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Research Article

An inverse source Cauchy-weighted time-fractional diffusion problem

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Abstract

In the present paper, we are concerned with an inverse source Cauchy weighted problem involving a one-dimensional diffusion equation with a time-fractional Riemann-Liouville derivative with $0 < \alpha < 1$. We start with results on the existence and regularity of the weak solution of the direct problem. Then, we investigate the invertibility of the input-output mapping defined by the additional over-determination integral data in order to the determination of the unknown time-dependent source coefficient.

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1. Introduction

Let us consider the following inverse source problem for the time-fractional diffusion equation

$$\partial^{\alpha}_{0+,t}u(x,t) - u_{xx}(x,t) = c(t)f(x); \quad (x,t) \in \Omega_T \tag{1.1}$$

along with the weighted initial condition

$$\lim_{t \longrightarrow 0+} t^{1-\alpha}u(x,t) = \varphi(x); \quad x \in [0,1] \tag{1.2}$$

the Dirichlet boundary conditions

$$u(0,t) = 0 = u(1,t); \quad t \in (0,T]; \tag{1.3}$$

and the overdetermination condition

$$\int_0^1 xu(x,t)dx = g(t); \quad t \in (0,T] \tag{1.4}$$

where $\Omega_T = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$; $\varphi(x)$, $f(x)$ and $g(t)$ are given functions while $c(t)$ is the unknown source term to be determined and $\partial^{\alpha}_{0+,t}$ stands for Riemann-Liouville time partial derivative of order $0 < \alpha < 1$.

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Our first objective is the theoretical study of the direct problem (1.1)-(1.3) by analyzing the weak solution. The second one is the uniquely recover of the time-dependant source term $c(t)$ from the additional data of integral type (1.4). To achieve this, we define an input-output mapping $G(c) = g$ on some admissible parameters space $\mathcal{H}$ via the overdetermination data which allows us to reduce the inverse source problem to the problem of the invertibility of the input-output mapping, see for example [2, 5].

Fractional differential equations become an important tool in modeling many real-life problems due to the nonlocal character of fractional order derivatives. They are used to describe memory and hereditary properties of various phenomena and processes in science and engineering such as mechanics, control and viscoelasticity. Also, fractional partial differential equations are a good alternative in some cases when the standard diffusion equations have a disagreement with experimental data due to non Gaussian diffusion and this in some important applications of the anomalous diffusion processes; to know more about see [7, 18]. In this context, we define a time-fractional diffusion equation as a parabolic-like partial differential equation with the partial time derivative of fractional order. It is called subdiffusion equation when $0 < \alpha < 1$. There are many works on the direct problems for subdiffusion equations such as an initial-boundary value problem [4, 10–12]. Likewise, studies of inverse problems of time-fractional partial differential equations have increased, [1–3, 8, 13, 14, 16]. Most of these works have dealt with the fractional time derivative in the Caputo sense and to the best of our knowledge, there are only a few works about inverse source subdiffusion problem involving Riemann-liouville derivative with initial weighted condition. The main reasons are the difficulty to give physical meaning to the related initial conditions and also to work in a suitable space. In [6], the autors gave a series of examples that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives. They attest that it does not require a direct experimental evaluation of these initial conditions.

The remainder of this paper is organized as following. In Section 2, we recall some preliminaries about fractional calculus. In Section 3, we obtain existence and regularity results for the unique weak solution of direct problem (1.1)-(1.3) using Fourier's method and Duhamel's principle in the fractional case (see [17]). In section 4, we investigate the inverse source problem by the obtained input-output mapping from the additional data $g(t)$.

2. Preliminaries

In the following, we recall some useful knowldge of fractional calculus which can be found in these books [9, 15].

Definition 2.1. The left-sided Riemann-Liouville fractional integral of order $0 < \alpha < 1$ of $f \in L^1(a, T)$ is defined by

$$I^{\alpha}_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Proposition 2.2. *For the Euler gamma function $\Gamma(z)$ the following hold:*

$$\Gamma(z+1) = z\Gamma(z); \quad \int_0^1 s^{z-1}(1-s)^{w-1} ds = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}; \; z, w \in \mathbb{R}^+.$$

Definition 2.3. The left-sided Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$D^{\alpha}_{a^+} f(t) := \frac{d}{dt} I^{1-\alpha}_{a^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds,$$

for all $f \in L^1(a,T)$ such that $I_{a+}^{1-\alpha} f \in W^{1,1}(a,T)$, a Sobolev space.

Proposition 2.4. *For* $0 < \alpha < 1$, $f \in L^1[a,T]$, $I_{a+}^{1-\alpha} f \in W^{1,1}(a,T)$,

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(t) = f(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a^+}^{1-\alpha} f\left(a^+\right); \; D_{a+}^{\alpha} I_{a+}^{\alpha} f(t) = f(t).$$

Lemma 2.5. *Let* $0 < \alpha < 1$,*the functions* $f \in L^1[0,T]$ *and* $K(t)$ *has a measurable derivative* $K'(t)$ *almost everywhere on* $[0,T]$, *then for any* $t \in [0,T]$,

$$D_{0+}^{\alpha} \int_0^t f(s) K(t-s)\, ds = \int_0^t f(t-s) D_{0+,s}^{\alpha} K(s)\, ds + f(t) \lim_{s \to 0+} I_{0^+,s}^{1-\alpha} K(s). \tag{2.1}$$

Now, let us define $C_\gamma[a,T]$, $0 \leq \gamma < 1$ the weighted space of defined functions on $(a,T]$ such that $(t-a)^\gamma f \in C[a,T]$ which is a Banach space with the norm $\|u\|_{C_\gamma[a,T]} = \|(t-a)^\gamma u\|_{C[a,T]}$. Then, we introduce the Mittag-Leffler functions.

Definition 2.6. A two-parameter Mittag-Leffler function is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}; \; \alpha, \beta, z \in \mathbb{C} \text{ with } \Re\alpha > 0. \tag{2.2}$$

In particular, for $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$ and for $\alpha = \beta = 1$; $E_1(z) = e^z$.

Lemma 2.7. *The following properties hold on* $[0,T]$ *for* $\alpha > 0, \beta > 0$, $\lambda \in \mathbb{R}^+$:

(1) $E_{\alpha,\beta}(-\lambda t^\alpha)$ *is bounded positive completely monotonic function*;

(2) $\lim_{t \to 0^+} E_{\alpha,\beta}(-\lambda t^\alpha) = \dfrac{1}{\Gamma(\beta)}$ *and*

$$I_{0+}^{\alpha}\left(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right) = t^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(-\lambda t^\alpha).$$

Definition 2.8. We define the α-Exponential function by

$$e_\alpha(\lambda, z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha), \alpha, \lambda \in \mathbb{C}, z \in \mathbb{C} \backslash \{0\} \text{ with } \Re\alpha > 0.$$

Lemma 2.9. $e_\alpha(\lambda,t)$ *is bounded positive completely monotonic function and satisfies*

$$\begin{aligned} \int_0^t e_\alpha(\lambda, s)\, ds &< \infty; \\ D_{0+}^{\alpha}(\, e_\alpha(\lambda,t)) &= \lambda\, e_\alpha(\lambda,t), \text{ for } 0 < \alpha < 1. \end{aligned}$$

Theorem 2.10. *For a sequence of functions* $(f_i(t))_{i \geq 0}$ *defined on* $(0,T]$, *suppose the following conditions are fulfilled:*

(i) For a given $\alpha > 0$ *the* α- *derivatives* $D_{0+}^{\alpha} f_i(t), i \geq 0$; $t \in (0,T]$ *exists.*

(ii) $\sum_{i=1}^\infty f_i(t)$ *and* $\sum_{i=1}^\infty D_{0+}^{\alpha} f_i(t)$ *are uniformly convergent on the interval* $[\varepsilon, T]$ *for any* $\varepsilon > 0$.

Then the function defined by the series $\sum_{i=1}^\infty f_i(t)$ *is* $\alpha-$ *differentiable and satisfies*

$$D_{0+}^{\alpha} \sum_{i=1}^{\infty} f_i(t) = \sum_{i=1}^{\infty} D_{0+}^{\alpha} f_i(t).$$

Theorem 2.11. *The fractional problem of weighted Cauchy type*

$$\left\{ \begin{array}{c} D_{a+}^{\alpha} u(t) = \lambda u(t), \; 0 < \alpha < 1, \; t \in (a,T], \; \lambda \in \mathbb{R}^*, \\ \lim\limits_{t \longrightarrow a+} (t-a)^{1-\alpha} u(t) = c, \end{array} \right. \tag{2.3}$$

has a unique solution $u \in C_{1-\alpha}[a,T]$ *given by* $u(t) = c\Gamma(\alpha)\, e_\alpha(\lambda, t-a)$.

3. Fractional problem of weighted Cauchy type

A crucial step in our approch is to establish the spectral representation of the solution. Then, we etablish existence, uniqueness and some regularity results for the weak solution of (1.1)-(1.3). Precisely, under the following assumptions

(A1) $c \in C_{1-\alpha}[0,T] \cap L^1(0,T)$ is positive and $c(t) \neq 0$ for each $t \in (0,T]$.

(A2) $f \in H^2(0,1) \cap H_0^1(0,1)$ with $f'(0) = f'(1)$.

(A3) $\varphi \in H^2(0,1) \cap H_0^1(0,1)$ with $\varphi'(0) = \varphi'(1)$,

$H^2(0,1)$ and $H_0^1(0,1)$ denote known Sobolev spaces.

The formal solution of initial boundary value problem (1.1)-(1.3) is written in the form of a Fourier series

$$u(x,t) = \sum_{n\geq 0} u_n(t) X_n(x); \quad (x,t) \in \overline{\Omega}_T \tag{3.1}$$

where $X_n(x) = \sin n\pi x; n \geq 1$ are the eigenfunctions corresponding to the eigenvalues $\lambda_n = (n\pi)^2, n \geq 1$ of the spectral problem

$$\begin{cases} -X"(x) = \lambda X(x); \quad x \in (0,1) \\ X(0) = 0 = X(1). \end{cases} \tag{3.2}$$

$(X_n(x))_{n\geq 1}$ forms a basis for the Hilbert space $L^2(0,1)$.

Definition 3.1. A function $u(.,t) \in H_0^1(0,1)$ is called a weak solution of problem (1.1)-(1.3) if it satisfies the following problem

$$\begin{cases} \int_0^1 \left(\partial_{0+,t}^{\alpha} u(x,t)\phi(x) + u_x(x,t)\phi_x(x) - c(t)f(x)\phi(x)\right) dx = 0, \text{ for any } t \in (0,T]; \\ \lim_{t\longrightarrow 0+} t^{1-\alpha} \int_0^1 (u(x,t) - \varphi(x))\phi(x)\,dx = 0; \end{cases} \tag{3.3}$$

for any function $\phi \in H_0^1(0,1)$ and for $\varphi \in H_0^1(0,1)$.

Remark 3.2. Problem (3.3) is a weak formulation of the direct problem (1.1)-(1.3).

Theorem 3.3. *Let assumptions (A1)-(A3) be satisfied. The time-fractional diffusion problem (1.1)-(1.3) has a unique weak solution given by (3.1) where the coefficients* $u_n(t), n \geq 1$ *are given in* $C_{1-\alpha}[0,T]$ *by*

$$u_n(t) = \varphi_n \Gamma(\alpha) e_\alpha(-\lambda_n, t) + f_n \int_0^t c(s) e_\alpha(-\lambda_n, t-s) ds; \tag{3.4}$$

with

$$\varphi_n = 2\int_0^1 \varphi(x) X_n(x)\,dx, n \geq 1,$$

$$f_n = 2\int_0^1 f(x) X_n(x)\,dx, n \geq 1.$$

Proof. Let us put

$$u(x,t) = v(x,t) + w(x,t) \tag{3.5}$$

where $w(x,t)$ satisfies

$$\begin{cases} \partial^{\alpha}_{0+,t} w(x,t) - w_{xx}(x,t) = 0; \quad (x,t) \in (0,1)\times(0,T] \\ \lim\limits_{t\longrightarrow 0+} t^{1-\alpha} w(x,t) = \varphi(x);\ x \in [0,1] \\ w(0,t) = 0 = w(1,t); \quad t \in (0,T] \end{cases} \tag{3.6}$$

and $v(x,t)$ satisfies

$$\begin{cases} \partial^{\alpha}_{0+,t} v(x,t) - v_{xx}(x,t) = c(t)f(x); \quad (x,t) \in (0,1)\times(0,T] \\ \lim\limits_{t\longrightarrow 0+} t^{1-\alpha} v(x,t) = 0;\quad x \in [0,1] \\ v(0,t) = 0 = v(1,t); \qquad t \in (0,T]. \end{cases} \tag{3.7}$$

Then, we get $w(x,t) = \sum_{n=1}^{\infty} w_n(t) X_n(x)$, where

$$w_n(t) = 2\int_0^1 w(x,t) X_n(x)\,dx, n \geq 1$$

the solutions of weighted Cauchy fractional problems

$$\begin{cases} D^{\alpha}_{0+} w_n(t) = -\lambda_n w_n(t), \quad t \in (0,T]\,; n \geq 1 \\ \lim\limits_{t\longrightarrow 0+} t^{1-\alpha} w_n(t) = \varphi_n, \end{cases} \tag{3.8}$$

are given in $C_{1-\alpha}[0,T]$ by

$$w_n(t) = \varphi_n \Gamma(\alpha) e_{\alpha}(-\lambda_n, t)\,;\ t \in (0,T]\,;\ n \geq 1. \tag{3.9}$$

Therefore, the solution of (3.6) has the series form

$$w(x,t) = \sum_{n\geq 1} \varphi_n \Gamma(\alpha) e_{\alpha}(-\lambda_n, t) X_n(x). \tag{3.10}$$

Next, We put $v(x,t) = \int_0^t V(x,t,s)\,ds$ in (3.7) and in view of Duhamel's principle we obtain the solution of (3.7) which has the following form $v(x,t) = \sum_{n\geq 1} v_n(t) X_n(x)$. By Lemma 2.5 we etablish that $V(x,t,s)$ is the solution of

$$\begin{cases} \partial^{\alpha}_{s+,t} V(x,t,s) - V_{xx}(x,t,s) = 0; \quad x \in (0,1);\ 0 < s < t \leq T \\ \lim\limits_{t\to s^{+}} (t-s)^{1-\alpha} V(x,t,s) = \dfrac{f(x)c(s)}{\Gamma(\alpha)}; \quad x \in [0,1] \\ V(0,t,s) = 0 = V(1,t,s); \quad 0 < s < t \leq T. \end{cases} \tag{3.11}$$

As previously, we get

$$V(x,t,s) = \sum_{n\geq 1} V_n(t,s) X_n(x) = \sum_{n\geq 1} f_n c(s) e_{\alpha}(-\lambda_n, t-s) X_n(x), \tag{3.12}$$

where $V_n(t,s) = 2\int_0^1 V(x,t,s) X_n(x)\,dx, n \geq 1$ are solutions of fractional differential problems of weighted Cauchy type

$$\begin{cases} D^{\alpha}_{s+} V_n(t,s) = -\lambda_n V_n(t,s), \quad t \in (s,T]\,; n \geq 1 \\ \lim\limits_{t\longrightarrow s+} (t-s)^{1-\alpha} V_n(t,s) = \dfrac{f_n c(s)}{\Gamma(\alpha)}, \end{cases} \tag{3.13}$$

given in $C_{1-\alpha}[s,T]$ by

$$V_n(t,s) = f_n c(s) e_{\alpha}(-\lambda_n, t-s)\,;\ t \in (s,T]\,;\ n \geq 1. \tag{3.14}$$

Consequently, $v_n(t), n \geq 1$ are in $C_{1-\alpha}[0,T]$ given by

$$v_n(t) = \int_0^t V_n(t,s)\,ds = f_n \int_0^t c(s)\, e_\alpha(-\lambda_n, t-s)\,ds, n \geq 1$$

and

$$v(x,t) = \sum_{n\geq 1} f_n \int_0^t c(s)\, e_\alpha(-\lambda_n, t-s)\,ds X_n(x). \tag{3.15}$$

Then, the sum of (3.10) and (3.15) gives the spectral representation of the solution of problem (1.1)- (1.3) which is of the form (3.1)-(3.4). Finally, the uniqueness of the spectral representation (3.1)- (3.4) of the weak solution $u(x,t)$ is obtained from the uniqueness of the solutions of (3.7) and (3.13) in view of Theorem 2.11 and (A1)-(A3). This completes the proof. □

In the following, we are interseted in the regularity properties of $u(x,t)$ for a fixed coefficient source $c(t)$ in $C_{1-\alpha}[0,T]$. Let us denote,

$$M := \sup_{n\geq 1} \sup_{0\leq s<t\leq T} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha); \tag{3.16}$$

$$Q_1 = \Gamma(\alpha) M; \;\; Q_2 = T^\alpha \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} M; \tag{3.17}$$

$$Q_3 = 2Q_1^2 \frac{T^{2\alpha-1}}{2\alpha-1}; \;\; Q_4 = 4\frac{T^{2\alpha-1}}{4\alpha-1}\left(Q_2^2+1\right). \tag{3.18}$$

Theorem 3.4. *The unique weak solution of (1.1)-(1.3) is in $C_{1-\alpha}([0,T]; L^2(0,1))$ and satisfies the following estimate*

$$\|u\|_{C_{1-\alpha}([0,T];L^2(0,1))} \leq Q_1 \|\varphi\|_{L^2(0,1)} + Q_2 \|c\|_{C_{1-\alpha}[0,T]} \|f\|_{L^2(0,1)}, \tag{3.19}$$

for some positive constants Q_1 and Q_2 given by (3.17).

Moreover, we have $\lim_{t\longrightarrow 0+} \|t^{1-\alpha} u(.,t) - \varphi\|_{L^2(0,1)} = 0.$

Remark 3.5. Note that $u(x,t)$ is in $C((0,T]; L^2(0,1))$ and has a singularity at $t=0$ and this justifies the use of weighted space.

Proof. It's known that the L^2-norm of $w(x,t)$ with respect to x, satisfies the parseval identity for each $t \in (0,T]$,

$$\|w(.,t)\|^2_{L^2(0,1)} = \frac{1}{2}\sum_{n\geq 1} |w_n(t)|^2.$$

We conclude that, for $t \in (0,T]$,

$$\begin{aligned} \|w(.,t)\|^2_{L^2(0,1)} &\leq \frac{\Gamma^2(\alpha)}{2} \sum_{n\geq 1} \varphi_n^2 \left[e_\alpha(-\lambda_n, t)\right]^2 \\ &\leq \frac{\Gamma^2(\alpha)}{2} \left[M t^{\alpha-1}\right]^2 \sum_{n\geq 1} \varphi_n^2 \leq t^{2(\alpha-1)} \Gamma^2(\alpha) M^2 \|\varphi\|^2_{L^2\,(0,1)}; \end{aligned}$$

where M is given by (3.16). In the same way, we get from (3.15) for each $t \in (0,T]$

$$\begin{aligned}
\|v(.,t)\|^2_{L^2(0,1)} &= \frac{1}{2}\sum_{n\geq 1}|v_n(t)|^2 \\
&\leq \frac{M^2}{2}\|c\|^2_{C_{1-\alpha}[0,T]}\sum_{n\geq 1}|f_n|^2\left[\int_0^t s^{\alpha-1}(t-s)^{\alpha-1}ds\right]^2 \\
&\leq \left(t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\right)^2\|c\|^2_{C_{1-\alpha}[0,T]}M^2\|f\|^2_{L^2(0,1)}.
\end{aligned}$$

In view of the L^2-norm of (3.5), we get for each $t \in (0,T]$,

$$\|u(.,t)\|_{L^2\ [0,1]} \leq t^{\alpha-1}\Gamma(\alpha)M\|\varphi\|_{L^2(0,1)} + t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\|c\|_{C_{1-\alpha}[0,T]}M\|f\|_{L^2\ (0,1)} \tag{3.20}$$

which leads to the result (3.19).

Now, for the initial condition, we have

$$\begin{aligned}
\left\|t^{1-\alpha}u(.,t)-\varphi\right\|^2_{L^2(0,1)} &= \frac{1}{2}\sum_{n\geq 1}\left|t^{1-\alpha}u_n(t)-\varphi_n\right|^2 \\
&\leq \sum_{n\geq 1}\left|\varphi_n\left(t^{1-\alpha}\Gamma(\alpha)e_\alpha(-\lambda_n,t)-1\right)\right|^2 \\
&\quad+\sum_{n\geq 1}\left|t^{1-\alpha}f_n\int_0^t c(s)e_\alpha(-\lambda_n,t-s)ds\right|^2.
\end{aligned}$$

From the defintion of the $\alpha-$exponential Mittag-Leffler function and its I^α_{0+} integral, we obtain

$$\begin{aligned}
\left\|t^{1-\alpha}u(.,t)-\varphi\right\|^2_{L^2(0,1)} &\leq \sum_{n\geq 1}|\varphi_n(\Gamma(\alpha)E_{\alpha,\alpha}(-\lambda_n t^\alpha)-1)|^2 \\
&\quad+\sum_{n\geq 1}\left[t^{1-\alpha}f_n\|c\|_{C_{1-\alpha}[0,T]}t^{2\alpha-1}E_{\alpha,2\alpha}(-\lambda_n t^\alpha)\right]^2 \\
&\leq \sum_{n\geq 1}|\varphi_n(\Gamma(\alpha)E_{\alpha,\alpha}(-\lambda_n t^\alpha)-1)|^2 \\
&\quad+t^{2\alpha}\|c\|^2_{C_{1-\alpha}[0,T]}\sum_{n\geq 1}[f_nE_{\alpha,2\alpha}(-\lambda_n t^\alpha)]^2.
\end{aligned}$$

By (A2)-(A3) and Lemma (2.7), applying the limit, we get

$$\lim_{t\longrightarrow 0+}\left\|t^{1-\alpha}u(.,t)-\varphi\right\|_{L^2(0,1)} = 0$$

which completes the proof. □

Remark 3.6. Remark that in (3.20) we have singularities when $t = 0$ and when $0 < \alpha < 1/2$ .This is overcome by using the weighted norm.

Next, we will give results concerning more regularity of the solution.

Theorem 3.7. *The unique weak solution $u(x,t)$ of (1.1)-(1.3) is in $C_{1-\alpha}([0,T];H^1_0(0,1)\cap H^2(0,1))$ such that $\partial^\alpha_{0+,t}u \in C_{1-\alpha}([0,T];L^2(0,1))$ and satisfies the following estimates*

$$\|u\|_{C_{1-\alpha}([0,T];H^1_0(0,1))} \leq Q_1\|\varphi\|_{H^1_0(0,1)} + Q_2\|c\|_{C_{1-\alpha}[0,T]}\|f\|_{H^1_0(0,1)} \tag{3.21}$$

and

$$\|u\|_{C_{1-\alpha}([0,T];H_0^1(0,1)\cap H^2(0,1))}+\left\|\partial_{0+,t}^{\alpha}u\right\|_{C_{1-\alpha}([0,T];L^2(0,1))}\leq 2Q_1\|\varphi\|_{H^2(0,1)} \quad (3.22)$$
$$+(2Q_2+1)\|c\|_{C_{1-\alpha}[0,T]}\|f\|_{H^2(0,1)},$$

for Q_1, Q_2 given by (3.17)

Moreover, $\partial_{0+,t}^{\alpha}u\in L^2((0,T)\times(0,1))$ for $\frac{1}{2}<\alpha<1$ and satisfies

$$\left\|\partial_{0+,t}^{\alpha}u\right\|^2_{L^2((0,T)\times(0,1))}\leq Q_3\|\varphi\|^2_{L^2(0,1)}+Q_4\|c\|^2_{C_{1-\alpha}[0,T]}\|f\|_{H^2(0,1)}, \quad (3.23)$$

for some positive constants Q_3 , Q_4 given by (3.18).

Proof. Using the spectral form (3.1)-(3.4), we obtain the partial derivative $u_x(x,t)=w_x(x,t)+v_x(x,t)$. Then, under (A3), we get

$$\begin{aligned}\|w_x(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\geq1}\lambda_n w_n^2(t)\\ &\leq \frac{\Gamma^2(\alpha)}{2}\sum_{n\geq1}|\varphi_n|^2\lambda_n\left[e_\alpha(-\lambda_n,t)\right]^2\\ &\leq \frac{M^2\Gamma^2(\alpha)t^{2(\alpha-1)}}{2}\sum_{n\geq1}\lambda_n(\varphi_n)^2\\ &\leq t^{2(\alpha-1)}M^2\Gamma^2(\alpha)\|\varphi'\|^2_{L^2(0,1)};\end{aligned}$$

where M is given by (3.16). Also, we have

$$\begin{aligned}\|v_x(.,t)\|^2_{L^2\ (0,1)} &\leq \frac{1}{2}\sum_{n\geq1}\lambda_n|f_n|^2\|c\|^2_{C_{1-\alpha}[0,T]}\\ &\quad\times\left[\int_0^t s^{\alpha-1}e_\alpha(-\lambda_n,t-s)\,ds\right]^2\\ &\leq \left(t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\right)^2\|c\|^2_{C_{1-\alpha}[0,T]}M^2\|f'\|^2_{L^2(0,1)}.\end{aligned}$$

Recall that the norm of $H_0^1\ (0,1)$ is defined by

$$\|u\|_{H_0^1(0,1)}=\|u_x\|_{L^2(0,1)}.$$

Then, we obtain

$$\|u\|_{C_{1-\alpha}([0,T];H_0^1(0,1))}\leq M\Gamma(\alpha)\|\varphi'\|_{L^2(0,1)}+MT^{\alpha}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\|c\|_{C_{1-\alpha}[0,T]}\|f'\|_{L^2(0,1)}. \quad (3.24)$$

Thus, $u\in C_{1-\alpha}([0,T];H_0^1(0,1))$ and we get estimate (3.21).

Following the same argument, we get

$$\begin{aligned}\|w_{xx}(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\geq1}\lambda_n^2 w_n^2(t)\leq t^{2(\alpha-1)}M^2\Gamma^2(\alpha)\|\varphi"\|^2_{L^2(0,1)};\\ \|v_{xx}(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\geq1}\lambda_n^2 v_n^2(t)\leq\left(t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\right)^2\|c\|^2_{C_{1-\alpha}[0,T]}M^2\|f"\|^2_{L^2(0,1)}.\end{aligned}$$

Hence,

$$\|u_{xx}(.,t)\|_{L^2\ (0,1)}\leq t^{\alpha-1}M\Gamma(\alpha)\|\varphi"\|_{L^2(0,1)}+t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}M\|c\|_{C_{1-\alpha}[0,T]}\|f"\|^2_{L^2(0,1)}$$

which implies that $u(.,t) \in H^2(0,1)$ for $t \in (0,T]$ and we get

$$\|u_{xx}\|_{C_{1-\alpha}([0,T];L^2\ (0,1))} \leq M\Gamma(\alpha)\|\varphi"\|_{L^2(0,1)} + T^\alpha \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} M \|c\|_{C_{1-\alpha}[0,T]} \|f"\|^2_{L^2(0,1)} . \tag{3.25}$$

The $\alpha-$derivative of $u_n(t)$, $n \geq 1$ exist for $t \in (0,T]$ and satisfy

$$\begin{aligned} D^\alpha_{0+} u_n(t) &= \varphi_n \Gamma(\alpha) D^\alpha_{0+,t}\ e_\alpha(-\lambda_n, t) + \int_0^t f_n c(t-s) D^\alpha_{0+,s}\ e_\alpha(\lambda_n, s)\, ds \\ &\quad + f_n c(t) \lim_{s\to 0+} I^{1-\alpha}_{0+,s} e_\alpha(-\lambda_n, s) \\ &= -\varphi_n \Gamma(\alpha)\lambda_n\ e_\alpha(-\lambda_n, t) - \int_0^t f_n c(t-s)\lambda_n\ e_\alpha(\lambda_n, s)\, ds \\ &\quad + f_n c(t) \lim_{s\to 0+} E_{\alpha,1}(-\lambda_n s^\alpha). \end{aligned}$$

Consequently, we get

$$\begin{aligned} &\left|\sum_{n\geq 1} D^\alpha_{0+} u_n(t) X_n(x)\right| \\ \leq\ & t^{(\alpha-1)} M\Gamma(\alpha) \left|-\sum_{n\geq 1} \lambda_n \varphi_n X_n(x)\right| \\ & + \left|-\sum_{n\geq 1}\lambda_n f_n X_n(x)\right| M \int_0^t s^{\alpha-1} c(t-s)\, ds + c(t)\left|\sum_{n\geq 1} f_n X_n(x)\right| \\ \leq\ & t^{(\alpha-1)} M \left|\sum_{n\geq 1}\varphi"_n X_n(x)\right| \Gamma(\alpha) \\ & + t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} M \|c\|_{C_{1-\alpha}[0,T]} \left|\sum_{n\geq 1} f"_n X_n(x)\right| + t^{\alpha-1}\|c\|_{C_{1-\alpha}[0,T]} |f(x)| . \end{aligned}$$

Thus, the series $\sum_{n\geq 1} D^\alpha_{0+} u_n(t) X_n(x)$ is uniformly convergent for $t \in [\epsilon, T]$, $\epsilon > 0$ and in view of Theorem 2.10

$$\sum_{n\geq 1} D^\alpha_{0+} u_n(t) X_n(x) = D^\alpha_{0+} \sum_{n\geq 1} u_n(t) X_n(x) .$$

From equation (1.1), we have for $t \in (0,T]$

$$\left\|\partial^\alpha_{0+,t} u(.,t)\right\|_{L^2\ (0,1)} \leq \|u_{xx}(.,t)\|_{L^2(0,1)} + |c(t)|\, \|f\|_{L^2(0,1)} . \tag{3.26}$$

In view of the approximation of $u_{xx}(x,t)$ we get

$$\begin{aligned} \left\|\partial^\alpha_{0+,t} u(.,t)\right\|_{L^2\ (0,1)} \leq\ & t^{(\alpha-1)} M\Gamma(\alpha)\|\varphi"\|_{L^2(0,1)} \\ & + t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} M\|c\|_{C_{1-\alpha}[0,T]}\|f"\|_{L^2(0,1)} + |c(t)|\,\|f\|_{L^2(0,1)} , \end{aligned} \tag{3.27}$$

which yields

$$\left\|\partial^\alpha_{0+,t} u\right\|_{C_{1-\alpha}([0,T];L^2(0,1))} \leq Q_1\|\varphi"\|_{L^2(0,1)} + \left(\|f\|_{L^2(0,1)} + Q_2\|f"\|_{L^2(0,1)}\right)\|c\|_{C_{1-\alpha}[0,T]} . \tag{3.28}$$

Then, between (3.19), (3.24), (3.25), (3.28), we obtain

$$\begin{aligned}
&\|u\|_{C_{1-\alpha}([0,T];H_0^1(0,1)\cap H^2(0,1))} + \left\|\partial_{0+,t}^{\alpha} u\right\|_{C_{1-\alpha}([0,T];L^2(0,1))} \\
\leq\ & Q_1 \left(\|\varphi\|_{L^2(0,1)} + 2\|\varphi'\|_{L^2(0,1)} + 2\|\varphi"\|_{L^2(0,1)}\right) \\
&+Q_2 \|c\|_{C_{1-\alpha}[0,T]} \left(\|f\|_{L^2(0,1)} + 2\|f'\|_{L^2(0,1)} + 2\|f"\|_{L^2(0,1)}\right) \\
&+\|c\|_{C_{1-\alpha}[0,T]} \|f\|_{L^2(0,1)}.
\end{aligned}$$

Recall that the norm of $H^2\ (0,1)$ is

$$\|u\|_{H^2(0,1)} = \left(\|u\|^2_{L^2(0,1)} + \|u_x\|^2_{L^2(0,1)} + \|u_{xx}\|^2_{L^2(0,1)}\right)^{1/2};$$

then we conclude (3.22). Next, from (3.27), we get

$$\begin{aligned}
\int_0^T \left\|\partial_{0+,t}^{\alpha} u(.,t)\right\|^2_{L^2\ (0,1)} dt \ \leq\ & 2M^2\Gamma^2(\alpha)\frac{T^{2\alpha-1}}{2\alpha-1}\|\varphi\|^2_{H^2(0,1)} \\
&+4\frac{T^{4\alpha-1}}{4\alpha-1}\frac{\Gamma^4(\alpha)}{\Gamma^2(2\alpha)}M^2\|c\|^2_{C_{1-\alpha}[0,T]}\|f\|^2_{H^2(0,1)} \\
&+4\frac{T^{2\alpha-1}}{2\alpha-1}\|c\|^2_{C_{1-\alpha}[0,T]}\|f\|^2_{L^2(0,1)}.
\end{aligned}$$

We deduce that $\partial_{0+,t}^{\alpha} u \in L^2((0,T)\times(0,1))$ for $\alpha \geq \frac{1}{2}$and satisfies (3.23). □

4. Inverse source subdiffusion problem

In this section, we consider the inverse problem of finding the time dependent source term $c(t)$ of the subdiffusion equation (1.1) from the given data (1.4). We will study the monotonicity and distinguishability of the input-output mapping.

In addition to the assumptions (A1)-(A3), we assume the following :

(A4) $g \in C_{1-\alpha}[0,T] \cap L^1(0,T); D_{0+}^{\alpha} g \in C_{1-\alpha}[0,T] \cap L^1(0,T)$ and $\lim\limits_{t\to 0^+} t^{1-\alpha} g(t) = \int_0^1 x\varphi(x)dx$.

(A5) $(-1)^n f_n \geq 0; n \geq 1$ and $\int\limits_0^1 xf(x)dx = F \neq 0$.

(A6) $\varphi_n \geq 0; n \geq 1$ and $\int\limits_0^1 x\varphi(x)dx \neq 0$.

When multiplying (3.1) by x and integrating over $[0,1]$, $g(t)$ can be determined by an analytical series , for $t \in (0,T]$,

$$\begin{aligned}
g(t) \ =\ & \sum_{n\geq1} \frac{(-1)^{n+1}}{\sqrt{\lambda_n}} \varphi_n \Gamma(\alpha) e_\alpha(-\lambda_n, t) \\
&+\sum_{n\geq1} \frac{(-1)^{n+1}}{\sqrt{\lambda_n}} f_n \int\limits_0^t c(s)\, e_\alpha(-\lambda_n, t-s)\, ds,
\end{aligned} \tag{4.1}$$

Indeed, in view of (A1)-(A3) with $\lambda_n \geq \pi^2$, (A4) is satisfied and

$$\begin{aligned}
\|g\|_{C_{1-\alpha}[0,T]} \ \leq\ & \frac{M\Gamma(\alpha)}{\pi}\left[\sum_{n\geq1}|\varphi_n| + \|c\|_{C_{1-\alpha}[0,T]}\frac{T^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\sum_{n\geq1}|f_n|\right] \\
<\ & \infty
\end{aligned}$$

Next, we define the set of admissible time-dependent source terms $c(t)$ by

$$\mathcal{H} = \{c \in C_{1-\alpha}[0,T] :\ 0 < C_0 \leq c(t) \leq C_1, t \in (0,T]\ \} \subset C_{1-\alpha}[0,T]$$

and by $\mathcal{G} \subset C_{1-\alpha}[0,T] \cap L_1(0,T)$ the set of measured free noisy output data $g(t)$. Hence, we introduce the input-output mapping $G(.) : \mathcal{H} \to \mathcal{G}$ as follows :

$$G(c) = w_\varphi + v_c = g, \tag{4.2}$$

where w_φ and v_c are the part of (4.1) depending of φ and c respectively given by $t \in (0,T]$

$$w_\varphi(t) = \int_0^1 xw(x,t)\,dx = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} \varphi_n \Gamma(\alpha) e_\alpha(-\lambda_n, t)\,; \tag{4.3}$$

and

$$v_c(t) = \int_0^1 xv(x,t)\,dx = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} f_n \int_0^t c(s)\, e_\alpha(-\lambda_n, t-s)\,ds. \tag{4.4}$$

Remark 4.1. We can note $G(c) = w_\varphi + \Phi(c)$ where Φ is a linear well defined mapping on $\mathcal{H}$ and w_φ is constant with respect to c.

As a result, the inverse problem of determination of the time-source term $c(t)$ in (1.1)-(1.4) is reduced to the problem of invertibility of G.

Note that $w(x,t)$ the solution of (3.6) does not depend on the source term $c(t)$, but it influences its uniqueness compared to the initial data. So, we give a uniqueness result concerning w_φ related on the input data φ.

Lemma 4.2. *Let $w_1(x,t)$, $w_2(x,t)$ be the solutions of the direct problem (3.6) related to the initial conditions $\varphi(x)$, $\psi(x)$ and w_φ , w_ψ be the part of* (1.2) *related to the input data $\varphi(x)$, $\psi(x)$ respectively given by (4.3) where $\varphi(x)$, $\psi(x)$ satisfy assumptions (A3) and (A6). If*

$$w_\varphi(t) = w_\psi(t),\ \ 0 < t \leq T,$$

then,

$$\varphi(x) = \psi(x) \tag{4.5}$$

almost everywhere on $[0,1]$.

Proof. $w_1,\ w_2 \in C_{1-\alpha}([0,T]\,; L^2[0,1])$ *the solutions of the direct problem (3.6) related to the initial conditions* $\varphi(x)$ and $\psi(x)$ are *given by (3.10)*, respectively. It is clear that $w_\varphi(t), w_\psi(t)$ *are in* $C_{1-\alpha}[0,T]$. Let us suppose that $w_\varphi(t) = w_\psi(t),\ 0 < t \leq T$ and

$$\begin{aligned} \varphi(x) &> \psi(x)\,; \quad x \in (x_i, x_{i+1}), i = 1, ..., m \\ \varphi(x) &= \psi(x)\,;\ \text{otherwise,} \end{aligned}$$

for some $x_i \in [0,1], i = 1, ..., m+1$. Then, *we get*

$$\begin{aligned} 0 &< \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} x\,[\varphi(x) - \psi(x)]\,dx = \int_0^1 x\,[\varphi(x) - \psi(x)]\,dx \\ &\leq \int_0^1 x \sum_{n\geq 1} [\varphi_n - \psi_n]\, X_n(x)\,dx. \end{aligned}$$

Thus , by the fact that

$$\lim_{t\to 0^+} t^{1-\alpha} t^{\alpha-1} \Gamma(\alpha)\, E_{\alpha,\alpha}(-\lambda_n t^\alpha) = 1, \tag{4.6}$$

we get

$$\begin{aligned} 0 &< \int_0^1 x \sum_{n\geq 1} [\varphi_n - \psi_n] \lim_{t\to 0^+} t^{1-\alpha} t^{\alpha-1} \Gamma(\alpha) E_{\alpha,\alpha}(-\lambda_n t^\alpha) X_n(x)\, dx \\ &\leq \sup_{t\in[0,T]} t^{1-\alpha} \int_0^1 x\ |w_\varphi(x,t) - w_\psi(x,t)|\, dx = \|w_\varphi - w_\psi\|_{C_{1-\alpha}[0,T]} = 0. \end{aligned}$$

This gives us a contradiction. Then, $\varphi(x) = \psi(x)$ almost everywhere on $[0,1]$. □

Let $u(x,t)$, $\widetilde{u}(x,t)$ be two solutions of the direct problem (1.1)-(1.3) and $g(t)$, $\widetilde{g}(t)$ be the overdetermination data corresponding to the admissible time-source terms c, $\widetilde{c} \in \mathcal{H}$ respectively. Now, to ensure the existence of the solution of the inverse problem let's show the monotonicity of G.

Theorem 4.3. *Let assumptions (A1)-(A6) hold and $u(x,t,c)$ be a unique solution of the direct problem (1.1)-(1.3). If the admissible coefficients c, $\widetilde{c} \in \mathcal{H}$ satisfy the condition $< c(t) < \widetilde{c}(t)$; $t \in (0,T]$ then, the input-output mapping satisfies $G(c(t)) < G(\widetilde{c}(t))$; $t \in (0,T]$.*

Proof. Given c, $\widetilde{c} \in \mathcal{H}$ such that $0 < \widetilde{c}(t) < c(t)$ for each $t \in (0,T]$ and set $c(t) - \widetilde{c}(t) = \Delta c(t)$ then $z(x,t) = u(x,t,c) - \widetilde{u}(x,t,\widetilde{c})$ is a solution of the direct problem

$$\begin{cases} \partial^\alpha_{0+,t} z(x,t) - z_{xx} = \Delta c(t) f(x); & (x,t) \in \Omega(0,1) \times (0,T] \\ \lim\limits_{t\longrightarrow 0+} t^{1-\alpha} z(x,t) = 0; \quad x \in [0,1] \\ z(0,t) = 0 = z(1,t); & t \in (0,T]. \end{cases} \tag{4.7}$$

In view of (3.1)-(3.4), z is given by

$$z(x,t) = \sum_{n\geq 1} \int_0^t \Delta c(s) e_\alpha(-\lambda_n, t-s) ds f_n X_n(x). \tag{4.8}$$

Then, the corresponding input data $g(t) - \widetilde{g}(t) = \Delta g(t)$; $t \in (0,T]$, is given by

$$\Delta g(t) = \int_0^1 x z(x,t)\, dx \tag{4.9}$$

and satisfies for $t \in (0,T]$,

$$\begin{aligned} \Delta g(t) &= \int_0^t \Delta c(s) \sum_{n\geq 1} e_\alpha(-\lambda_n, t-s) f_n \int_0^1 x X_n(x)\, dx ds \\ &= \int_0^t \Delta c(s) \sum_{n\geq 1} e_\alpha(-\lambda_n, t-s) (-1)^n f_n ds > 0 \end{aligned}$$

This implies the monotonicity of G. The proof is complete. □

Next, we will study the distinguishability of the unknown function $c(t)$ via the input-output mapping G in the sense that $G(c) \neq G(\widetilde{c})$ implies $c \neq \widetilde{c}$ and this means the injectivity of its inverse G^{-1}. This yields to identify $c(t)$ uniquely

Theorem 4.4. *Assume that assumptions (A1)-(A6) hold. Then, the input-output mapping $G(c)$ corresponding to the additional data (1.4), has the distinguishability property in the class of admissible source parameters $\mathcal{H}$.*

Proof. Let $c, \widetilde{c} \in \mathcal{H}$ such that $c(t) - \widetilde{c}(t) = \Delta c(t) \neq 0$ then $z(x,t) = u(x,t,c) - \widetilde{u}(x,t,\widetilde{c})$ is a solution of the direct problem (4.7) and z is given by (4.8). Then, for the corresponding input data $\Delta g(t)$, we get for $t \in (0,T]$,

$$|\Delta g(t)| \leq \left| \int_0^1 x \sum_{n\geq 1} f_n X_n(x)\, dx \right| \|\Delta c\|_{C_{1-\alpha}[0,T]} M \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} ds.$$

Therefore, in view of (A2) and (A5) we obtain

$$\|G(c) - G(\widetilde{c})\|_{C_{1-\alpha}[0,T]} \leq M \frac{T^{\alpha}\Gamma^2(\alpha)}{\Gamma(2\alpha)} |F| \|c - \widetilde{c}\|_{C_{1-\alpha}[0,T]} .$$

Hence, $G(c)$ is Lipschitz continuous on $\mathcal{H}$.

Consequently, if $G(c) \neq G(\widetilde{c})$ for each $c, \widetilde{c} \in \mathcal{H}$ then $c(t) \neq \widetilde{c}(t)$ on $(0,T]$ which is the wanted property. □

Theorem 4.5. *The source term c can be determined uniquely by the input data (1.4).*

Proof. For any $c, \widetilde{c} \in \mathcal{H}$ *such that* $c(t) > \widetilde{c}(t) > 0$ on $(0,T]$, *we have in view of* (4.7)

$$D^{\alpha}_{0+}(\Delta g(t)) - z_x(1,t) = \Delta c(t) \int_0^1 x f(x) dx$$

with

$$z_x(1,t) = \sum_{n\geq 1} (-1)^n \sqrt{\lambda_n} f_n \int_0^t \Delta c(s)\, e_{\alpha}(-\lambda_n, t-s)\, ds > 0$$

then

$$\Delta c(t) \int_0^1 x f(x) dx \leq z_x(1,t) + \Delta c(t) \int_0^1 x f(x) dx = D^{\alpha}_{0+}(\Delta g(t)) .$$

Then, we conclude that

$$\|\Delta c\|_{C_{1-\alpha}[0,T]} \leq \frac{1}{|F|} \left\| D^{\alpha}_{0+}(\Delta g) \right\|_{C_{1-\alpha}[0,T]} . \tag{4.10}$$

We note that the data of $\Delta g(t) = 0$ implies that $D^{\alpha}_{0+}(\Delta g(t)) = 0$. So, $c(t)$ is obtained uniquely on $(0,T]$ by the input data $g(t)$ and the proof is complete. □

Conclusion. In this work, we considered an inverse source problem for the time-fractional of Riemann-Liouville type diffusion problem (1.1)-(1.3) with an integral over-detemination data (1.4).

In summary, the weighted initial condition required a suitable weighted space $C_{1-\alpha}[0,T]$ as a solution space and the unique determination of the unknown time dependent term source $c(t)$ was reduced to the invertibility of the input-output mapping $G(c)$ obtained explicitely from the series representation of the output data $g(t)$.

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