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ON SUBFLAT DOMAINS OF RD-FLAT MODULES

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Abstract. The concept of subflat domain is used to measure how close (or far away) a module is to be flat. A right module is flat if its subflat domain is the entire class of left modules. In this note, we focus on of RD-flat modules that have subflat domain which is exactly the collection of all torsion-free modules, shortly tf-test modules. Properties of subflat domains and of tf-test modules are studied. New characterizations of left P-coherent rings and torsion-free rings by subflat domains of cyclically presented left R-modules are obtained.

1. Introduction

The rings R in this note are associative with identity, and every module is, if not specified otherwise, right R-module. We use $Mod - R$ ($R - Mod$) to denote the class of right (left) R-modules.

There are important subclasses of $Mod - R$ that shed light on the whole of $Mod - R$. The classes of all projectives, all injective modules and all flat modules are the prominent ones. Recently, many authors have studied on alternative ways to test projectivity, injectivity and flatness of modules. In general, they are trying to find test module whose test projectivity (injectivity or flatness) of modules ($[1, 2, 4, 10, 11, 18]$). In this paper, we test the flatness of the RD-flat modules by torsion-free modules.

Inspired by homological properties of torsion-free modules over an integral domain, Hattori in [9] defined and studied torsion-free modules over non-commutative rings. A right R-module X is called torsion-free if $\mathbf{T}or_1(X, R/Ra) = 0$ for all $a \in R$. Flat modules are torsion-free, but the converse is not true in general. Torsion-free modules are intimately related to relatively divisible (RD) exact sequences. A short exact sequence $0 \to K \to L \to M \to 0$ is called RD-exact if, for every $a \in R$, the

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induced homomorphism $Hom_R(R/Ra, L) \to Hom_R(R/Ra, M) \to 0$ is surjective, or equivalently, the induced map $(R/aR) \otimes K \to (R/aR) \otimes L$ is monic ([19, Proposition 2]). An R-module T (respectively, D) is torsion-free (respectively, divisible) if and only if every short exact sequence $0 \to D \to B \to T \to 0$ is RD-exact ([13]). Note that torsion-free (respectively, divisible) modules are called P-flat (rspectively, P-injective) by some authors. By the standard adjoint isomorphism, a module B is torsion-free if and only if its character module B^+ is a divisible left R-module. Obviously, every pure exact sequence is RD-exact. Moreover, every flat and fp-injective module is respectively torsion-free and divisible.

An R-module N is called RD-injective (respectively, RD-projective, RD-flat) if it has the injective (respectively, projective, flat) property with respect to every RD-exact sequence. The notions of RD-projective, RD-injective and RD-flat module were used by Stenström in [17]. Commutative rings for which each Artinian module is RD-injective (RD-flat) were completely characterized in [5]. In [13], the author studied main properties of RD-projective, RD-injective and RD-flat modules.

Inspired and motivated by Whitehead injective test modules (shortly, i-test modules) in [7, 18], f-test modules is defined and studied in [2], through $\mathbf{T}or$ functor. A module F is called f-test provided that for every left R-module K, $\mathbf{T}or(F, K) = 0$ implies that K is flat. In the same vein as f-test module, the main objective of the present paper is to study test modules for torsion-freeness. A module K_R is said to be ${}_RL$*-subflat* if for every short exact sequence $0 \to U \to D \to L \to 0$ of left R-modules, the sequence $0 \to K \otimes U \to K \otimes D \to K \otimes L \to 0$ is exact. For any $K \in Mod - R$, we denote by $\mathfrak{F}^{-1}(K)$ the class $\{L \in R - Mod :$ K is L-subflat$\}$. Clearly, K_R is flat if and only if $\mathfrak{F}^{-1}(K) = R - Mod$. As can be seen from the definitions, all flat left R-modules are contained in $\mathfrak{F}^{-1}(K)$ for each module K. In particular, if M_R is RD-flat and ${}_RN$ is torsion-free, then M_R is ${}_RN$-subflat. So, the smallest possible subflat domain for an RD-flat module is the class of torsion-free modules. We call a left module K test module for torsion-free (shortly, tf-test) module if $\mathfrak{F}^{-1}(K)$ is exactly the class of torsion-free modules. We show that every ring has a tf-test module.

In Section 2, we first obtain elementary properties of subflat domains of modules. We present new characterizations for P-coherent rings and torsion-free rings by subflat domains. For example, a ring R is torsion-free if and only if the subflat domain of any cyclically presented left (or right) R-module is closed under submodules. In Section 3, we discuss tf-test modules.

In what follows, we write $\mathcal{TF}_R$(respectively, $\mathcal{F}_R, \mathcal{N}_R$) for the family of torsion-free (respectively, flat, nonsingular) modules. For a right R-module M, the character module $Hom_{\mathbb{Z}}(U, \mathbb{Q}/\mathbb{Z})$ is denoted by U^+. Given R-modules U and H, $Hom(U, H)$ (resp. $Ext^n(U, H)$) means $Hom_R(U, H)$ (resp. $Ext^n_R(U, H)$), and similarly $U \otimes H$ (resp. $\mathbf{T}or_n(U, H)$) denotes $U \otimes_R H$ (resp. $\mathbf{T}or^R_n(U, H)$) for an integer $n \geq 1$ unless otherwise specified.

2. Subflat Domains

This section is devoted to obtain some elementary properties of subflat domains of modules that will be needed later in the paper.

Given a left module X, a module T is X-subflat if and only if $\mathbf{T}or_1^R(T,X)=0$ by [2, Proposition 2.3]. Moreover, if $T \leq M$ and, T and M/T are N-subflat, then M is N-subflat.

Lemma 1. *Let $Y \in Mod-R$ and X be a pure submodule of Y. $\mathfrak{F}^{-1}(Y) \subseteq \mathfrak{F}^{-1}(X)$.*

Proof. Let $A \in \mathfrak{F}^{-1}(Y)$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes F_0 & \xrightarrow{\epsilon} & Y \otimes F_0 & \xrightarrow{\delta} & (Y/X) \otimes F_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & X \otimes F_1 & \xrightarrow{\eta} & Y \otimes F_1 & \xrightarrow{\vartheta} & (Y/X) \otimes F_1 & \longrightarrow & 0, \end{array}$$

where $0 \to F_0 \to F_1 \to A \to 0$ is any short exact sequence. Since $A \in \mathfrak{F}^{-1}(Y)$, γ is monic. On the other hand, since X is pure submodule of Y, the rows are exact. Then, α is a monomorphism, because $\eta\alpha = \gamma\epsilon$ is monomorphism. □

For $Y \in Mod-R$, the flat dimension of Y (fd(Y)) ≤ 1 if and only if $\mathbf{T}or_2^R(Y,B) = 0$, $\forall B \in R-Mod$ ([15, pp.239]).

Lemma 2. *Let $Y \in Mod-R$ and W be a submodule of Y. If $fd(Y/W) \leq 1$, then $\mathfrak{F}^{-1}(Y) \subseteq \mathfrak{F}^{-1}(W)$.*

Proof. Recall that $fd(Y/W) \leq 1$ if and only if then $\mathbf{T}or_2^R(Y/W,A)=0$ for every left R-module A. If $A \in \mathfrak{F}^{-1}(Y)$, then $\mathbf{T}or_1^R(Y,A)=0$ by [2, Proposition 2.3]. So the sequence $0 \to W \to Y \to \frac{Y}{W} \to 0$ implies that $0 = \mathbf{T}or_2^R(\frac{Y}{W},A) \to \mathbf{T}or_1^R(W,A) \to \mathbf{T}or_1^R(Y,A)=0$ is exact. Therefore, W is A-subflat by [2, Proposition 2.3].

□

In general, for any R-module M, $\mathfrak{F}^{-1}(M)$ is closed under pure submodules.

Theorem 1. *Let $T \in Mod-R$. $fd(T) \leq 1$ if and only if $\mathfrak{F}^{-1}(T)$ is closed under submodules.*

Proof. Let $Z \in \mathfrak{F}^{-1}(T)$ and $H \subseteq Z$ be any submodule. From the sequence $0 \to H \to Z \to Z/H \to 0$, we have that $0 = \mathbf{T}or_2^R(T,Z/H) \to \mathbf{T}or_1^R(T,H) \to \mathbf{T}or_1^R(T,Z)=0$. Then, T is H-subflat by [2, Proposition 2.3]. For the converse, let $Z \in R-Mod$ and consider the short exact sequence $0 \to H \to U \to Z \to 0$ with U projective. Since $U \in \mathfrak{F}^{-1}(T)$, $\mathbf{T}or_1^R(T,H)=0$ by our hypothesis. By the exactness of $0 = \mathbf{T}or_2^R(T,U) \to \mathbf{T}or_2^R(T,Z) \to \mathbf{T}or_1^R(T,H)=0$, $\mathbf{T}or_2^R(T,Z)=0$. Therefore, $fd(T) \leq 1$. □

$wD(R) \leq 1$ if and only if $fd(X) \leq 1$ for all right (or left) modules X ([15, pp. 240]).

Corollary 1. *$wD(R) \leq 1$ if and only if $\mathfrak{F}^{-1}(X)$ is closed under submodules for every (finitely presented) left (or right) R-module X.*

We say R is torsion-free if all its (finitely generated) right (or left) ideals of R are torsion-free. The concept of a torsion-free ring is left and right symmetric ([6]). It is easy to see that a cyclic module is torsion-free if and only if it is flat. So, a ring is torsion-free if and only if it is a pf-ring, i.e. each principal ideal is flat. A cyclic module $M \cong R/I$ is called cyclically presented if $I = aR$ for some $a \in R$.

Corollary 2. *R is torsion-free ring if and only if the subflat domain of any cyclically presented (or RD-flat) right (or left) R-module is closed under submodules.*

Theorem 2. *Let U be a finitely presented module and $0 \to K \to H \to U \to 0$ be a short exact sequence with finitely generated projective module H. $\mathfrak{F}^{-1}(U)$ is closed under direct products if and only if K is finitely presented*

Proof. ($\Rightarrow$) $\mathbf{T}or_1^R(U, \prod R) = 0$ by our assumption. Consider the following commutative diagram

$$\begin{array}{ccccccc}
K \otimes (\prod R) & \xrightarrow{\beta} & H \otimes (\prod R) & \xrightarrow{\delta} & U \otimes (\prod R) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta} & & \\
\prod K & \xrightarrow{\eta} & \prod H & \xrightarrow{\vartheta} & \prod U & \longrightarrow & 0
\end{array}$$

γ and θ are isomorphisms by [8, Theorem 3.2.22]. Then α is an isomorphism by the Five Lemma, therefore K is finitely presented by [8, Theorem 3.2.22].

($\Leftarrow$) Let $A \in \mathfrak{F}^{-1}(U)$, i.e. $\mathbf{T}or_1^R(U, A) = 0$. By the adjoint isomorphism, $Ext_R^1(U, A^+) = 0$. Note that $\mathbf{T}or_1^R(N, B^+) = Ext_R^1(N, B)^+$ for every $B \in R-Mod$ if a module N has a projective resolution $P_2 \to P_1 \to P_0 \to N \to 0$, where P_i is finitely generated for $i = 0, 1, 2$ (see [15, Remark, pp. 257]). Thus this implies that $\mathbf{T}or_1^R(U, A^{++}) = 0$, that is U is A^{++}-subflat.

Let $\{M_i\}_{i \in J}$ be a family of left R-modules in $\mathfrak{F}^{-1}(U)$. Then $\bigoplus_{i \in J} M_i \in \mathfrak{F}^{-1}(U)$ by main properties of $\mathbf{T}or$. So $(\bigoplus_{i \in J} M_i)^{++} \cong (\prod_{i \in J} M_i^+)^+$ is in $\mathfrak{F}^{-1}(U)$ by the preceding paragraph. But $\bigoplus_{i \in J} M_i^+$ is a pure submodule of $\prod_{i \in J} M_i^+$ by [12, Example 4.84(d)], hence $(\prod_{i \in J} M_i^+)^+ \to (\bigoplus_{i \in J} M_i^+)^+ \to 0$ is a splitting epimorphism. Therefore $(\bigoplus_{i \in J} M_i^+)^+ \cong \prod_{i \in J} M_i^{++}$ is in $\mathfrak{F}^{-1}(U)$. Since $\prod_{i \in J} M_i$ is a pure submodule of $\prod_{i \in J} M_i^{++}$ and $\mathfrak{F}^{-1}(U)$ is closed under pure submodules, $\prod_{i \in J} M_i$ is in $\mathfrak{F}^{-1}(U)$. □

R is called a right coherent (respectively, P-coherent) ring if every finitely generated (respectively, principal) right ideal is finitely presented ([14]).

Corollary 3. *R is right coherent(respectively, P-coherent) ring if and only if $\mathfrak{F}^{-1}(U)$ is closed under direct products for every finitely presented (respectively, cyclically presented) module U.*

3. RD-Flat Modules Having a Restricted Subflat Domain

In this section, we study existence of test modules for torsion-freeness. If U is RD-flat and N is torsion-free left R-module, then U is N-subflat. The next proposition shows that the subflat domain of any RD-flat module must contain at least the torsion-free modules. The following fact can be easily verified.

Proposition 1. *${}_R\mathcal{TF} = \bigcap_{M\in\Omega} \mathfrak{F}^{-1}(M)$, where Ω is the class of all RD-flat modules.*

Definition 1. *An RD-flat module K is called tf-test module if $\mathfrak{F}^{-1}(K) = \mathcal{TF}$, i.e. $\mathbf{T}or(K, X) \neq 0$ for every non-torsion-free left module X.*

Set $\mathfrak{CP} := \oplus_{C_i\in\Gamma} C_i$, where Γ is a set of representatives for cyclically presented right R-modules. Clearly, $\mathfrak{CP}$ is an RD-flat module.

Proposition 2. *$\mathfrak{CP}$ is a tf-test module.*

Proof. Let $U \in R - Mod$. Assume that $\mathbf{T}or_1^R(\mathfrak{CP}, U) = 0$. Since $\mathbf{T}or_1^R(\mathfrak{CP}, U) \cong \oplus_{C_i\in\Gamma}\mathbf{T}or_1^R(C_i, U)$, $\mathbf{T}or_1^R(C_i, U) = 0$ for each $C_i \in \Gamma$. This means that U is torsion-free. □

By Lemma 1 and Proposition 2, we get:

Corollary 4. *Any pure extension of the module $\mathfrak{CP}$ is a tf- test module.*

By Proposition 2 and Lemma 2, we get:

Corollary 5. *If $wD(R) \leq 1$, then $E(\mathfrak{CP})$ is a tf- test module.*

Remark 1. *Let K be a finitely presented module and $F_0 \to F_1 \to K \to 0$ be a minimal free resolution of K. The transpose of K, denoted by $Tr(K)$, is defined as the cokernel of dual map $Hom_R(F_1, R) \to Hom_R(F_0, R)$. The isomorphism classes of $Tr(K)$ do not depend on our choice of the minimal resolution. $Tr(K)$ is a finitely presented left R-module. ([3, 16]).*

A module K is said to be U-subprojective if the map $Hom_R(K, P) \to Hom_R(K, U)$ is an epimorphism for every epimorphism $P \to U$. The family of all modules U such that K is U-subprojective is called the subprojectivity domain of K, and is denoted by $\underline{\mathfrak{Pr}}^{-1}(K)$ ([11]). [16, Theorem 8.3] presents a double-sided path between subprojectivity domain and subflat domain.

Corollary 6. *For a finitely presented module K, $\underline{\mathfrak{Pr}}^{-1}(K) = \mathfrak{F}^{-1}(Tr(K))$ and $\underline{\mathfrak{Pr}}^{-1}(Tr(K)) = \mathfrak{F}^{-1}(K)$.*

Corollary 7. *For a finitely presented module K, the following are hold.*

1. *K is RD-flat if and only if $Tr(K)$ is RD-projective module.*
2. *$Tr(K)$ is RD-flat if and only if K is RD-projective module.*

By Corollary 6 and Corollary 7, we have the following.

Corollary 8. *A finitely presented RD-flat module U is tf-test if and only if* $\mathcal{TF} = \underline{\mathfrak{Pr}}^{-1}(Tr(U))$.

Lemma 3. *If an RD-flat module U is tf-test, then $Hom_R(C,U) \neq 0$ for each nonprojective finitely presented RD-flat module C.*

Proof. Assume contrarily that $Hom_R(C,U) = 0$ for some nonprojective finitely presented RD-flat module C. Given a short exact sequence $0 \to F_0 \to F_1 \to U \to 0$ where F_1 is projective, we have $0 \to Hom_R(C,F_0) \to Hom_R(C,F_1) \to Hom_R(C,U) = 0$. Then, by [16, Theorem 8.3], $0 \to F_0 \otimes Tr(C) \to F_1 \otimes Tr(C) \to U \otimes Tr(C) \to 0$ is exact, and hence $\mathbf{T}or(U,Tr(C)) = 0$. Since U is tf-test, $Tr(C)$ is torsion-free. But $Tr(C)$ is RD-flat, and so it is flat by [13, Corollary 2.5]. Again by [16, Theorem 8.3], C is projective. This contradicts with our hypothesis. Therefore, $Hom_R(C,U) \neq 0$. □

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