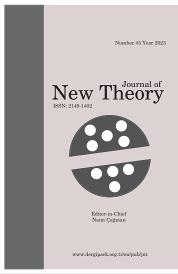
Journal of
New Theory
ISSN: 2149-1402

43 (2023) 1-10
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Statistical Convergence of Double Sequences in Intuitionistic Fuzzy Metric Spaces

Ahmet Özcan[1], **Gökay Karabacak**[2], **Sevcan Bulut**[3], **Aykut Or**[4]

Article Info
Received: 6 Jan 2023
Accepted: 3 May 2023
Published: 30 Jun 2023
doi:10.53570/jnt.1230368
Research Article

Abstract — Statistical convergence has been a prominent research area in mathematics since this concept was independently introduced by Fast and Steinhaus in 1951. Afterward, the statistical convergence of double sequences in metric spaces and fuzzy metric spaces has been widely studied. The main goal of the present study is to introduce the concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy metric spaces. Moreover, this study characterizes the statistical convergence of a double sequence by an ordinary convergent of a subsequence of the double sequence. Besides, the current study theoretically contributes to the mentioned concepts and investigates some of their basic properties. Finally, the paper handles whether the aspects should be further investigated.

Keywords *Statistical convergence, statistical Cauchy sequences, double sequences, intuitionistic fuzzy metric spaces*

Mathematics Subject Classification (2020) 40A05, 40A35

1. Introduction

Statistical convergence, a generalization of ordinary convergence, is based on the natural density of a subset of $\mathbb{N}$, the set of all the natural numbers. Fast [1] and Steinhaus [2] have established this concept separately in 1951. Many mathematicians particularly Salat [3], Freedman and Sember [4], Fridy [5], Connor [6], Kolk [7], Fridy and Orhan [8], and Bulut and Or [9], have contributed to the development of statistical convergence. Pringshem [10] has introduced the convergence and Cauchy sequence of double sequences. After that, Mursaleen and Edely [11] have studied double sequences' statistical convergence.

Fuzzy sets, defined by Zadeh [12], have been used in many fields, such as artificial intelligence, decision-making, image analysis, probability theory, and weather forecasting. In addition, Kramosil and Michalek [13] and Kaleva and Seikkala [14] have first examined fuzzy metric spaces (FMSs). Further, George and Veeramani [15] have redefined the concept of fuzzy metrics to construct the Hausdorff topology. Lately, Mihet [16] has studied the point convergence (p-convergence), a weaker concept than ordinary convergence. Moreover, Gregori et al. [17] have put forward the concept of s-convergence. Morillas and Sapena [18] have defined the standard convergence (std-convergence). Gre-

[1]ozcannaahmet@gmail.com; [2]gokaykarabacak@klu.edu.tr; [3]sevcanbulut7@gmail.com; [4]aykutor@comu.edu.tr (Corresponding Author)
[1,4]Department of Mathematics, Faculty of Science, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye
[2]Department of Mathematics, Faculty of Arts and Sciences, Kırklareli University, Kırklareli, Türkiye
[3]Department of Computer Engineering, Faculty of Engineering and Natural Sciences, İstanbul Rumeli University, İstanbul, Türkiye

gori and Miňana [19] have introduced the concept of strong convergence (st-convergence), a stronger concept than ordinary convergence. Li et al. [20] have proposed the statistical convergence and statistical Cauchy sequence in FMSs and examined some of their basic properties. After that, Park [21] has recently introduced the intuitionistic fuzzy metric spaces (IFMSs) by intuitionistic fuzzy sets, defined by Atanassov [22], and triangular norms and triangular conorms [23]. Moreover, Park [21] studied the convergence sequence concerning in IFMSs. Varol [24] has suggested the statistical convergence in IFMSs and analyzed statistical Cauchy sequences in IFMSs. Besides, Savaş [25] has introduced the statistical convergence and statistical Cauchy sequences for the double sequences in FMSs. Motivated the article [25] and the studies done in the literature on this subject, this paper defines statistical convergence and statistical Cauchy sequences for double sequences in IFMSs.

Section 2 of the handled study provides some basic definitions and properties to be needed in the following sections. Section 3 describes statistical convergence and statistical Cauchy sequences for double sequences in IFMSs. Finally, it discusses the need for further research.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following sections.

Definition 2.1. [4] The natural density of a set $A \subseteq \mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

where $|\{k \in A : k \leq n\}|$ denotes the number of elements of A that do not exceed n. It can be observed that if the set A is finite, then $\delta(A) = 0$.

Throughout this paper, $\mathbb{Y}$ denotes $\mathbb{N} \times \mathbb{N}$.

Definition 2.2. [11] The double natural density of a set $A \subseteq \mathbb{Y}$ is defined by

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|}{mn}$$

where $|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|$ denotes the number of elements of A, whose the first and second components do not exceed m and n, respectively. It can be observed that if the set A is finite, then $\delta_2(A) = 0$.

Definition 2.3. [11] Let (x_{jk}) be a double sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Then, (x_{jk}) is called statistically convergent to x_0, if, for all $\varepsilon > 0$, $\delta_2(\{(j,k) \in \mathbb{Y} : |x_{jk} - x_0| \geq \varepsilon\}) = 0$ and is denoted by $st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_0$.

Definition 2.4. [23] Let $\oplus : [0,1]^2 \to [0,1]$ be a binary operation. Then, $\oplus$ is called a triangular norm (t-norm), if it satisfies the following conditions:

i. $\oplus$ is associative and commutative.

ii. $a \oplus 1 = a$, for all $a \in [0,1]$.

iii. If $a_1 \leq a_3$ and $a_2 \leq a_4$, for each $a_1, a_2, a_3, a_4 \in [0,1]$, then $a_1 \oplus a_3 \leq a_2 \oplus a_4$.

Definition 2.5. [23] Let $\otimes : [0,1]^2 \to [0,1]$ be a binary operation. Then, $\otimes$ is referred to as a triangular conorm (t-conorm), if it satisfies the following conditions:

i. $\otimes$ is associative and commutative

ii. $a \otimes 0 = a$, for all $a \in [0,1]$.

iii. If $a_1 \leq a_3$ and $a_2 \leq a_4$, for each $a_1, a_2, a_3, a_4 \in [0,1]$, then $a_1 \otimes a_3 \leq a_2 \otimes a_4$.

Example 2.6. [21] The following operators are basic examples of t-norms and t-conorms, respectively:

i. $a_1 \oplus a_2 = a_1 a_2$

ii. $a_1 \oplus a_2 = \min\{a_1, a_2\}$

iii. $a_1 \otimes a_2 = \max\{a_1, a_2\}$

iv. $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$

Definition 2.7. [21] Let $\mathbb{B}$ be an arbitrary set, $\oplus$ be a continuous t-norm, $\otimes$ be a continuous t-conorm, and φ, ϑ be fuzzy sets on $\mathbb{B}^2 \times (0, \infty)$. For all $x_1, x_2, x_3 \in \mathbb{B}$ and $u, s > 0$, if φ and ϑ satisfy the following conditions:

i. $\varphi(x_1, x_2, u) + \vartheta(x_1, x_2, u) \leq 1$

ii. $\varphi(x_1, x_2, u) > 0$

iii. $\varphi(x_1, x_2, u) = 1 \Leftrightarrow x_1 = x_2$

iv. $\varphi(x_1, x_2, u) = \varphi(x_2, x_1, u)$

v. $\varphi(x_1, x_3, u + s) \geq \varphi(x_1, x_2, u) \oplus \varphi(x_2, x_3, s)$

vi. The function $\varphi_{x_1 x_2} : (0, \infty) \to (0, 1]$, defined by $\varphi_{x_1 x_2} = \varphi(x_1, x_2, u)$, is continuous

vii. $\vartheta(x_1, x_2, u) > 0$

viii. $\vartheta(x_1, x_2, u) = 0 \Leftrightarrow x_1 = x_2$

ix. $\vartheta(x_1, x_2, u) = \vartheta(x_2, x_1, u)$

x. $\vartheta(x_1, x_3, u + s) \leq \vartheta(x_1, x_2, u) \otimes \vartheta(x_2, x_3, s)$

xi. The function $\vartheta_{x_1 x_2} : (0, \infty) \to (0, 1]$, defined by $\vartheta_{x_1 x_2} = \vartheta(x_1, x_2, u)$, is continuous

then a 5-tuple $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is said to be an intuitionistic fuzzy metric space (IFMS).

The values $\varphi(x_1, x_2, u)$ and $\vartheta(x_1, x_2, u)$ represent the degree of nearness and non-nearness of x_1 and x_2 concerning u, respectively.

Example 2.8. [21] Let $(\mathbb{B}, d)$ be a metric space. Define $a_1 \oplus a_2 = a_1 a_2$ and $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$, for all $a_1, a_2 \in [0, 1]$, and suppose that φ and ϑ are fuzzy sets on $\mathbb{B}^2 \times (0, \infty)$ defined by

$$\varphi(x_1, x_2, u) = \frac{u}{u + d(x_1, x_2)} \text{ and } \vartheta(x_1, x_2, u) = \frac{d(x_1, x_2)}{u + d(x_1, x_2)}$$

for all $x_1, x_2 \in \mathbb{B}$ and $u > 0$. Then, $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS.

Remark 2.9. [24] If $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS, then $(\mathbb{B}, \varphi, \oplus)$ is an FMS. Moreover, if $(\mathbb{B}, \varphi, \oplus)$ is an FMS, then $(\mathbb{B}, \varphi, 1 - \varphi, \oplus, \otimes)$ is an IFMS such that $a_1 \otimes a_2 = 1 - [(1 - a_1) \oplus (1 - a_2)]$, for all $a_1, a_2 \in [0, 1]$.

Definition 2.10. [21] Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{B}$ is said to be convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $n \geq n_\varepsilon$ implies that

$$\varphi(x_n, x_0, u) > 1 - \varepsilon \text{ and } \vartheta(x_n, x_0, u) < \varepsilon$$

or equivalently

$$\lim_{n \to \infty} \varphi(x_n, x_0, u) = 1 \text{ and } \lim_{n \to \infty} \vartheta(x_n, x_0, u) = 0$$

and is denoted by $\frac{\varphi}{\vartheta} - \lim_{n \to \infty} x_n = x_0$ or $x_n \xrightarrow{\frac{\varphi}{\vartheta}} x_0$ as $n \to \infty$.

Definition 2.11. [21] Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS. Then, a sequence (x_n) is referred to as a Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ), if, for all $u>0$ and $\varepsilon\in(0,1)$, there exists $n_\varepsilon\in\mathbb{N}$ such that $n,N\geq n_\varepsilon$ implies that

$$\varphi(x_n,x_N,u)>1-\varepsilon \text{ and } \vartheta(x_n,x_N,u)<\varepsilon$$

or equivalently

$$\lim_{n,N\to\infty}\varphi(x_n,x_N,u)=1 \text{ and } \lim_{n,N\to\infty}\vartheta(x_n,x_N,u)=0$$

Definition 2.12. [24] Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS. Then, a sequence (x_n) in $\mathbb{B}$ is called statistically convergent to $x_0\in\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ), if, for all $\varepsilon\in(0,1)$ and $u>0$,

$$\delta(\{n\in\mathbb{N}:\varphi(x_n,x_0,u)\leq 1-\varepsilon \text{ or } \vartheta(x_n,x_0,u)\geq\varepsilon\})=0$$

or equivalently

$$\lim_{n\to\infty}\frac{|\{k\leq n:\varphi(x_k,x_0,u)\leq 1-\varepsilon \text{ or } \vartheta(x_k,x_0,u)\geq\varepsilon\}|}{n}=0$$

Definition 2.13. [24] Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS. Then, a sequence (x_n) is said to be a statistically Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ), if, for all $\varepsilon\in(0,1)$ and $u>0$, there exists $N\in\mathbb{N}$ such that

$$\delta(\{n\in\mathbb{N}:\varphi(x_n,x_N,u)\leq 1-\varepsilon \text{ or } \vartheta(x_n,x_N,u)\geq\varepsilon\})=0$$

Definition 2.14. [25] Let $(\mathbb{B},\varphi,\oplus)$ be an FMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is called statistically convergent to $x_0\in\mathbb{B}$ concerning fuzzy metric φ, if, for all $u>0$ and $\varepsilon\in(0,1)$,

$$\delta_2(\{(j,k)\in\mathbb{Y}:\varphi(x_{jk},x_0,u)\leq 1-\varepsilon\})=0$$

and is denoted by $st_2s-\lim_{j,k\to\infty}x_{jk}=x_0$.

Definition 2.15. [25] Let $(\mathbb{B},\varphi,\oplus)$ be an FMS. Then, a double sequence (x_{jk}) is referred to as a statistically Cauchy sequence in $\mathbb{B}$ concerning fuzzy metric φ, if, for all $u>0$ and $\varepsilon\in(0,1)$, there exists $m,n\in\mathbb{N}$ such that

$$\delta_2(\{(j,k)\in\mathbb{Y}:\varphi(x_{jk},x_{mn},u)\leq 1-\varepsilon\})=0$$

3. Main Results

This section defines the concepts of statistical convergence and statistical Cauchy sequences for double sequences in IFMSs. In addition, it provides some of their basic properties.

Definition 3.1. Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is said to be convergent to $x_0\in\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ), if, for all $\varepsilon\in(0,1)$ and $u>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $j,k\geq n_\varepsilon$ implies that

$$\varphi(x_{jk},x_0,u)>1-\varepsilon \text{ and } \vartheta(x_{jk},x_0,u)<\varepsilon$$

or equivalently

$$\lim_{j,k\to\infty}\varphi(x_{jk},x_0,u)=1 \text{ and } \lim_{j,k\to\infty}\vartheta(x_{jk},x_0,u)=0$$

and is denoted by $\frac{\varphi}{\vartheta}-\lim_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{\frac{\varphi}{\vartheta}}x_0$ as $j,k\to\infty$.

Definition 3.2. Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS. Then, a double sequence (x_{jk}) in $\mathbb{B}$ is called statistically convergent to $x_0\in\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ), if, for all $\varepsilon\in(0,1)$ and

$u > 0$,

$$\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, x_0, u) \geq \varepsilon\}\right) = 0$$

or equivalently

$$\lim_{m,n\to\infty} \frac{|\{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } (\varphi(x_{jk}, x_0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, x_0, u) \geq \varepsilon)\}|}{mn} = 0$$

and is denoted by ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{{}^{\varphi}_{\vartheta}st_2} x_0$ as $j,k \to \infty$.

Example 3.3. Let $\mathbb{B} = \mathbb{R}$, $a_1 \oplus a_2 = a_1 a_2$ and $a_1 \otimes a_2 = \min\{a_1 + a_2, 1\}$ for all $a_1, a_2 \in [0,1]$. Define φ and ϑ by

$$\varphi(x_1, x_2, u) = \frac{u}{u + |x_1 - x_2|} \text{ and } \vartheta(x_1, x_2, u) = \frac{|x_1 - x_2|}{u + |x_1 - x_2|}$$

for all $x_1, x_2 \in \mathbb{R}$ and $u > 0$. Then, $(\mathbb{R}, \varphi, \vartheta, \oplus, \otimes)$ is an IFMS [21]. Moreover, define a sequence (x_{jk}) by

$$x_{jk} := \begin{cases} 1, & j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

Then, for all $\varepsilon \in (0,1)$ and for any $u > 0$, let

$$K = \{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } (\varphi(x_{jk}, 0, u) \leq 1-\varepsilon \text{ or } \vartheta(x_{jk}, 0, u) \geq \varepsilon)\}$$

Hence,

$$\begin{aligned} K &= \left\{(j,k) \in \mathbb{Y} : (j \leq m \text{ and } k \leq n) \text{ and } \left(\tfrac{u}{u+|x_{jk}|} \leq 1-\varepsilon \text{ or } \tfrac{|x_{jk}|}{u+|x_{jk}|} \geq \varepsilon\right)\right\} \\ &= \{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n,\ \text{and } x_{jk} = 1\} \\ &= \{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n,\ \text{and } j \text{ and } k \text{ are squares }\} \end{aligned}$$

and thus

$$\frac{|K|}{mn} = \frac{|\{(j,k) \in \mathbb{Y} : j \leq m,\ k \leq n,\ \text{and } j \text{ and } k \text{ are squares }\}|}{mn} \leq \frac{\sqrt{m}\sqrt{n}}{mn} \to 0 \text{ as } m,n \to \infty$$

Consequently, (x_{jk}) is statistically convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ).

Lemma 3.4. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS, (x_{jk}) be a double sequence in $\mathbb{B}$, and $x_0 \in \mathbb{B}$. Then, for all $\varepsilon \in (0,1)$ and $u > 0$, the following statements are equivalent:

i. ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_0$

ii. $\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) > 1-\varepsilon\}\right) = \delta_2\left(\{(j,k) \in \mathbb{Y} : \vartheta(x_{jk}, x_0, u\} < \varepsilon\}\right) = 1$

iii. $\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1-\varepsilon\}\right) = \delta_2\left(\{(j,k) \in \mathbb{Y} : \vartheta(x_{jk}, x_0, u) \geq \varepsilon\}\right) = 0$

Proof.

The proof is straightforward using Definition 3.2 and the density function's properties. □

Theorem 3.5. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. If a double sequence (x_{jk}) in $\mathbb{B}$ is statistically convergent concerning intuitionistic fuzzy metric (φ, ϑ), then the statistically limit is unique.

Proof.

Suppose that ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_1$, ${}^{\varphi}_{\vartheta}st_2 - \lim\limits_{j,k\to\infty} x_{jk} = x_2$, and $x_1 \neq x_2$. For a given $\varepsilon \in (0,1)$, choose $\eta \in (0,1)$ such that $(1-\eta) \oplus (1-\eta) > 1-\varepsilon$ and $\eta \otimes \eta < \varepsilon$. Then, define the following sets, for any $u > 0$,

$$K_1(\eta, u) := \{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_1, u) \leq 1-\eta\}$$

$$K_2(\eta, u) := \{(j,k) \in \mathbb{Y} : \varphi(x_{jk}, x_2, u) \leq 1-\eta\}$$

$$T_1(\eta, u) := \{(j,k) \in \mathbb{Y} : \vartheta\left(x_{jk}, x_1, u\right) \geq \eta\}$$

and

$$T_2(\eta, u) := \{(j,k) \in \mathbb{Y} : \vartheta\left(x_{jk}, x_2, u\right) \geq \eta\}$$

By Lemma 3.4, since (x_{jk}) is statistically convergent to x_1 and x_2 concerning intuitionistic fuzzy metric (φ, ϑ), for $u > 0$,

$$\delta_2\left(K_1(\eta, u)\right) = \delta_2\left(T_1(\eta, u)\right) = 0 = \delta_2\left(K_2(\eta, u)\right) = \delta_2\left(T_2(\eta, u)\right)$$

Let

$$A(\eta, u) := \left(K_1(\eta, u) \cup K_2(\eta, u)\right) \cap \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$$

for $u > 0$. Hence, $\delta_2\left(A(\eta, u)\right) = 0$ which implies that $\delta_2\left(\mathbb{Y} \backslash A(\eta, u)\right) = 1$. If $(j,k) \in \mathbb{Y} \backslash A(\eta, u)$, then

$$(j,k) \in \mathbb{Y} \backslash \left(K_1(\eta, u) \cup K_2(\eta, u)\right) \text{ or } (j,k) \in \mathbb{Y} \backslash \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$$

Let $(j,k) \in \mathbb{Y} \backslash \left(K_1(\eta, u) \cup K_2(\eta, u)\right)$. Then,

$$\varphi\left(x_1, x_2, u\right) \geq \varphi\left(x_1, x_{jk}, \frac{u}{2}\right) \oplus \varphi\left(x_{jk}, x_2, \frac{u}{2}\right) > (1 - \eta) \oplus (1 - \eta) > 1 - \varepsilon$$

Therefore, $\varphi\left(x_1, x_2, u\right) > 1 - \varepsilon$. Since $\varepsilon \in (0,1)$ is arbitrary, for all $u > 0$, $\varphi\left(x_1, x_2, u\right) = 1$ and thus $x_1 = x_2$.

Let $(j,k) \in \mathbb{Y} \setminus \left(T_1(\eta, u) \cup T_2(\eta, u)\right)$. Then,

$$\vartheta\left(x_1, x_2, u\right) \leq \vartheta\left(x_1, x_{jk}, \frac{u}{2}\right) \otimes \vartheta\left(x_{jk}, x_2, \frac{u}{2}\right) < \eta \otimes \eta < \varepsilon$$

Therefore, $\vartheta\left(x_1, x_2, u\right) < \varepsilon$. Since $\varepsilon \in (0,1)$ is arbitrary, for all $u > 0$, $\vartheta\left(x_1, x_2, u\right) = 0$ and thus $x_1 = x_2$. □

Theorem 3.6. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. If (x_{jk}) is convergent to $x_0 \in \mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), then it is statistically convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ).

PROOF.
Let (x_{jk}) be convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ). Then, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that $j, k \geq n_0$ implies that $\varphi\left(x_{jk}, x_0, u\right) > 1 - \varepsilon$ and $\vartheta\left(x_{jk}, x_0, u\right) < \varepsilon$. Hence, the set

$$\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, u\right) \leq 1 - \varepsilon \text{ or } \vartheta\left(x_{jk}, x_0, u\right) \geq \varepsilon\}$$

has a finite number of terms. Therefore,

$$\delta_2\left(\{(j,k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, u\right) \leq 1 - \varepsilon \text{ or } \vartheta\left(x_{jk}, x_0, u\right) \geq \varepsilon\}\right) = 0$$

Consequently, (x_{jk}) is statistically convergent to x_0 concerning intuitionistic fuzzy metric (φ, ϑ). □

The converse of Theorem 3.6 is not always correct.

Example 3.7. For the IFMS provided in Example 3.3, define a double sequence (x_{jk}) by

$$x_{jk} = \begin{cases} jk, & j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

(x_{jk}) is statistically convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ). However, (x_{jk}) is not convergent to 0 concerning intuitionistic fuzzy metric (φ, ϑ).

Theorem 3.8. Let $(\mathbb{B},\varphi,\vartheta,\oplus,\otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. Then, (x_{jk}) is statistically convergent to $x_0\in\mathbb{B}$ concerning intuitionistic fuzzy metric (φ,ϑ) if and only if there exists a subset $K\subset\mathbb{Y}$ such that

$$\delta_2(K)=1 \quad \text{and} \quad {}^{\varphi}_{\vartheta}-\lim_{\substack{m,n\to\infty\\(m,n)\in K}} x_{mn}=x_0$$

Proof.
($\Rightarrow$:) Let ${}^{\varphi}_{\vartheta}st_2-\lim\limits_{j,k\to\infty} x_{jk}=x_0$ and

$$K_r(u)=\left\{(j,k)\in\mathbb{Y}:\varphi\left(x_{jk},x_0,u\right)>1-\frac{1}{r}\text{ and }\vartheta\left(x_{jk},x_0,u\right)<\frac{1}{r}\right\},\quad r\in\mathbb{N}$$

Then, according to Definition 3.2,

$$\delta_2\left(K_r(u)\right)=1 \tag{1}$$

such that $r\in\mathbb{N}$. From the definition of $K_r(u)$, it is clear that

$$K_1(u)\supset K_2(u)\supset\cdots\supset K_r(u)\supset K_{r+1}(u)\supset\cdots \tag{2}$$

such that $r\in\mathbb{N}$. Choose an arbitrary element $(t_1,s_1)\in K_1(u)$. According to Equation 1, there exists such that a $(t_2,s_2)\in K_2(u)$ satisfying the conditions $t_2>t_1$ and $s_2>s_1$, for each (m,n) such that $m>t_2$ and $n>s_2$,

$$\frac{K_2(u)(m,n)}{mn}>\frac{1}{2}$$

where

$$K_2(u)(m,n)=\sum_{\substack{k\leq m\\ l\leq n\\ (k,l)\in K_2(u)}}1$$

Further, according to Equation 1, there exists such that a $(t_3,s_3)\in K_3(u)$ satisfying the conditions $t_3>t_2$ and $s_3>s_2$, for each (m,n) such that $m>t_3$ and $n>s_3$,

$$\frac{K_3(u)(m,n)}{mn}>\frac{2}{3}$$

etc. Therefore, by induction, construct a sequence (t_r,s_r) of the set $\mathbb{Y}$ such that

$$t_1<t_2<\ldots<t_r<\ldots\text{ and }s_1<s_2<\ldots<s_r<\ldots$$

$(t_r,s_r)\in K_r(u)$, for all $r\in\mathbb{N}$, and

$$\frac{K_r(u)(m,n)}{mn}>\frac{r-1}{r},\quad r\in\mathbb{N} \tag{3}$$

for each (m,n) where $m\geq t_r$ and $n\geq s_r$. Form the set K as follows: Each element between $(1,1)$ and (t_1,s_1) belongs to the set K, further, any element between (t_r,s_r) and (t_{r+1},s_{r+1}) belongs to K if and only if it belongs to $K_r(u)$ such that $r\in\mathbb{N}$. According to Equations 1 and 3, for each (m,n) such that $t_r\leq m<t_{r+1}$ and $s_r\leq n<s_{r+1}$,

$$\frac{K(m,n)}{mn}>\frac{K_r(u)(m,n)}{mn}>\frac{r-1}{r}$$

Thus, it is obvious that $\delta_2(K)=1$. Let $u>0$ and $\varepsilon\in(0,1)$. Choose an r such that $\frac{1}{r}<\varepsilon$. Let $(m,n)\in K$ such that $m\geq t_r$ and $n\geq s_r$. Then, there exists a number $l\geq r$ such that $t_l\leq m<t_{l+1}$, $s_l\leq n<s_{l+1}$, and $(m,n)\in K_l$. Hence,

$$\varphi\left(x_{mn},x_0,u\right)>1-\frac{1}{l}\geq 1-\frac{1}{r}>1-\varepsilon\text{ and }\vartheta\left(x_{mn},x_0,u\right)<\frac{1}{l}\leq\frac{1}{r}<\varepsilon$$

Thereby, $\varphi(x_{mn}, x_0, u) > 1 - \varepsilon$ and $\vartheta(x_{mn}, x_0, u) < \varepsilon$, for each $(m, n) \in K$ where $m \geq t_r$ and $n \geq s_r$, i.e.,

$$\tfrac{\varphi}{\vartheta} - \lim_{\substack{m,n\to\infty \\ (m,n)\in K}} x_{mn} = x_0$$

($\Leftarrow$:) Suppose that there exists a set $K = \{(m, n) \in \mathbb{Y} : m, n = 1, 2, \ldots\}$ such that $\delta_2(K) = 1$ and $\tfrac{\varphi}{\vartheta} - \lim_{\substack{m,n\to\infty \\ (m,n)\in K}} x_{mn} = x_0$, i.e., for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $m, n \geq n_\varepsilon$ implies that $\varphi(x_{mn}, x_0, u) > 1 - \varepsilon$ and $\vartheta(x_{mn}, x_0, u) < \varepsilon$. Hence,

$$A(\varepsilon, u) := \{(j, k) \in \mathbb{Y} : \varphi(x_{jk}, x_0, u) \leq 1 - \varepsilon \text{ or } \vartheta(x_{jk}, x_0, u) \geq \varepsilon\} \subseteq \mathbb{Y} \setminus \{(j_{n_\varepsilon+1}, k_{n_\varepsilon+1}), (j_{n_\varepsilon+2}, k_{n_\varepsilon+2}), \cdots\}$$

Therefore, $\delta_2(A(\varepsilon, u)) = 0$. Consequently, $\tfrac{\varphi}{\vartheta} st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$. $\square$

Definition 3.9. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) is referred to as a Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $u > 0$ and $\varepsilon \in (0, 1)$, there exists $n_\varepsilon \in \mathbb{N}$ such that $j \geq p \geq n_\varepsilon$ and $k \geq q \geq n_\varepsilon$ implies that

$$\varphi(x_{jk}, x_{pq}, u) > 1 - \varepsilon \text{ and } \vartheta(x_{jk}, x_{pq}, u) < \varepsilon$$

or equivalently

$$\lim_{p,q\to\infty} \varphi(x_{jk}, x_{pq}, u) = 1 \text{ and } \lim_{p,q\to\infty} \vartheta(x_{jk}, x_{pq}, u) = 0$$

Definition 3.10. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS. Then, a double sequence (x_{jk}) is called a statistically Cauchy sequence in $\mathbb{B}$ concerning intuitionistic fuzzy metric (φ, ϑ), if, for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $(p, q) \in \mathbb{Y}$ such that

$$\delta_2(\{(j, k) \in \mathbb{Y} : \varphi(x_{jk}, x_{pq}, u) \leq 1 - \varepsilon \text{ or } \vartheta(x_{jk}, x_{pq}, u) \geq \varepsilon\}) = 0$$

Theorem 3.11. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. If (x_{jk}) is statistically convergent concerning intuitionistic fuzzy metric (φ, ϑ), then it is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

Proof.
Let $\tfrac{\varphi}{\vartheta} st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$. Then, for all $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that $(1 - \varepsilon_2) \oplus (1 - \varepsilon_2) > 1 - \varepsilon_1$ and $\varepsilon_2 \otimes \varepsilon_2 < \varepsilon_1$, and for all $u > 0$,

$$\delta_2\left(\left\{(j, k) \in \mathbb{Y} : \varphi\left(x_{jk}, x_0, \frac{u}{2}\right) \leq 1 - \varepsilon_1 \text{ or } \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \geq \varepsilon_1\right\}\right) = 0$$

In particular, for $j = p$ and $k = q$,

$$\delta_2\left(\left\{(p, q) \in \mathbb{Y} : \varphi\left(x_{pq}, x_0, \frac{u}{2}\right) \leq 1 - \varepsilon_1 \text{ or } \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right) \geq \varepsilon_1\right\}\right) = 0$$

Since

$$\varphi(x_{jk}, x_{pq}, u) \geq \varphi\left(x_{jk}, x_0, \frac{u}{2}\right) \oplus \varphi\left(x_{pq}, x_0, \frac{u}{2}\right) \geq (1 - \varepsilon_2) \oplus (1 - \varepsilon_2) > 1 - \varepsilon_1$$

and

$$\vartheta(x_{jk}, x_{pq}, u) \leq \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \otimes \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right) < \varepsilon_2 \otimes \varepsilon_2 < \varepsilon_1$$

then

$$\delta_2(\{(j, k) \in \mathbb{Y} : \varphi(x_{jk}, x_{pq}, u) \leq 1 - \varepsilon_1 \text{ or } \vartheta(x_{jk}, x_{pq}, u) \geq \varepsilon_1\}) = 0$$

That is, (x_{jk}) is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ). $\square$

Theorem 3.12. Let $(\mathbb{B}, \varphi, \vartheta, \oplus, \otimes)$ be an IFMS and (x_{jk}) be a double sequence in $\mathbb{B}$. Then, the following statements are equivalent.

i. (x_{jk}) is a statistically Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

ii. There exists a subset $K \subset \mathbb{Y}$ such that $\delta_2(K) = 1$ and the subsequence (x_{mn}), indexed by elements in K, of the sequence (x_{jk}) is a Cauchy sequence concerning intuitionistic fuzzy metric (φ, ϑ).

Proof.
The proof is similar to the proof of Theorem 3.8. □

4. Conclusion

This paper investigated the concept of statistical convergence, a generalization of ordinary convergence, for the double sequences in IFMSs. Additionally, it researched statistical Cauchy sequences and revealed characterizations of these concepts for double sequences. In further works, researchers can study the concept of Lacunary convergence in an IFMS using the concepts and results herein and analyze some of its basic properties.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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