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A Note on The Some Class of Symmetric Numerical Semigroups

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Abstract

In this article, we will examine some classes of symmetric numerical semigroups and half of them, which are a class of irreducible numerical semigroups such that $U =< a, a+1 >$ and $U =< a, a+2 >$ symmetric numerical semigroups, respectively.

Keywords: Numerical semigroup; Gap; Perfect; Symmetric; Irreducible numerical semigroup.

Bazı Simetrik Sayısal Yarı Grup Sınıflarına İlişkin Bir Not

Öz

Bu makalede, indirgenemez sayısal yarıgrupların bir sınıfı olan sırasıyla $U =< a, a+1 >$ ve $U =< a, a+2 >$ şeklindeki simetrik sayısal yarıgrupları ve onların yarısı olan sayısal yarıgrupları inceleyeceğiz.

Anahtar Kelimeler: Sayısal yarıgrup; Boşluk; Mükemmel; Simetrik; İndirgenemez sayısal yarıgrup.

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1. Introduction

Numerical semigroups naturally arose as the set of values of d which have nonnegative integer solutions to Diophantine equations of the form $c_1x_1 + c_2x_2 + \ldots + c_nx_n = d$, where $c_1, c_2, \ldots, c_n \in \mathbb{N}$ (here $\mathbb{N}$ denotes the nonnegative integers). We reduce to the case $gcd(c_1, c_2, \ldots, c_n) = 1$. Frobenius asked what is the largest integer d such that a given equation has no solutions over the nonnegative integers. Sylvester and others solved the $n = 2$ case, and since then finding the largest such d has been known as the Frobenius problem (The case $n = 2$ corresponds to a numerical semigroup S produced with two elements, which is called a symmetric numerical semigroup). A through introduction to the Frobenius problem and related topics are given in [1].

The numerical semigroups have an important role in commutative algebra and algebraic geometry. In addition, Bertin and Carbonne [2], Delorme [3], Watanabe [4] et al. have successfully described the properties of numerical semigroups that are compatible with numerical semigroup rings related to various classifications in ring theory.

Numerical semigroups are also used to analyze singularities on planar algebraic curves. The numerical semigroups associated with these planar curves are irreducible.

A numerical semigroup is irreducible if it cannot be expressed as an intersection of two numerical semigroups containing it properly. There are two families of numerical semigroups of particular importance in the theory of irreducible numerical semigroups. These are symmetric and pseudo-symmetric numerical semigroups (for details, see [5-7]).

Recently, some applications to coding theory and cryptography have emerged. The reason for this is the use of algebraic codes and Weierstrass numerical semigroups. The goal here is to find the properties of the codes in terms of a corresponding numerical semigroup (for details see [8, 9]).

Let $\mathbb{N}$ and $\mathbb{Z}$ be non-negative integers and integers set, respectively. $U \subseteq \mathbb{N}$ is called a numerical semigroup if it satisfies following conditions

1) $0 \in U$,

2) $u_1 + u_2 \in U$ for all$u_1, u_2 \in U$,

3) $Card(\mathbb{N} \backslash U)$ is finite.

The Frobenius number of S is

$f(U) = max\{x \in \mathbb{Z}: x \notin U\}$

and the element number of U is

$n(U) = Card(\{0,1,2,\dots,f(U)\} \cap U)$.

If $U$is a numerical semigroup such that $U =< u_1, u_2, \dots, u_n >$, then we write that

$$U =< u_1, u_2, \dots, u_n >= \{c_0 = 0, c_1, c_2, \dots, c_{n-1}, c_n = f(U) + 1, \rightarrow \dots\}$$

where $u_t < u_{t+1}, n = n(U)$, and the arrow means that every integer greater than $f(U) + 1$ belongs to U, for $t = 1,2, \dots, n = n(U)$. Here, we say the number $C(U) = f(U) + 1$ is conductor of U ([10, 11]).

Let $U =< u_1, u_2, \dots, u_k >$ be a numerical semigroup. Then the numbers $m(U) = min\{x \in U: x \neq 0\} = u_1$ and $e(U) = k$ are called multiplicity and embedding dimension of U, respectively. On the other hand, for numerical semigroup S it is known that $e(U) \leq m(U)$. If $e(U) = m(U)$, then we say U has maximal embedding dimension. The numerical semigroup U is called symmetric if $f(U) - b \in U$ for all $b \in \mathbb{Z} \backslash U$. It is known that $U =< u_1, u_2 >$ is symmetric numerical semigroup and $f(U) = u_1 u_2 - u_1 - u_2$. Also, a numerical semigroup U is pseudo-symmetric if $f(U)$ is even and the only integer such that $b \in \mathbb{Z} \backslash U$and $f(U) - b \notin U$ is $b = \frac{f(U)}{2}$ (for details see, [12, 13, 14]). On the other hand, we define the numerical semigroup $\frac{U}{q} = \{x \in \mathbb{N}: qx \in U\}$. It is clear that $U \subseteq \frac{U}{q}$, for $q > 0, q \in \mathbb{N}$. If $q = 2$ then the numerical semigroup $\frac{U}{2}$ is called half of U. If $v|m$ then $\frac{U}{m} \subseteq \frac{U}{v}$ for $v, m \in \mathbb{N} \backslash \{0\}$ and also if $q \in U$ then we find $\frac{U}{q} = \mathbb{N}$ ([15]).

Let U be a numerical semigroup. The element z is a gap of U if $z \in \mathbb{N}$ but $z \notin U$, we denote the set of gaps of U, by $H(U)$, i.e. $H(U) = \{z \in \mathbb{N}: z \notin U\}$. Here, $z \in H(U)$ is called isole gap if $z - 1, z + 1 \in U$. The set of isole gaps of U is denoted by $I(U)$, i.e. $I(U) = \{z \in H(U): z - 1, z + 1 \in U\}$. Also, the numerical semigroup U is called perfect if $I(U) = \phi$ (for details see [16, 17, 18]).

In this study, we will examine some classes of symmetric numerical semigroups and half of them, which are a class of irreducible numerical semigroups such that $U =< a, a + 1 >$ and $U =< a, a + 2 >$ symmetric numerical semigroups, respectively.

2. Main Results

Theorem 2.1. ([5]) Let U be a numerical semigroup. Then,

(a) If $f(U)$ is a odd integer then U is irreducible if and only if for all $h, h' \in \mathbb{Z}$, such that $h + h' = f(U)$, we have that either $h \in U$ or $h' \in U$ (that is, U is symmetric).

(b) If $f(U)$ is an even integer then U is irreducible if and only if for all $h, h' \in \mathbb{Z} \setminus \left\{\frac{f(U)}{2}\right\}$ such that $h + h' = f(U)$, we have that either $h \in U$ or $h' \in U$ (that is, U is pseudo-symmetric).

Theorem 2.2. If $U =< a, a + 1 >$, where $a > 1$ and $a \in \mathbb{N}$, then we have $I(U) = \{f(U)\}$.

Proof. Let $U =< a, a + 1 >$ be a symmetric numerical semigroup. Then, we write

$$U =< a, a + 1 >= \{0, a, a + 1, 2a, 2a + 1, 2a + 2, \ldots, a^2 - a - 2, a^2 - a, \rightarrow \ldots\},$$
$$f(U) = a^2 - a - 1$$

and

$$H(U) = \{1,2,3, \ldots, a - 1, a + 2, \ldots, a^2 - a - 1\}.$$

In this case, we have

$$I(U) = \{z \in H(U): z - 1, z + 1 \in U\} = \{a^2 - a - 1\} = \{f(U)\}.$$

Corollary 2.3. If $U =< a, a + 1 >$, where $a > 1$ and $a \in \mathbb{N}$, then U is not perfect.

Theorem 2.4. Let $U =< a, a + 1 >$ be a symmetric numerical semigroup, where $a > 1$ and $a \in \mathbb{N}$. In this case,

(1) If $a > 2$ is even number then $\frac{U}{2} =< \frac{a}{2}, a + 1 >$,

(2) If $a > 1$ is odd number then $\frac{U}{2} =< \frac{a+1}{2}, a >$.

Proof. Let $U =< a, a + 1 >$ be a symmetric numerical semigroup, where $a > 1$ and $a \in \mathbb{N}$. In this case,

(1) If $a > 2$ is even number then $\frac{a}{2} \in \mathbb{N}$. Thus,

$$x \in < \frac{a}{2}, a + 1 > \Leftrightarrow \exists p_1, p_2 \in \mathbb{N}, x = (\frac{a}{2})p_1 + (a + 1)p_2$$

$$\Leftrightarrow 2x = (a)p_1 + (a + 1)2p_2 \in < a, a + 1 >= U$$

$$\Leftrightarrow x \in \frac{U}{2}.$$

(2) If $a > 1$ is odd number then $\frac{a+1}{2} \in \mathbb{N}$. So,

$$y \in < \frac{a+1}{2}, a > \Leftrightarrow \exists v_1, v_2 \in \mathbb{N}, y = (\frac{a+1}{2})v_1 + (a)v_2$$

$$\Leftrightarrow 2y = (a+1)v_1 + (a)2v_2 \in < a, a+1 >= U$$

$$\Leftrightarrow y \in \frac{U}{2}.$$

Theorem 2.5. If $U =< a, a+1 >$ be a symmetric numerical semigroup, where $a > 1$ and $a \in \mathbb{N}$, then the numerical semigroup $\frac{U}{2}$ is not perfect.

Proof. Let $U =< a, a+1 >$ be a symmetric numerical semigroup, where $a > 1$ and $a \in \mathbb{N}$. Then, from Theorem 2.4we write that

(1) If $a > 2$ is even number then $\frac{U}{2} =< \frac{a}{2}, a+1 >$,

(2) If $a > 1$ is odd number then $\frac{U}{2} =< \frac{a+1}{2}, a >$.

In this case,

(1) If $a > 2$ is even number then

$$\frac{U}{2} =< \frac{a}{2}, a+1 >= \left\{0, \frac{a}{2}, a, a+1, \frac{3a}{2}, \frac{3a}{2} + 1, 2a, 2a+1, \ldots, \frac{a^2}{2} - a - 2, \frac{a^2}{2} - a, \rightarrow \ldots\right\}$$

and

$$H(\frac{U}{2}) = \left\{1,2,3, \ldots, \frac{a}{2} - 1, \frac{a}{2} + 1, a - 1, \ldots, \frac{a^2}{2} - a - 1\right\}.$$

Thus, we obtain

$$f(\frac{U}{2}) = \frac{a^2}{2} - a - 1 \in I(\frac{U}{2})$$

since

$$I(\frac{U}{2}) = \left\{x \in H(\frac{U}{2}) : x - 1, x + 1 \in \frac{U}{2}\right\}.$$

So, $I(\frac{U}{2}) \neq \phi$.

(2) If $a > 1$ is odd number then

$$\frac{U}{2} =< \frac{a+1}{2}, a >$$

$$= \left\{0, \frac{a+1}{2}, a, a+1, \frac{3a}{2}+1, 2a, 2a+2, \ldots, \frac{a^2-1}{2}-a-1, \frac{a^2-1}{2}-a+1, \rightarrow \cdots\right\}$$

and

$$H(\frac{U}{2}) = \left\{1,2,3, \ldots, \frac{a+1}{2}-1, \frac{a+1}{2}+1, a-1, \ldots, \frac{a^2-1}{2}-a\right\}.$$

Thus, we obtain

$$f\left(\frac{U}{2}\right) = \frac{a^2-1}{2} - a \in I\left(\frac{U}{2}\right),$$

since

$$I(\frac{U}{2}) = \left\{x \in H(\frac{U}{2}) : x-1, x+1 \in \frac{U}{2}\right\}$$

and $I(\frac{U}{2}) \neq \phi$. So, we find that the numerical semigroup $\frac{U}{2}$ is not perfect.

Theorem 2.6. Let $U =< a, a+2 >$ be a numerical semigroup, where $a \in \mathbb{N}$ and $a > 1$ is odd. Then U is not perfect numerical semigroup.

Proof. Let $U =< a, a+2 >$ be a symmetric numerical semigroup. Then, we write

$$U =< a, a+2 >= \{0, a, a+2, 2a, 2a+2, \ldots, a^2-3, a^2-1, \rightarrow \ldots\},$$

$$f(U) = a^2 - 2$$

and

$$H(U) = \{1,2,3, \ldots, a-1, a+1, 2a+1, \ldots, a^2-2\}.$$

In this case, we have,

$$f(U) = a^2 - 2 \in I(U)$$

since

$$I(U) = \{z \in H(U) : z-1, z+1 \in U\}.$$

Thus, U is not perfect numerical semigroup.

Theorem 2.7. Let $U =< a, a + 2 >$ be a symmetric numerical semigroup for $a \in \mathbb{N}, a > 1$ is odd. Then, we have $\frac{U}{2} =< a, a + 1, a + 2 >$.

Proof. Let $U =< a, a + 2 >$ be a symmetric numerical semigroup, for $a \in \mathbb{N}, a > 1$ is odd. Then,

$$u \in < a, a + 1, a + 2 > \Leftrightarrow \exists v_1, v_2, v_3 \in \mathbb{N}, u = (a)v_1 + (a + 1)v_2 + (a + 2)v_3$$

$$\Leftrightarrow 2u = (a)2v_1 + (a + 1)2v_2 + (a + 2)2v_3$$

$$\Leftrightarrow 2u = 2av_1 + 2av_2 + 2v_2 + 2av_3 + 4v_3 = a(2v_1 + v_2) + (a + 2)(2v_3 + v_2)$$

$$\Leftrightarrow 2u \in < a, a + 2 >= U \Leftrightarrow u \in \frac{U}{2}.$$

Theorem 2.8. Let $U =< a, a + 2 >$ be a symmetric numerical semigroup for $a \in \mathbb{N}, a > 1$ is odd. Then, the numerical semigroup $\frac{U}{2}$ is perfect.

Proof. Let $U =< a, a + 2 >$ be symmetric numerical semigroup for $a \in \mathbb{N}, a > 1$is odd. Then, we have $\frac{U}{2} =< a, a + 1, a + 2 >$ from Theorem 2.7. In this case, we write that

$$\frac{U}{2} =< a, a + 1, a + 2 > = \left\{0, a, a + 1, a + 2, 2a, 2a + 2, 2a + 3, \ldots, (\frac{a-1}{2})a - 3, (\frac{a-1}{2})a, \rightarrow \ldots\right\}$$

and

$$H(\frac{U}{2}) = \left\{1,2,3, \ldots, a - 1, a + 3, a + 4, 2a - 1, \ldots, f(\frac{U}{2}) = (\frac{a-1}{2})a - 1\right\}.$$

Thus, we obtain

$$I(\frac{U}{2}) = \left\{x \in H(\frac{U}{2}) : x - 1, x + 1 \in \frac{U}{2}\right\} = \phi.$$

So, we find that the numerical semigroup $\frac{U}{2}$ is perfect.

Example 2.9. Let $U =< 4,5 >= \{0,4,5,8,9,10,12, \rightarrow \ldots\}$. Then we have

$$f(U) = 11, m(U) = 4, n(U) = 6, e(U) = 2, H(U) = \{1,2,3,6,7,11\}$$

and

$$I(U) = \{z \in H(U): z-1, z+1 \in U\} = \{f(U) = 11\}.$$

Thus, the numerical semigroup U is not perfect. Also,

$$\frac{U}{2} = \{u \in \mathbb{N}: 2u \in U\} = \{0,2,4,\rightarrow\ldots\} =< 2,5 >, H(\frac{U}{2}) = \{1,3\}$$

and

$$I(\frac{U}{2}) = \left\{g \in H(\frac{U}{2}): g-1, g+1 \in \frac{U}{2}\right\} = \{1,3\} \neq \phi.$$

So, $\frac{U}{2}$ is not perfect.

Example 2.10. Let $U =< 5,6 >= \{0,5,6,10,11,12,15,16,17,18,20,\rightarrow\ldots\}$. Then we write

$$H(U) = \{1,2,3,4,7,8,9,13,14,19\}$$

and

$$I(U) = \{z \in H(U): z-1, z+1 \in U\} = \{f(U) = 19\}.$$

Thus, the numerical semigroup U is not perfect. Also,

$$\frac{U}{2} = \{u \in \mathbb{N}: 2u \in U\} = \{0,3,5,6,8,\rightarrow\ldots\} =< 3,5 >, H(\frac{U}{2}) = \{1,2,4,7\}$$

and

$$I(\frac{U}{2}) = \left\{g \in H(\frac{U}{2}): g-1, g+1 \in \frac{U}{2}\right\} = \{4,7\} \neq \phi.$$

So, $\frac{U}{2}$ is not perfect.

Example 2.11. Let's $U =< 5,7 >= \{0,5,7,10,12,14,15,17,19,20,21,22,24,\rightarrow\ldots\}$. Then we write

$$H(U) = \{1,2,3,4,6,8,9,11,13,16,18,23\}$$

and

$$I(U) = \{z \in H(U): z-1, z+1 \in U\} = \{6,11,13,16,18,23\}.$$

Thus, the numerical semigroup U is not perfect. Also,

$$\frac{U}{2} = \{u \in \mathbb{N}: 2u \in U\} = \{0,5,6,7,10,\rightarrow\ldots\} =< 5,6,7 >, H(\frac{U}{2}) = \{1,2,3,4,8,9\}$$

and

$$I(\frac{U}{2}) = \left\{g \in H(\frac{U}{2}): g-1, g+1 \in \frac{U}{2}\right\} = \phi.$$

So, $\frac{U}{2}$ is perfect numerical semigroup.

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