathbb{N}^n$ that are contained in the union of finite number of sets such that $\{\mathbf{j} \in \mathbb{N}^n : \mathbf{j} = (j_1, j_2, \ldots, j_{k-1}, j_k = \text{constant}, j_{k+1} \ldots, i_n),\ 1 \leq k \leq n\}$ and $\mu(B) = 0$ if the set B is contained in the union of finite number of sets such that $\{\mathbf{j} \in \mathbb{N}^n : \mathbf{j} = (j_1, j_2, \ldots, j_{k-1}, j_k = \text{constant}, j_{k+1} \ldots, j_n),\ 1 \leq k \leq n\}$ as in Das's papers [4, 5].

We say that a n-tuple sequence $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^n}$ is μ-statistically convergent to ξ if for each $c \gg \theta$,

$$\mu\left(\{\mathbf{j} \in \mathbb{N}^n : d(x_{\mathbf{j}}, \xi) \succeq c\}\right) = 0 \tag{4.1}$$

and $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^n}$ is convergent to $\xi \in X$ in μ density if there exists a set $M \in \Gamma$ with $\mu(M) = 1$ such that $\{x_{\mathbf{j}}\}_{\mathbf{j}\in M}$ is convergent to ξ in tvs cone metric space (X, d).

Furthermore, we say that $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^n}$ is $\mu-$statistically Cauchy if for each $c \in E$ $c \gg 0$ there exists $\mathbf{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ and

$$\mu\left(\{\mathbf{j} \in \mathbb{N}^n : d(x_{\mathbf{j}}, x_{\mathbf{k}}) \succeq c\}\right) = 0, \tag{4.2}$$

and $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{N}^n}$ is called as a Cauchy sequence in $\mu-$density if and only if there exists an $B \subset \mathbb{N}^n$ with $\mu(B) = 1$ such that $\{x_{\mathbf{j}}\}_{\mathbf{j}\in B}$ is a ordinary Cauchy $n-$tuple sequence in tvs cone metric space (X, d).

Subsequently, it is possible to generalize the Theorems 1, 2, 3, 4, 5 and 6 to $n-$tuple sequences which can be done by replacing the double indices by $n-$tuple indices.

5 Conclusion

In the study, the concepts of $\mu-$statistical convergence and $\mu-$density convergence of a multiple sequence in a tvs valued cone metric space. Moreover, the notion of $\mu-$statistical Cauchy sequence and Cauchy sequence in -density is introduced and some important results are established regarding these concepts.

The question of summability of multiple sequences in tvs valued cone metric spaces left for future studies.

Conflict of Interest Declaration

The authors declare that there is no conflict of interest statement.

Ethics Committee Approval and Informed Consent

The authors declare that there is no ethics committee approval and/or informed consent statement.

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Ahmet Sahiner, ORCID: https://orcid.org/0000-0002-4945-2476
Nurullah Yilmaz, ORCID: https://orcid.org/0000-0001-6429-7518