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The Geometry of Vector Fields and Two Dimensional Heat Equation

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

Abstract

The geometry of orbits of families of smooth vector fields was studied by many mathematicians due to its importance in applications in the theory of control systems, in dynamic systems, in geometry and in the theory of foliations. In this paper it is studied geometry of orbits of vector fields in four dimensional Euclidean space. It is shown that orbits generate singular foliation every regular leaf of which is a surface of negative Gauss curvature and zero normal torsion.
In addition, the invariant functions of the considered vector fields are used to find solutions of the two-dimensional heat equation that are invariant under the groups of transformations generated by these vector fields.
In the present paper, smoothness is understood as smoothness of the class C^∞.

Keywords: *Vector field, orbit of vector fields, foliation, infinitesimal generator, heat equation, symmetry group.*
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1. Introduction

Let M be a smooth manifold of dimension n.

Definition 1.1. A partition F of a riemannian manifold M by connected immersed submanifolds (the leaves) is called a singular foliation of M if it verifies condition:

F is singular, i.e., the module XF of smooth vector fields on M that are tangent at each point to the corresponding leaf acts transitively on each leaf. In other words, for each leaf L and each $v \in T_pL$ with foot point p, there is $X \in XF$ with $X(p) = v$ [1-5].

Let F be a singular foliation on a riemannian manifold M. A leaf L of F (and each point in L) is called regular if the dimension of L is maximal, otherwise L is called singular.

It is known that orbits of vector fields generate singular foliation (see [1],[2],[5].

Let $V(M)$ is the set of all smooth (class C^∞) vector fields defined on M. The set $V(M)$ is a linear space over the field of real numbers and Lie algebra in which a binary operation is Lie bracket $[X, Y]$ of vector fields $X, Y \in V(M)$.

Let us consider a set $D \subset V(M)$ and denote the smallest Lie subalgebra containing D by $A(D)$. The family D may contain finitely or infinitely many smooth vector fields. For a point $x \in M$, by $t \to X^t(x)$ we denote integral curve of the vector field X passing through the point x for $t = 0$. The map $t \to X^t(x)$ is defined in some region $I(x)$, which in the general case depends not only from the field X, but also from the starting point x.

Definition 1.2. The orbit $L(x)$ of a family D of vector fields through a point x is the set of points y in M such that there exist real numbers $t_1, t_2, ..., t_k$ and vector fields $X_{i_1}, X_{i_2}, ..., X_{i_k}$ in D (where k is an arbitrary positive integer) such that

$$y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(...(X_{i_1}^{t_1})...)).$$

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Numerous investigations have been devoted to the study of geometry of orbits of vector fields [1]-[6].

The fundamental result in study of orbits is Sussmann theorem [5], which asserts that every orbit of smooth vector fields with Sussmann topology has differential structure with respect to which it is a immersed submanifold of M.

Recall that a mapping P that takes each point $x \in M$ to some subspace $P(x) \subset T_xM$ is called a distribution. If dim $P(x) = k$ for all $x \in M$, then P is called a k-dimensional distribution. A distribution P is said to be smooth if, for each point $x \in M$, there exists a neighborhood $U(x)$ of the point and smooth vector fields $X_1, X_2, ..., X_m$ defined on $U(x)$ such that the vectors

$$X_1(y), X_2(y), ..., X_m(y)$$

form a basis of the subspace $P(y)$ for each $y \in U(x)$.

A family D of smooth vector fields naturally generates the smooth distribution that takes each point $x \in M$ to the subspace $P(x)$ of the tangent space T_xM spanned by the set

$$D(x) = \{X(x) : X \in D\}.$$

Obviously, the dimension of the subspace $P(x)$ can vary from point to point.

A distribution P is said to be completely integrable if, for each point $x \in M$, there exists a connected submanifold N_x of the manifold M such that $T_yN_x = P(y)$ for all $y \in N_x$.

The submanifold N_x is callled an integral submanifold of the distribution P. For a vector field X, we write $X \in P$ if $X(x) \in P(x)$ for all $x \in M$.

A distribution P is said to be involutive if the inclusion $X, Y \in P$ implies that $[X, Y] \in P$, where $[X, Y]$ is the Lie bracket of the fields X and Y.

The Frobenius theorem [2] provides a necessary and suffcient condition for the completely integrability of a distribution of constant dimension.

Theorem 1.1. *(Frobenius) A distribution P on a manifold M is completely integrable if and only if it is involutive.*

Let $A(D)$ be the smallest Lie algebra containing the set D. By setting $A_x(D) = \{X(x) : X \in A(D)\}$, we obtain an involutive distribution $P_D : x \to Ax(D)$. If the dimension $dimA_x(D)$ is independent of x, then the distribution $P_D : x \to Ax(D)$ is completely integrable by the Frobenius theorem.

If the dimension $dimA_x(D)$ depends on x, then, as examples show, the distribution $P_D : x \to A_x(D)$ is not necessarily completely integrable.

The Frobenius theorem generalized by Hermann to distributions of variable dimension provides a necessary and sufficient condition for the complete integrability of distributions which is finitely generated [2].

Definition 1.3. A system of vector fields

$$D = \{X_1, X_2, ..., X_k\}$$

on M is in involution if there exist smooth real-valued functions $f^l_{ij}(x), x \in M,\ i, j, l = 1, ..., k$ such that for each (i, j) it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f^l_{ij}(x)X_l.$$

Theorem 1.2. *The system*

$$D = \{X_1, X_2, ..., X_k\}$$

of smooth vector fields on M generates completely integrable distribution if and only if it is in involution.

2. Main Part

2.1. Geometry of Orbits of Vector Fields

Let us consider a family of $D = \{X_1, X_2\}$ vector fields on four-dimensional Euclidean space E^4 the with cartesian coordinates x_1, x_2, t, u, where

$$X_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - 2u\frac{\partial}{\partial u}. \tag{2.1}$$

These vector fields are infinitesimal generators of the symmetry group of the two-dimensional heat equation.

Theorem 2.1. *Orbits of the family of vector fields (1) generate singular foliation regular leaf of which is a surface with a negative Gauss curvature and zero normal torsion. .*

Proof.
First of all we calculate Lie brocket and find $[X_1, X_2] = 0$. It follows from Hermann theorem the family D is completely integrable.
Vector field

$$X_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \tag{2.2}$$

generate following one parametrical group of transformations

$$(x_1, x_2, t, u) \to (x_1, x_2, te^s, ue^s), s \in R \tag{2.3}$$

Vector field

$$X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - 2u\frac{\partial}{\partial u}. \tag{2.4}$$

generate following one parametrical group of transformations

$$(x_1, x_2, t, u) \to (x_1e^s, x_2e^s, t, ue^{-2s}), s \in R \tag{2.5}$$

We find the invariant functions of these transformations.
It is known that [12, p. 117] a smooth function $f : M \to R$ is an invariant function of the transformation group G, acting on M if and only if $Xf = 0$ for each infinitesimal generator X of the group G.
Using this criterion, we find that the functions

$$I_1(t, x_1, x_2, u) = \frac{t}{x_1^2 u}, I_2(t, x_1, x_2, u) = \frac{x_2}{x_1}, \tag{2.6}$$

are invariant functions of the transformation groups (2.3), (2.5), which follows from the following equalities

$$X_1(I_1) = 0, X_1(I_2) = 0, X_2(I_1) = 0, X_2(I_2) = 0. \tag{2.7}$$

This invariant functions give us a family of two dimensional surfaces

$$\frac{t}{x_1^2 u} = C_1, \frac{x_2}{x_1} = C_2. \tag{2.8}$$

where C_1, C_2 are constants.
For given C_1, C_2 let us denote by F^C the surface which is defined by system of equations (8). For definiteness, we will assume that $C_1 > 0$.
If $p^0(x_1^0, x_2^0, t^0, u^0) \in F^C$, it follows from equalities (2.7) the orbit $L(p^0)$ of a family D of vector fields through the point p^0 is contained in the surface F^C.
If the point p^0 is the origin of a coordinate system, then it is a fixed point for vector fields, and in this case the orbit $L(p^0)$ is the point itself.
If $t^0 = 0$ for then $u^0 = 0$. It follows the vector field X_1 vanish at the point p^0. In this case the orbit $L(p^0)$ is a integral line of the the vector field X_2.
Let $t^0 > 0$ for definiteness. It follows that $u^0 > 0$. If $x_1^0 = 0$, we have $x_2^0 = 0$. In this case the orbit $L(p^0)$ is a domain in the plane Otu.
We assume that $x_1^0 \neq 0$ and show that the orbit $L(p^0)$ is a two dimensional surface in F^C.
Let $p(x_1, x_2, t, u) \in F^C$ be any point of different form the point p^0 with $t > 0$. At the integral curve of the vector field (4) trough the point p^0 there is the point $p^1(x_1^0e^{s_1}, x_2^0e^{s_1}, t^0, u^0e^{-2s_1})$, where $s_1 = \ln\sqrt{\frac{u^0 t}{ut^0}}$.
From the point p^1 by the integral line of the vector field X_1 over 'time' s_2 we can get the point p, where $s_2 = \ln\frac{t}{t^0}$.
Really

$$x_i^0 e^{s_1} = x_i^0\sqrt{\frac{u^0 t}{ut^0}} = x_i^0\sqrt{\frac{x_i^2}{(x_i^0)^2}} = x_i,$$

$$t^0 e^{s_2} = t, u^0 e^{s_2 - 2s_1} = u.$$

Thus we can assert that the orbit $L(p^0)$ consists of the points $p(x_1, x_2, t, u) \in F^C$, with $t > 0, u > 0$ and x_1 has the same sign as x_1^0. At any $p(x_1, x_2, t, u)$ of the orbit $L(p^0)$ vectors $X_1(p), X_1(p)$ linearly independent which shows the orbit $L(p^0)$ is a two dimensional manifold.

Let us recall some characteristics of two dimensional surface F in four-dimensional Euclidean space E^4.

Consider on the surface F at the point x some direction given by the nonzero vector ξ.

The vector ξ and the normal plane N of the surface at the point x define a hyperplane $E^3(x, \xi N)$ in E^4 which intersects the surface F along some curve γ. The curve γ is called the normal section of the surface F at the point x along the direction ξ. By its construction, the γ curve is a three-dimensional curve. Curvature $k_N(x, \xi)$ and torsion $\chi_N(x, \xi)$ of the curve γ at the point x are called, respectively, the normal curvature and the normal torsion of the surface at the point x in the direction ξ.

Geometry of two dimensional surfaces four-dimensional Euclidean space E^4 is a very important part of differential geometry and studied by many authors [9-11],[13].

Let S be the set of two-dimensional surfaces in the space E^4 whose normal torsion is equal to zero at any point in any direction. It is known that two-dimensional hyperplane surfaces belong to the set S, but do not exhaust it. Thus, the two-dimensional torus $S^1 \times S^1$ on the hypersphere S^3 in E^4 belongs to the set S, but is not hyperplane. The description of hyperplane two-dimensional surfaces in the set S is given in [11].

Now one can check metric characteristics of the orbit $L(p^0)$.

We find first and second quadratic forms of the orbit $L(p^0)$.

Let us parameterize the surface F^C by following equations

$$\begin{cases} x_1 = s \\ x_2 = C_2 s \\ t = q \\ u = C_1 \frac{q}{s^2} \end{cases}$$

Now we find

$$\frac{\partial r}{\partial s} = r_1 = \{1; -C_2; 0; -2C_1 \frac{q}{s^3}\}$$

$$\frac{\partial r}{\partial q} = r_2 = \{0; 0; 1; \frac{C_1}{s^2}\}$$

and coefficients of first quadratic form

$$g_{11} = \langle r_1, r_1 \rangle = 1 + C_2^2 + 4C_1^2 \frac{q^2}{s^6}$$

$$g_{12} = g_{21} = \langle r_1, r_2 \rangle = -\frac{2C_1^2 q}{s^5}$$

$$g_{22} = \langle r_2, r_2 \rangle = 1 + \frac{C_1^2}{s^4}$$

We need two normal vectors to find coefficients of second quadratic forms

$$n_1 = \{C_2; -1; 0; 0\}$$

and

$$n_2 = \{1; C_2; -\frac{s(1 - C_2^2)}{2q}; \frac{s^3(1 - C_2^2)}{2C_1 q}\}.$$

Now we can find two second quadratic forms of the orbit $L(p^0)$.

Coefficients of first of them calculated by formula

$$b_{ij} = -\frac{1}{|n_1|} \langle \partial_i r, \partial_j n_1 \rangle.$$

As normal vector n_1 is a constant vector we have

$$b_{11} = b_{22} = b_{12} = 0$$

Coefficients of second of them calculated by formula

$$c_{ij} = -\frac{1}{|n_2|}\langle \partial_i r, \partial_j n_2\rangle$$

It follows from here

$$c_{11} = -\frac{1}{|n_2|}\frac{3(1-C_2^2)}{s}, c_{12} = c_{21} = \frac{1}{|n_2|}\frac{(1-C_2^2)}{q}, c_{22} = 0.$$

It is known that Gauss curvature of two dimensional surface in E^4 is calculated by the formula [13]:

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

From this formula we have

$$K = -\frac{1}{t^2|n_2|^2}\frac{(1-C_2^2)^2}{g_{11}g_{22} - g_{12}^2}.$$

It follows Gauss curvature is a negative.

Now we consider normal torsion. Let ξ be a vector in the tangent plane $T_pL(p^0)$ at the point $p(x_1, x_2, t, u)$. The vector ξ and the normal plane N of the surface at the point p define a hyperplane $E^3(p, \xi N)$ in E^4 which intersects the surface $L(p^0)$ along some curve γ. The curve γ lies in the half hyperplane $x_2 = C_2x_1$ too. It follows from here the curve γ is a plane curve and has zero torsion. It follows from here the orbit is surface $L(p^0)$ with zero normal torsion.**Theorem-2.1 is proved.**

2.2. Invariant Solutions of Two Dimensional Heat Equation

In this section we use geometry of vector fields (1) to find invariant solutions of two dimensional heat equation.

Let us consider two dimensional heat equation

$$u_t = \sum_{i=1}^{2}\frac{\partial}{\partial x_i}(k_i(u)\frac{\partial u}{\partial x_i}) + Q(u) \tag{2.9}$$

where $u = u(x_1, x_2, t) \geq 0$ is the temperature function, $k_i(u) \geq 0$, $Q(u)$ are functions of temperature u. The function $Q(u)$ describes the heat release process if $Q(u) > 0$ and the heat absorption process if $Q(u) < 0$. Studies show that the thermal conductivity coefficients $k_1(u), k_2(u)$ in a fairly wide range of parameter changes can be described by a power-law function of temperature, i.e. it has the form $k(u) = u^\sigma$.

For conductivity coefficients $k_1(u) = k_2(u) = u^\sigma$ at $\sigma \geq 0$ invariant solutions of two dimensional heat equation were studied in the papers [6-7].

We will consider the case $\sigma = -1$. In this case symmetry group of the two dimensional heat equation is very "large" i.e. it is infinite-dimensional group.

Consider the case $k_1(u) = k_2(u) = u^{-1}$, $Q(u) = 0$. In this case the equation (2.9) has following form:

$$u_t = u^{-1}\Delta u - u^{-2}(\nabla u)^2 \tag{2.10}$$

where $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ is Laplace operator,$\nabla u = \{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\}$ is the gradient of the function u.

As shown in [8], the vector fields (2.1) are infinitesimal generators of the subgroup of the symmetry group for equation (2.10). This means that the flows of these vector fields (1) generate a group of transformations of the space of variables (t, x_1, x_2, u), which transform the solutions of equation (2.10) into solutions.

Theorem 2.2. *The invariant solutions of equation (2.10) with respect to the group of transformations (2.3),(2.5) generated by vector fields (2.1) are the functions*

$$u = \frac{t}{x_1^2}V(\xi)$$

where

$$V(\xi) = \frac{x_1^2}{2}\frac{tan(\frac{(arctan\frac{x_2}{x_1} - C_2)^2}{4C_1^2} + 1)}{C_1^2(x_1^2 + x_2^2)}, \tag{2.11}$$

C_1, C_2 are arbitrary constants.

Proof. We know the functions

$$I_1(t,x_1,x_2,u)=\frac{t}{x_1^2u}, I_2(t,x_1,x_2,u)=\frac{x_2}{x_1}, \tag{2.12}$$

are invariant functions of the group of transformations (2.3),(2.5).

These invariant functions allow us to seek solution of equation (2.10) in the form

$$u(x_1,x_2,t,u)=\frac{t}{x_1^2}V(\xi), \tag{2.13}$$

where

$$\xi(x_1,x_2)=\frac{x_2}{x_1},, \tag{2.14}$$

Substituting in (2.10) for the function $V(\xi)$ we obtain the following second-order differential equation:

$$(1+\xi^2)\frac{V''V-V'^2}{V^2}+2\xi\frac{V'}{V}=V-2$$

Be integrating this equation, we obtain the general solution in the form (2.11). **The theorem-2.2 is proved.**

Now we consider the case $k_1(u)=k_2(u)=u^{-1}$, $Q(u)=2u$.

In this case the equation (2.9) has following form:

$$u_t=u^{-1}\Delta u-u^{-2}(\nabla u)^2+2u \tag{2.15}$$

We will use following vector fields

$$X_1=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}-2u\frac{\partial}{\partial u}, X_2=e^t\frac{\partial}{\partial t}+e^tu\frac{\partial}{\partial u} \tag{2.16}$$

to find invariant solutions of the equation (2.15).

Theorem 2.3. *The invariant solutions of equation (2.15) with respect to the group of transformations, generated by vector fields (2.16) are the functions*

$$u=C_2\frac{e^{2t}}{(x_1^2+x_2^2)^2}e^{C_1\frac{x_2}{x_1^2+x_2^2}}$$

where C_1, C_2 are arbitrary constants.

Proof. Following functions

$$I_1(t,x_1,x_2,u)=\frac{e^t}{x_2\sqrt{u}}, \tag{2.17}$$

$$I_2(t,x_1,x_2,u)=\frac{x_2}{x_1^2+x_2^2}, \tag{2.18}$$

are invariant functions for the group transformations generated by vector fields (2.16), which follows from equalities

$$X_i(I_j(t,x_1,x_2,u))=0, i=1,2, j=1,2. \tag{2.19}$$

These invariant functions allow us to seek solution of equation (2.15) in the form

$$u(x_1,x_2,t)=\frac{e^{2t}}{x_2^2}V(\xi), \tag{2.20}$$

where

$$\xi(x_1,x_2)=\frac{x_2}{x_1^2+x_2^2} \tag{2.21}$$

Substituting $u(x_1,x_2,t)$ from (2.20) to (2.15) for the function $V(\xi)$ we obtain the following second-order differential equation:

$$\frac{\xi^2V''}{V}-\frac{\xi^2V'^2}{V^2}+2=0. \tag{2.22}$$

By introducing new function $p(\xi) = \frac{V'}{V}$ we have the differential equation for the function $p(\xi)$:

$$\xi^2 p' + 2 = 0. \tag{2.23}$$

We find

$$p(\xi) = -\frac{2}{\xi^2} \tag{2.24}$$

and

$$V = C_2 \xi^2 e^{C_1 \xi}. \tag{2.25}$$

It follows from here

$$u = C_2 \frac{e^{2t}}{(x_1^2 + x_2^2)^2} e^{C_1 \frac{x_2}{x_1^2 + x_2^2}} \tag{2.26}$$

where C_1, C_2 are arbitrary constants. **The theorem-2.3 is proved.**

As flows of the vector fields (2.16) translate solutions (2.26) of the equation (2.15) into solutions of (2.15) from the theorem-2.3 follows

Corollary 2.1. *The vector fields (2.16) are infinitesimal generators of the symmetry group for equation (2.15).*

3. Conclusions

The paper studies the geometry of the singular foliation generated by the orbits of vector fields. It is shown that regular leaves are two-dimensional manifolds of negative Gaussian curvature and zero normal curvature.

In the second part, the geometry of vector fields is used to find invariant solutions of the two-dimensional heat equation. Using vector fields that are infinitesimal generators symmetry group of the heat equation allows you to find a whole family of solutions that are invariant under the symmetry group. For this case, a family of exact solutions is found, depending on arbitrary constants.

In the case when there is no heat source, as formula (11) shows, at each moment of time it remains uniformly limited.

In the case when there is a heat source, the temperature increases exponentially with increasing time.

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