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The Flow-geodesic Curvature and the Flow-evolute of Hyperbolic Plane Curves

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We introduce a new type of curvature function and associated evolute curve for a given curve in the hyperboloid model of plane hyperbolic geometry. A special attention is devoted to the examples, particularly to a horocycle provided by the null Lorentzian rotation.

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1. Introduction

The strong relationship between the hyperbolic geometry and one of the major field of Physics, namely the relativity, is excellent illustrated in several papers of Prof. Dr. Krishan Lal Duggal, e.g. [8]-[10]. Recall also that the trajectory of a material point (or particle) in a given setting is a (smooth) curve in that setting and the main tool in studying such a curve is its curvature(s). Hence, motivated by this argument, we introduce a new curvature function for curves belonging to 2D hyperbolic geometry.

Let $M^2(c)$ be the two-dimensional space form with the constant curvature $c \in \{-1, 0, 1\}$:
i) $M^2(0)$ is the Euclidean plane $E^2 := (\mathbb{R}^2, \langle\cdot,\cdot\rangle)$, for $\langle \bar{x} = (x^1, x^2), \bar{y} = (y^1, y^2)\rangle := x^1y^1 + x^2y^2$,
ii) $M^2(1)$ is the unit sphere S^2 of $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ where in the inner product expression above we add the term x^3y^3,
iii) $M^2(-1)$ is the upper hyperboloid H^2 of unit time-like vectors of $\mathbb{R}^{2,1}$. Specifically, $\mathbb{R}^{2,1} := (\mathbb{R}^3, \langle\cdot,\cdot\rangle_L)$ with the Lorentzian product: $\langle \bar{x} = (x^1, x^2, x^3), \bar{y} = (y^1, y^2, y^3)\rangle_L := x^1y^1 + x^2y^2 - x^3y^3$ and $H^2 := \{\bar{x} \in \mathbb{R}^{2,1} : \langle \bar{x}, \bar{x}\rangle_L = -1, x^3 > 0\}$. An advantage of this model of hyperbolic geometry is that its isometries are restrictions of linear maps of $\mathbb{R}^{2+1}$ which preserves $\langle\cdot,\cdot\rangle_L$ and H^2.

Fix now $\gamma : t \in I \subseteq \mathbb{R} \to M^2(c) \subset \mathbb{R}^3$ be a smooth regular curve with I an open real interval. Being a curve in a smooth 2-dimensional manifold, γ is characterized by its *geodesic curvature* $k_g : I \to \mathbb{R}$, $k_g \in C^\infty(I)$ and: i) for $c = 0$ we have $k_g = k$=the usual (Frenet) curvature; ii) for $c = 1$ we have $k = \sqrt{k_g^2 + 1} \geq 1$. Let us point out that we do not suppose that γ is parametrized by its arc-length $s \in (0, l(\gamma) \leq +\infty)$ for $l(\gamma)$ the length of γ.

This geodesic curvature appears naturally in the moving equation of a frame $\mathcal{F} = \mathcal{F}(t)$ adapted to the geometry of the pair $(\gamma, M^2(c))$. In the papers [5] ($c = 0$) and [6] ($c = 1$) we rotate this frame obtaining a new one, called *the flow-frame of* γ and denoted $\mathcal{F}^f$. Consequently, we introduce a new curvature k_g^f, called *the flow-geodesic curvature* of γ. The present paper concerns with this approach in the remaining framework, namely the case $c = -1$ and a main interest is to study curves with a vanishing k_g^f. The flow-curvature of spacelike parametrized curves in the Lorentz plane is studied in the paper [4].

The contents of this note is as follows. Firstly, we present the general theory of curves in the hyperbolic plane choosing as model the upper sheet of the hyperboloid with two sheets. Our new curvature function and its associated evolute curve is introduced and studied in the following section; we point out that the section two ends with a large discussion of a particular horocycle as an example of flat-hyperbolic geodesic, a notion introduced here. The last section deals with other non-flat examples.

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2. The geometry of hyperbolic plane curves

Recall that $\bar{x} \in \mathbb{R}^{2,1}$ is called *space-like*, *time-like* or *null* if $\langle \bar{x}, \bar{x} \rangle_L > 0$, $\langle \bar{x}, \bar{x} \rangle_L < 0$ or $\langle \bar{x}, \bar{x} \rangle_L = 0$ respectively. In addition to $\langle \cdot, \cdot \rangle_L$ we will also need the Lorentz version of the cross-product:

$$\bar{x} \times_L \bar{y} := \begin{vmatrix} \bar{i} & \bar{j} & -\bar{k} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}, \quad \bar{i} = (1, 0, 0), \quad \bar{j} = (0, 1, 0), \quad \bar{k} = (0, 0, 1) \in H^2. \tag{2.1}$$

It follows immediately:

$$\begin{cases} \langle \bar{x}, \bar{x} \times_L \bar{y} \rangle_L = 0 = \langle \bar{y}, \bar{x} \times_L \bar{y} \rangle_L \\ \bar{x} \times_L (\bar{y} \times_L \bar{z}) = \langle \bar{x}, \bar{y} \rangle_L \bar{z} - \langle \bar{x}, \bar{z} \rangle_L \bar{y}. \end{cases} \tag{2.2}$$

Returning to the given curve $\gamma : I \to H^2$ its regularity means that $\gamma'(t) \neq \bar{0} = (0, 0, 0)$ and it results the space-like character of its velocity vector field $\langle \gamma'(t), \gamma'(t) \rangle_L > 0$. Then we define the Lorentz norm:

$$\|\gamma'(t)\|_L := \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_L} \tag{2.3}$$

yielding two unit space-like vector fields:
1) the tangent $T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|_L}$; 2) the normal $\bar{n}(t) := \gamma(t) \times_L T(t)$.
It results the adapted frame (with h from the word "hyperbolic"):

$$\mathcal{F}_h(t) := \begin{pmatrix} \gamma \\ T \\ \bar{n} \end{pmatrix} (t) \tag{2.4}$$

with the moving equation:

$$\frac{d}{dt} \mathcal{F}_h(t) = \|\gamma'(t)\|_L \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g(t) \\ 0 & -k_g(t) & 0 \end{pmatrix} \mathcal{F}_h(t). \tag{2.5}$$

The expression of the geodesic curvature is:

$$k_g(t) = \frac{\langle \gamma''(t), \gamma(t) \times_L \gamma'(t) \rangle_L}{\|\gamma'(t)\|_L^3} = \frac{\det(\gamma''(t), \gamma(t), \gamma'(t))}{\|\gamma'(t)\|_L^3} \tag{2.6}$$

and choosing $k_g > 0$ there are three types of curves with constant geodesic curvature $k_g = K$ ([11]): a) circles, for $K > 1$; b) horocycles, with $K = 1$; c) equidistant curves (i.e. curves of finite distance from a hyperbolic geodesic), for $K \in (0, 1)$. An approach to curves of constant geodesic curvature directly in the Bolyai-Lobachevskian plane is proposed in [12].

When the plane hyperbolic geometry is treated in the Poincaré upper half-plane model ($\mathcal{H}^2 := \{z = X + iY \in \mathbb{C}; Y > 0\}, g_h = \frac{1}{Y^2}(dX^2 + dY^2)$) and a point $p \in \mathbb{R} \cup \{\infty\}$ is chosen then the horocycles with *center* p are: i) Euclidean circles in $\mathcal{H}^2$ passing through p if p is real; ii) horizontal Euclidean lines if $p = \infty$; moreover the group $PSL(2, \mathbb{R}) = Isom^+(\mathcal{H}^2)$ preserves the set of horocycles. The relationship between the H^2-model and $\mathcal{H}^2$-model of plane hyperbolic geometry is provided by the celebrated Cayley map:

$$Cay : H^2 \to \mathcal{H}^2, \quad (x, y, z) \to (X, Y) := \frac{1}{z - y}(x, 1). \tag{2.7}$$

Example 2.1 The curve $\gamma(s) = (\sinh s, \cosh s, \sqrt{2} \cosh s) \in H^2$ is an unit speed geodesic having $k_g = 0$ and the space-like constant normal $\bar{n} = (0, \sqrt{2}, 1)$. Its Cayley image is the Euclidean semi-circle centered in the origin O of the plane E^2 (hence a geodesic in the Poincaré model) and having the radius $r = \sqrt{2} + 1$. $\square$

At the end of this section we adopt the approach of the paper [1] concerning the theory of evolutes for plane hyperbolic curves. Fix $\gamma = \gamma(s)$ a non-horocycle curve parametrized by arc-length and with $k_g \neq -1$. Then its *h-evolute* is the curve:

$$E_\gamma(s) := \frac{k_g(s)}{\sqrt{|k_g^2(s) - 1|}} \gamma(s) + \frac{1}{\sqrt{|k_g^2(s) - 1|}} \bar{n}(s). \tag{2.8}$$

A motivation for this choice is the existence of the Darboux vector field $\Omega = \Omega(t)$ satisfying similar relations to the Euclidean space curves theory:

$$\Omega \times_L \gamma = \gamma', \quad \Omega \times_L T = T', \quad \Omega \times_L \bar{n} = \bar{n}' \tag{2.9}$$

and having the expression:

$$\Omega(t) = \|\gamma'(t)\|[k_g(t)\gamma(t) + \bar{n}(t)]. \tag{2.10}$$

It is worth to mention that for the elements of $\mathcal{F}_h$ it holds:

$$\bar{n} \times_L \gamma = T, \quad \bar{n} \times_L T = \gamma. \tag{2.11}$$

3. The flow-geodesic curvature and the flow-evolute of hyperbolic plane curves

We recall after [2, p. 473] that the group of isometries of $\mathbb{R}^{2,1}$ has three one-parameter sub-groups that fixes an axis, *called rotations*:
i) if the axis is space-like; e.g. the x^1-axis:

$$R_1(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \in Sym(3), \quad t \in \mathbb{R},$$

ii) if the axis is time-like; e.g. the x^3-axis:

$$R^3(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in [0, 2\pi],$$

iii) if the axis is null; e.g. the positive (or first) bisectrix of the (x^2x^3)-plane:

$$R(+23; t) := \begin{pmatrix} 1 & t & -t \\ -t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\ -t & -\frac{t^2}{2} & 1 + \frac{t^2}{2} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Inspired by these expressions we introduce now a second frame:

$$\mathcal{F}_h^f(t) = \begin{pmatrix} \gamma \\ E_f^1 \\ E_f^2 \end{pmatrix}(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \mathcal{F}_h(t) \tag{3.1}$$

and since the involved 3×3 matrix belongs to the group $\{1\} \times SO(2)$ it follows that E_1^f, E_2^f are also space-like vector fields.

Definition 3.1 The *flow-geodesic curvature of* γ is the smooth function $k_g^f : I \to \mathbb{R}$ provided by the moving equation of $\mathcal{F}^f$:

$$\frac{d}{dt}\mathcal{F}_h^f(t) = \|\gamma'(t)\|_L \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g^f(t) \\ 0 & -k_g^f(t) & 0 \end{pmatrix} \mathcal{F}_h^f(t). \tag{3.2}$$

If k_g^f is constant zero then we call γ a *flat-hyperbolic geodesic*. If $k_g^f \notin \{-1, 1\}$ for a parametrized by arc-length curve γ then its *flow-h-evolute* is:

$$E_\gamma^f(s) := \frac{k_g^f(s)}{\sqrt{|(k_g^f)^2(s) - 1|}}\gamma(s) + \frac{1}{\sqrt{|(k_g^f)^2(s) - 1|}}E_2^f(s). \tag{3.3}$$

We derive directly a computational formula which coincide (modulo L) with the previous cases $c \in \{0, 1\}$:

Theorem 3.2 *The formula expressing* k^f_g *is:*

$$k^f_g(t) = k_g(t) - \frac{1}{\|\gamma'(t)\|_L} < k_g(t). \tag{3.4}$$

The given curve is a flat-hyperbolic geodesic if and only if the following relation holds for any $t \in I$*:*

$$\langle \gamma''(t), \gamma(t) \times_L \gamma'(t)\rangle_L = \langle \gamma'(t), \gamma'(t)\rangle_L. \tag{3.5}$$

In particular, along a flat-hyperbolic horocycle parametrized by arc-length i.e. $\|\gamma'(s)\|_L = 1$ *for all* $s \in (0, l(\gamma))$ *we have a conservation law:*

$$\langle \gamma''(s), \gamma(s) \times_L \gamma'(s)\rangle_L = 1. \tag{3.6}$$

while the moving equation (1.5) *reads*:

$$\frac{d}{ds}\mathcal{F}_h(s) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \mathcal{F}_h(s), \quad \Gamma_h := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Gamma^3_h = O_3. \tag{3.7}$$

Example 3.3 The matrix Γ_h has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 0$ with the null vector $\bar{v} = (1, 0, -1)$ as eigenvector; this vector belongs to the asymptotic cone $AC : x^2 + y^2 - z^2 = 0$ of H^2. This null vector is the negative (or second) bisectrix of the (x^1x^3)-plane and the corresponding rotation is:

$$R(-13; t) := \begin{pmatrix} 1 - \frac{t^2}{2} & t & -\frac{t^2}{2} \\ t & 1 & t \\ \frac{t^2}{2} & -t & 1 + \frac{t^2}{2} \end{pmatrix}, \quad t \in \mathbb{R}, \quad \frac{d}{dt}R(-13; t)|_{t=0} = \Gamma_h, \tag{3.8}$$

having the characteristic polynomial $-P_{R(-13;t)}(\lambda) = (\lambda - 1)^3$; if $t \neq 0$ then $R(-13; t)$ is not diagonalizable. The first column of this last matrix is a space-like curve, the second column is a null curve with $\bar{v}$ as constant tangent vector field while the third column suggests the following horocycle parametrized by arc-length as example of flat-hyperbolic geodesic:

$$\gamma(s) = \left(-\frac{s^2}{2}, s, 1 + \frac{s^2}{2}\right) \in H^2, \quad T(s) = (-s, 1, s), \quad \bar{n}(s) = \left(\frac{s^2}{2} - 1, -s, -\frac{s^2}{2}\right). \tag{3.9}$$

This curve is the intersection of H^2 with the (Euclidean) plane $\pi : z = 1 - x$, and hence is the Euclidean parabola $y^2 = -2x$ in this plane; its Euclidean curvature as space curve is $k(s) = \frac{\sqrt{2}}{(1+2s^2)^{\frac{3}{2}}} \in (0, \sqrt{2}]$. Applying the Cayley map (2.7) gives the following horocycle in $\mathcal{H}^2$:

$$Cay \circ \gamma(s) = \frac{1}{1 + (s-1)^2}(-s^2, 2) \tag{3.10}$$

which is the Euclidean circle $\mathcal{C}$ with the center $z_0 = (-1, 1) = -1 + i \in \mathcal{H}^2$ and radius 1. Hence $\mathcal{C}$ is tangent to $\mathbb{R}$ to the point $p = (-1, 0)$. The hyperbolic distance between z_0 and i in $(\mathcal{H}^2, g_h)$ is:

$$d_h(z_0, z_1 = i) = \ln \frac{|z_0 - \overline{z_1}| + |z_0 - z_1|}{|z_0 - \overline{z_1}| - |z_0 - z_1|} = \ln \frac{3 + \sqrt{5}}{2} = 2 \ln \Phi \simeq 0.96 < 1 = d_{Euclidean}(z_0, i) \tag{3.11}$$

with $\Phi = \frac{1+\sqrt{5}}{2}$ the well-known golden ratio ([7]).

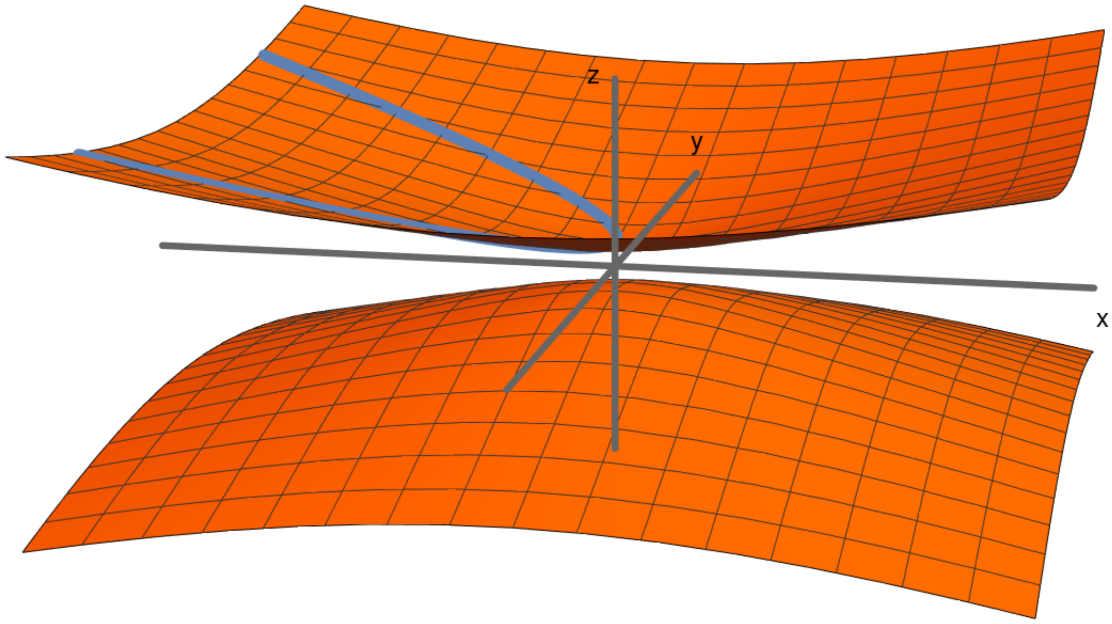
The flow-h-evolute of a horocycle γ parametrized by arclength is the curve:

$$E^f_\gamma(s) = E^f_2(s) = \sin s T(s) + \cos s \bar{n}(s). \tag{3.12}$$

For our curve (3.9) we have the unit space-like vector field:

$$E^f_\gamma(s) = \left((\frac{s^2}{2} - 1) \cos s - s \sin s, \sin s - s \cos s, s \sin s - \frac{s^2}{2} \cos s \right) \tag{3.13}$$

with $(E^f_\gamma)'(s) = \sin s \cdot \gamma(s)$. □



The horocycle γ on H^2

4. Other examples

In this section we present some curves which are not flat-hyperbolic geodesics.

Example 4.1 For $C > 0$ let the space-like curve:

$$\gamma_C(t) = (C\cos t, C\sin t, \sqrt{1+C^2}) \in H^2, \|\gamma_C'(t)\|_L = C, (R^3(u) \circ \gamma_C)(t) = \gamma_C(u+t). \tag{4.1}$$

A straightforward calculus gives that k_g is the constant $K = \frac{\sqrt{1+C^2}}{C} > 1$ which means that γ_C is a circle having also a constant flow-geodesic curvature:

$$k_g^f = \frac{\sqrt{1+C^2}}{C} - \frac{1}{C} = \frac{\sqrt{1+C^2}-1}{C} > 0. \tag{4.2}$$

The evolute of γ_C is the constant unit time-like vector:

$$E_C(s) = \Omega(s) = \sqrt{1+C^2}\gamma_C(s) + C\bar{n}_C(s) =$$

$$= \sqrt{1+C^2}\left(C\cos\frac{s}{C}, C\sin\frac{s}{C}, \sqrt{1+C^2}\right) + C\left(-\sqrt{1+C^2}\cos\frac{s}{C}, -\sqrt{1+C^2}\sin\frac{s}{C}, -C\right) = \bar{k}. \tag{4.3}$$

The Cayley image of the curve γ_C is the Euclidean circle $\tilde{C}$ with the center $z_2 = (0, \sqrt{1+C^2}) = \sqrt{1+C^2}i \in Oy$ and radius C. Recall that for a circle $\hat{C}$ with center $(0, H) \in Oy$ and radius $R < H$ its hyperbolic length is:

$$L_h(\hat{C}) = \frac{2\pi R}{\sqrt{H^2 - R^2}}. \tag{4.4}$$

For our circle $\tilde{C}$ two lengths are equal: $L_h(\tilde{C}) = L_{Euclidean}(\hat{C}) = 2\pi C$. Also, the hyperbolic area of the disc having $\hat{C}$ as boundary is:

$$\mathcal{A}_h(\hat{C}) = 2\pi\left(\frac{H}{\sqrt{H^2-R^2}} - 1\right) \tag{4.5}$$

which for our disk becomes: $\mathcal{A}_h(\hat{C}) = 2\pi(\sqrt{1+C^2} - 1)$. □

Example 4.2 If we modify the horocycle of the example 3.3 into the curve parametrized by arc-length:

$$\gamma_-(s) = \left(\frac{s^2}{2}, s, 1+\frac{s^2}{2}\right) \in H^2, \quad T_-(s) = (s, 1, s), \quad \bar{n}_-(s) = \left(\frac{s^2}{2} - 1, s, \frac{s^2}{2}\right). \tag{4.6}$$

it results the constant negative curvatures: $k_g = -1, \quad k_g^f = -2$ and the flow-h-evolute is:

$$E_\gamma^f(s) = \left((\frac{s^2}{2} - 1)\cos s + s\sin s, \sin s + s\cos s, s\sin s + \frac{s^2}{2}\cos s \right) \tag{4.7}$$

its Cayley image is the Euclidean circle centered in $(1,1)$ and having the radius $r = 1$. $\square$

Example 4.3 Let us consider the space-like curve:

$$\gamma(t) = (\cos t \sinh t, \sin t \sinh t, \cosh t) \in H^2, \quad \|\gamma'(t)\|_L = \cosh t. \tag{4.8}$$

Then its data is:

$$\begin{cases} T(t) = (\cos t - \sin t \tanh t, \sin t + \cos t \tanh t, \tanh t), \\ \bar{n}(t) = \left(-\frac{\sin t}{\cosh t} - \cos t \sinh t, \frac{\cos t}{\cosh t} - \sin t \sinh t, -\frac{\sinh^2 t}{\cosh t} \right), \\ k_g(t) = 1 + \frac{1}{\cosh^2 t} \in (1,2], \quad k_g^f(t) = 1 + \frac{1}{\cosh^2 t} - \frac{1}{\cosh t}, \\ \Omega(t) = \left(\frac{\cos t \sinh t}{\cosh t} - \sin t, \frac{\sin t \sinh t}{\cosh t} + \cos t, 2 \right). \end{cases} \tag{4.9}$$

The arc-length parametrization of this curve is:

$$\begin{cases} \gamma(s) = (s\cos(\sinh^{-1} s), s\sin(\sinh^{-1} s), \sqrt{1+s^2}), \\ k_g(s) = \frac{s^2+2}{s^2+1}, \quad k_g^f(s) = \frac{s^2+2}{s^2+1} - \frac{1}{\sqrt{s^2+1}}. \end{cases} \tag{4.10}$$

with:

$$\begin{cases} T(s) = \left(\cos(\sinh^{-1} s) - \frac{s}{\sqrt{1+s^2}} \sin(\sinh^{-1} s), \sin(\sinh^{-1} s) + \frac{s}{\sqrt{1+s^2}} \cos(\sinh^{-1} s), \frac{s}{\sqrt{1+s^2}} \right), \\ \bar{n}(s) = \left(-\frac{\sin(\sinh^{-1} s)}{\sqrt{1+s^2}} - s\cos(\sinh^{-1} s), \frac{\cos(\sinh^{-1} s)}{\sqrt{1+s^2}} - s\sin(\sinh^{-1} s), \frac{s^2}{\sqrt{1+s^2}} \right). \end{cases} \tag{4.11}$$

The hyperbolic plane geometry is studied sometimes through the Lobachevsky's angle of parallelism function Π defined by ([3, p. 141]): $\sin \Pi(x) = \frac{1}{\cosh x}$. Then:

$$\begin{cases} k_g(t) = 1 + \sin^2 \Pi(t), \quad \int_0^{2\pi} (1 + \sin^2 u) du = 3\pi, \\ k_g^f(t) = 1 + \sin^2 \Pi(t) - \sin \Pi(t) = \left(\frac{1}{2} - \sin \Pi(t) \right)^2 + \frac{3}{4} \geq \frac{3}{4} \end{cases} \tag{4.12}$$

and therefore the minimum value $\frac{3}{4}$ of the function k_g^f corresponds to $\cosh t_0 = 2$ and $s_0 = \sinh t_0 = \sqrt{3}$ i.e. $t_0 = \ln(2 + \sqrt{3})$; hence $k_g(t_0) = \frac{5}{4}$. $\square$

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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