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Research Article

Statistical Relative A−summability for Double Sequences of Positive Linear Operators

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Abstract

In this work, we study the concept of statistical relative A-summability. Based upon this definition and A-statistical relative uniform convergence for double sequences of functions, we prove a Korovkin-type approximation theorem and give a strong example. Also, we compute the rates of convergence of positive linear operators via statistical relative A−summability.

Key Words: Statistical relative convergence, statistical A−summability, the Korovkin theorem, positive linear operator.

Pozitif Lineer Operatörlerin Double Dizileri İçin İstatistiksel Relative A-toplanabilme

Öz

Bu makalede, istatistiksel relative A-toplanabilme kavramı tanıtılmıştır. Fonksiyonların double dizileri için bu tanım ve A-istatistiksel relative düzgün yakınsaklık kullanılarak, Korovkin tipi yaklaşım teoremi ispatlandı ve kuvvetli bir örnek verildi. Ayrıca, pozitif lineer operatörlerin istatistiksel relative A-toplanabilme oranı çalışıldı.

Anahtar Kelimeler: İstatistiksel relative yakınsaklık, istatistiksel A−toplanabilme, Korovkin teoremi, pozitif lineer operatör.

Introduction

Firstly, Korovkin [11] researched the necessary and sufficient conditions for the uniform convergence of $L_{mn}(f)$ to a function f considering the test functions f_i defined by $f_i(x)=x^i$, (i=0,1,2) for a sequence (L_{mn}) of positive linear operators defined on $C(D)$ which is the space of all continuous real valued functions on a compact subset D of $\mathbb{R}$ (all the real numbers) (see, for instance, [1]). Later many researchers have investigated these conditions for various operators defined on different spaces. Furthermore, in recent years, various Korovkin-type approximation theorems have been proved using the concept of statistical convergence [2,3]. Recall that every convergent sequence (in the usual sense) is statistical convergent but its converse is not always true. Also, statistical

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convergent sequences do not need to be bounded. So, the usage of this convergence method in the approximation theory provides us many advantages. Now, we remind some definitions that we will use. A double sequence $x=(x_{mn})$ is said to be convergent in the Pringsheim's sense if, for every $\varepsilon>0$ there exists $N\in\mathbb{N}$, the set of all natural numbers, such that $|x_{mn}-L|<\varepsilon$ whenever $m,n>N$. Here, L is called the Pringsheim limit of x and denoted by $P-\lim x_{mn}=L$ [15]. We shall call such an x more briefly as "$P-$convergent". By a bounded double sequence we mean there exists a positive number K such that $|x_{mn}|<K$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$, two-dimensional set of all positive integers. Recall that a P −convergent double sequence does not have to be bounded. Let $A:=(a_{jkmn})$ be a four-dimensional summability method. For a given double sequence $x=(x_{mn})$, the $A-$transform of x, denoted by $Ax:=((Ax)_{jk})$, is given by

$$(Ax)_{jk}:=\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn}$$

provided the double series converges in the Pringsheim's sense for $(j,k)\in\mathbb{N}^2$. We say that a double sequence $x=(x_{mn})$ is A-summable to L if the A-transform of x exists for all $(j,k)\in\mathbb{N}^2$ and is convergent in the Pringsheim's sense.

In 1926 Robison [16] introduced a four dimensional analog of regularity for double sequences in which he gave an additional assumption of boundedness. This assumption was necessary because a P −convergent double sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices are known as Robison-Hamilton conditions (briefly, RH−regularity) [8,16]. Express that a four dimensional matrix $A=(a_{jkmn})$ is said to be RH−regular if it maps every bounded P −convergent sequence into a P −convergent sequence with the same P −limit. The Robison- Hamilton conditions state that a four dimensional matrix $A=(a_{jkmn})$ is RH−regular if and only if

(i) $P-\lim_{j,k} a_{jkmn}=0$ for each m and n,

(ii) $P-\lim_{j,k}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}=1$,

(iii) for each $n\in\mathbb{N}$, $P-\lim_{j,k}\sum_{m\in\mathbb{N}}|a_{jkmn}|=0$,

(iv) for each $m\in\mathbb{N}$, $P-\lim_{j,k}\sum_{n\in\mathbb{N}}|a_{jkmn}|=0$,

(v) $\sum_{(m,n)\in\mathbb{N}^2}|a_{jkmn}|$ is P-convergent,

(vi) There exists finite positive integers A and B such that $\sum_{m,n>B}|a_{jkmn}|<A$ holds for every $(j,k)\in\mathbb{N}^2$.

Now let $A=(a_{jkmn})$ be a nonnegative $RH-$regular summability matrix, and let $K\subset\mathbb{N}^2$. Then $A-$density of K is given by

$$\delta^2_{(A)}(K)=P-\lim_{j,k}\sum_{(m,n)\in K} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x=(x_{mn})$ is said to be $A-$statistical convergent to L if, for every $\varepsilon>0$,

$$\delta^2_{(A)}\{(m,n)\in\mathbb{N}^2:|x_{m,n}-L|\geq\varepsilon\}=0.$$

In this case, we write $st^2_{(A)}-\lim x=L$. Clearly, each P −convergent double sequence is $A-$statistical convergent to the same value but its converse it is not always true. Also, note that an $A-$statistical convergent double sequence need not to be bounded. We note that if we replace *C(1,1)*, which is double Cesàro matrix, by

A matrix, then we obtain *C(1,1)*-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in ([12],[14]). Finally, if we replace the matrix *A* by the identity matrix for four-dimensional matrices, then $A-$statistical convergence reduces to the Pringsheim convergence. E. H. Moore [13] introduced the concept of relative uniform convergence of a function sequences. Then, E. W. Chittenden [5] stated the definition convergence which is equivalent to the definition given by Moore. Similarly, a double sequence $\left(f_{mn}\right)$ of functions, defined on any compact subset of $\mathbb{R}^2$, converges *relative uniformly to a limit function f* if there exists a function $\sigma(x,y)$ called a scale function $\sigma(x,y)$ such that for every $\varepsilon>0$ there are two integers $n_\varepsilon, m_\varepsilon$ such that for every $n>n_\varepsilon$ and $m>m_\varepsilon$ the inequality

$$|f_{mn}(x,y)-f(x,y)|<\varepsilon|\sigma(x,y)|$$

holds uniformly in (*x,y*). The double sequence (f_{mn}) is said to converge *uniformly relative to the scale function σ* or more simply, *relative uniformly*. It will be observed that if we take the scale function from a non-zero constant, we obtain the concept of uniform convergence of function sequences. (for more properties and details, see also [4-6, 10, 17]). Let *f* and f_{mn} belong to $C(D)$ which is the space of all continuous real valued functions on a compact subset *D* of $\mathbb{R}^2$ and $\|f\|_{C(D)}$ denotes the usual supremum norm of *f* in *C*(*D*).

Definition 1. (f_{mn}) is said to be statistical *relative uniform* convergent to *f* on *D* if there exists a function $\sigma(x,y)$ called a scale function satisfying $|\sigma(x,y)|>0$ such that for every $\varepsilon>0$,

$$\delta^2_{(C(1,1))}\left(\left\{(m,n):\sup_{(x,y)\in D}\left|\frac{f_{mn}(x,y)-f(x,y)}{\sigma(x,y)}\right|\geq\varepsilon\right\}\right)=0.$$

This limit is denoted by

$$(st)^2_{(C(1,1))}-f_{mn}\rightrightarrows f(D;\sigma).$$

Definition 2. Let $A=\left(a_{jkmn}\right)$ be a nonnegative $RH-$regular summability matrix. (f_{mn}) is said to be statistical relative $A-$summable to *f* on *D* if there exists a function $\sigma(x,y)$, $|\sigma(x,y)|>0$, such that for every $\varepsilon>0$,

$$\delta^2_{(C(1,1))}\left(\left\{(m,n):\sup_{(x,y)\in D}\left|\frac{(Af_{mn})(x,y)-f(x,y)}{\sigma(x,y)}\right|\geq\varepsilon\right\}\right)=0.$$

This limit is denoted by

$$(st)^2_{(C(1,1))}-\left(Af_{mn}\right)\rightrightarrows f(D;\sigma).$$

Remark 1. If we replace the matrix *A* in Definition 2 by the identity double matrix, then we immediately get the Definition 1.

Remark 2. If we take *A* = *C(1,1)* = *C* (double Cesàro matrix), then statistical relative $A-$summability is reduced to statistical relative $C-$summability.
Now we present an example as follows :

Example 1. Let *A* be double Cesàro matrix, i.e.

$$c_{jkmn}=\begin{cases}\frac{1}{jk}, & 1\leq m\leq j\, and\, 1;\leq n\leq k,\\ 0, & d.d.\end{cases}$$

and for each $(m,n)\in\mathbb{N}^2$, define $g_{mn}:[0,1]\times[0,1]\to\mathbb{R}$ by

$$g_{mn}(x,y)=\frac{2n^2m^2xy}{1+n^3m^3x^2y^2}. \quad (1)$$

Then observe that

$$(st)^2_{(C(1,1))}-\left(Cg_{mn}\right)\rightrightarrows g=0\ (D;\sigma),\ \text{where}$$

$$\sigma(x,y)=\begin{cases}\frac{1}{xy}, & (x,y)\in(0,1]\times(0,1]\\ 0, & (x,y)=(0,0)\end{cases},$$

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however (g_{mn}) is not statistical (or ordinary) uniform convergent to the function $g=0$ on the interval $[0,1]\times[0,1]$.

2. A Korovkin-type approximation theorem

In this section we prove a Korovkin-type approximation theorem on $C(D)$ using the concept of statistical relative A-summability. Let L be a linear operator on $C(D)$. Then, as usual, we say that L is positive linear operator provided that $f\geq 0$ implies $L(f)\geq 0$. Also, we denote the value of $L(f)$ at a point $(x,y)\in D$ by $L(f(u,v);x,y)$ or, briefly, $L(f;x,y)$.
First, Dirik and Demirci in [7] have studied the statistical version of the Korovkin-type theorem;

Theorem 1.

Let $A=(a_{jkmn})$ be a nonnegative RH-regular summability matrix method. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f\in C(D)$,

$$(st)^2_{(A)}-\left\|L_{mn}f-f\right\|_{C(D)}=0$$

if and only if

$$(st)^2_{(A)}-\left\|L_{mn}(f_i)-f_i\right\|_{C(D)}=0,\quad (i=0,1,2,3)$$

where $f_0(x,y)=1,\ f_1(x,y)=x$, $f_2(x,y)=y,\ f_3(x,y)=x^2+y^2$.
Then we recall the following Korovkin-type result introduced in [9];

Theorem 2.

Let $A=(a_{jkmn})$ be a nonnegative RH-regular summability matrix method. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into $C(D)$. Then, for all $f\in C(D)$,

$$(st)^2_{(C(1,1))}-\left\|\sum_{(m,n)\in\mathbb{N}^2}a_{jkmn}L_{mn}(f)-f\right\|_{C(D)}=0 \qquad (2)$$

if and only if

$$(st)^2_{(C(1,1))}-\left\|\sum_{(m,n)\in\mathbb{N}^2}a_{jkmn}L_{mn}(f_i)-f_i\right\|_{C(D)}=0,\ (i=0,1,2,3). \qquad (3)$$

Theorem 3.

Let $A=(a_{jkmn})$ be a nonnegative RH-regular summability matrix method. Let (L_{mn}) be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f\in C(D)$,

$$(st)^2_{(C(1,1))}-\left(AL_{mn}(f)\right)\rightrightarrows f\ (D;\sigma), \qquad (4)$$

if and only if

$$(st)^2_{(C(1,1))}-\left(AL_{mn}(f_i)\right)\rightrightarrows f_i(D;\sigma),\ (i=0,1,2,3) \qquad (5)$$

where $\sigma(x,y)=\max\{|\sigma_i(x,y)|;i=0,1,2,3\}$, $|\sigma_i(x,y)|>0$ and $\sigma_i(x,y)$ is unbounded, *i=0,1,2,3.*

Proof.

Since each $f_i\in C(D)$, *(i=0,1,2,3),* the implication (4) is obvious. Now we prove the converse part. By the continuity of f on compact set D, we can write

$$|f(x,y)|\leq M$$

where $M=\|f\|_{C(D)}$.
Also, since f is continuous on D, we write that for every $\varepsilon>0$, there exists a number $\delta>0$ such that $|f(u,v)-f(x,y)|<\varepsilon$ for all $(u,v)\in D$ satisfying $|u-x|<\delta$ and $|v-y|<\delta$. Hence, we get

$$|f(u,v)-f(x,y)|<\varepsilon+\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\}. \qquad (6)$$

Using the linearity and the positivity of the operators (L_{mn}) and considering (6), we obtain

$$\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;x,y)-f(x,y)\right|$$
$$\leq\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(\left|f(u,v)-f(x,y)\right|;x,y)\right|$$
$$+\left|f(x,y)\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;x,y)-f_0(x,y)\right|$$
$$\leq\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(\varepsilon+\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\};x,y)\right|$$
$$+M\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;x,y)-f_0(x,y)\right|$$
$$\leq\varepsilon+\left(\varepsilon+M+\frac{2M}{\delta^2}\|f_3\|_{C(D)}\right)\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;x,y)-f_0(x,y)\right|$$
$$+\frac{4M}{\delta^2}\|f_1\|_{C(D)}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_1;x,y)-f_1(x,y)\right|$$
$$+\frac{4M}{\delta^2}\|f_2\|_{C(D)}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_2;x,y)-f_2(x,y)\right|$$
$$+\frac{2M}{\delta^2}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_3;x,y)-f_3(x,y)\right|.$$

Then, we get

$$\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\leq\sup_{(x,y)\in D}\frac{\varepsilon}{|\sigma(x,y)|}$$
$$+K\left\{\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;x,y)-f_0(x,y)}{\sigma_0(x,y)}\right|\right.$$
$$+\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_1;x,y)-f_1(x,y)}{\sigma_1(x,y)}\right|$$
$$+\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_2;x,y)-f_2(x,y)}{\sigma_2(x,y)}\right|$$
$$\left.+\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_3;x,y)-f_3(x,y)}{\sigma_3(x,y)}\right|\right\} \qquad (7)$$

where

$$K=\varepsilon+M+\frac{2M}{\delta^2}\left(\|f_3\|_{C(D)}+2\|f_2\|_{C(D)}+2\|f_1\|_{C(D)}+1\right)$$
$$\sigma(x,y)=\max\left\{\left|\sigma_i(x,y)\right|;i=0,1,2,3\right\}.$$
,

Now, for a given $r>0$, choose $\varepsilon>0$ such that $\sup_{(x,y)\in D}\frac{\varepsilon}{|\sigma(x,y)|}<r.$

Then,

$$R:=\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\geq r\right\}$$

and

$$R_i:=\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in D}\left|\frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_i;x,y)-f_i(x,y)}{\sigma_i(x,y)}\right|\geq\frac{r-\sup_{(x,y)\in D}\frac{\varepsilon}{|\sigma(x,y)|}}{4K}\right\}$$

,i=0,1,2,3.

It follows from (7) that

$$R\subset\bigcup_{i=0}^{3}R_i$$

and so

$$\sum_{(m,n)\in R}a_{jkmn}\subset\sum_{i=0}^{3}\sum_{(m,n)\in R_i}a_{jkmn}.$$

Then using the hypothesis (5), we get

$$(st)^2_{(C(1,1))}-\left(AL_{mn}\right)\rightrightarrows f(D;\sigma),$$

where

$\sigma(x,y)=\max\left\{\left|\sigma_i(x,y)\right|;i=0,1,2,3\right\}$. This completes the proof of the theorem.

Remark 3

If we replace the matrix *A* in Theorem 1 by the identity double matrix, then we get the following theorem.

Theorem 4

Let $A=\left(a_{jkmn}\right)$ be a nonnegative RH-regular summability matrix method. Let $\left(L_{mn}\right)$ be a double sequence of positive linear operators acting from *C(D)* into *C(D)*. Then, for all $f\in C(D)$,

$$(st)^2_{(C(1,1))}-L_{mn}\left(f\right)\rightrightarrows f(D;\sigma)$$

if and only if

$$(st)^2_{(C(1,1))}-L_{mn}\left(f_i\right)\rightrightarrows f_i(D;\sigma_i) \qquad i=0,1,2,3,$$

where

$\sigma(x,y)=\max\left\{\left|\sigma_i(x,y)\right|;i=0,1,2,3\right\}$, $\left|\sigma_i(x,y)\right|>0$

and $\sigma_i(x,y)$ is unbounded, *i=0, 1 , 2, 3.*
To see how our theorem works, we construct the following example:

Example 2:

Let $D=[0,A]\times[0,B]\subset\mathbb{R}^2$ and $A:=C(1,1)=(c_{jkmn})$ double Cesàro matrix, defined by

$$c_{jkmn}:=\begin{cases}\frac{1}{jk}, & 1\le m\le j \text{ and } 1\le n\le k,\\ 0, & \text{otherwise.}\end{cases}$$

Consider the double Szàsz polynomials

$$S_{mn}(f;x,y)=e^{-mx}e^{-ny}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}f\left(\frac{s}{m},\frac{t}{n}\right)\frac{(mx)^s}{s!}\frac{(ny)^t}{t!}$$

on *C(D)*. Using these polynomials, we introduce the foolowing positive linear operators *C(D);*

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))S_{mn}(f;x,y)$$

$$(x,y)\in D \text{ and } f\in C(D), \qquad (8)$$

where $g_{mn}(x,y)$ is given by (1). Then observe that

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))\,f_0(x,y)$$

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))\,f_1(x,y)$$

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))\,f_2(x,y)$$

$$P_{mn}(f;x,y)=(1+g_{mn}(x,y))\left(f_3(x,y)+\frac{x}{m}+\frac{y}{n}\right).$$

Since $(st)^2_{(C(1,1))}-(Cg_{mn})\rightrightarrows g=0\ (D;\sigma)$, where

$$\sigma(x,y)=\begin{cases}\frac{1}{xy}, & (x,y)\in(0,1]\times(0,1]\\ 0, & (x,y)=(0,0)\end{cases},$$

we conclude that

$$(st)^2_{(C(1,1))}-(CP_{mn}(f_i))\rightrightarrows f_i\ (D;\sigma),\quad i=0,1,2,3.$$

So, by Theorem 3, we see that

$$(st)^2_{(C(1,1))}-(CP_{mn}(f))\rightrightarrows f\,(D;\sigma),\quad \text{for all } f\in C(D).$$

However, since (g_{mn}) is not statistical uniform convergent to the function *g* = 0 on the compact set *D*, we can say that Theorem 4 does not work for our operators defined by (8). Furthermore, since (g_{mn}) is not uniformly convergent to the function *g* = 0 on *D*, the classical Korovkin theorem does not work either. Therefore, this application clearly shows that our Theorem 3 is a non-trivial generalization of the classical and the statistical cases of the Korovkin results introduced in [7] and [9], respectively.

2. Rates of convergence in Theorem 3

In this section, using the concept of statistical relative $A-$summability we study the rate of convergence of a sequence of positive linear operators defined on *C*(*D*) with the help of modulus of continuity. Let $f\in C(D)$. Then the modulus of continuity of a function *f* is defined by

$$w(f,\delta)=\sup\left\{|f(u,v)-f(x,y)|:(u,v),(x,y)\in D,\sqrt{(u-x)^2+(v-y)^2}\le\delta\right\}$$

$(\delta>0)$

Now we have the following result.

Theorem 5

Let $A=(a_{jkmn})$ be a nonnegative RH-regular summability matrix metho and let (L_{mn}) be a double sequence of positive linear operators acting from *C(D)* into *C(D)*. Suppose that

a) $(st)^2_{(C(1,1))}-(AL_{mn}(f_0))\rightrightarrows f_0(D;\sigma_0)$,

b) $(st)^2_{(C(1,1))}-w(f,\delta_{mn})\rightrightarrows 0(D;\sigma_1)$, where $\delta_{mn}=\sqrt{\|L_{mn}(\varphi)\|_{C(D)}}$ with

$$\varphi(u,v)=\varphi_{xy}(u,v)=(u-x)^2+(v-y)^2.$$

Then we have, for all $f\in C(D)$,

$$(st)^2_{(C(1,1))}-(AL_{mn}(f))\rightrightarrows f(D;\sigma),$$

where

$$\sigma(x,y)=\max\{|\sigma_0(x,y)|,|\sigma_1(x,y)|,|\sigma_0(x,y)\sigma_1(x,y)|\},\ |\sigma_i(x,y)|>0$$

and $\sigma_i(x,y)$ is unbounded, *i=0,1.*

Proof.

Let $f \in C(D)$ and $(x,y) \in D$. Using the liinearity and positivity of the operators L_{mn},
for all $(m,n) \in \mathbb{N}^2$ and $\delta_{mn} > 0$, we have

$$\left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;x,y) - f(x,y) \right|$$

$$\leq \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(\left| f(u,v) - f(x,y) \right|; x,y) \right|$$

$$+ \left| f(x,y) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y) \right|$$

$$\leq \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}((\frac{(u-x)^2+(v-y)^2}{\delta^2}+1) w(f,\delta); x,y) \right|$$

$$+ \left| f(x,y) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y) \right|$$

$$\leq \frac{w(f,\delta)}{\delta^2} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}((u-x)^2+(v-y)^2; x,y) \right|$$

$$+ w(f,\delta) \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) \right|$$

$$+ \left| f(x,y) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y) \right|$$

$$\leq \frac{w(f,\delta)}{\delta^2} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}((u-x)^2+(v-y)^2; x,y) \right|$$

$$+ w(f,\delta) \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y) \right|$$

$$+ w(f,\delta) + M \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y) \right|$$

where

$M = \| f \|_{C(D)}$ *and* $\delta := \delta_{mn} = \sqrt{\| L_{mn}(\varphi) \|_{C(D)}}$.

This yields that

$$\sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)} \right|$$

$$\leq \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{\left| \sigma_1(x,y) \right|} \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y)}{\sigma_0(x,y)} \right|$$

$$+ M \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y)}{\left| \sigma_0(x,y) \right|} \right|$$

$$+ 2 \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{\left| \sigma_1(x,y) \right|}. \qquad (9)$$

Now given $\varepsilon > 0$, define the following sets:

$$K := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)} \right| \geq \varepsilon \right\}$$

$$K_1 := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{\left| \sigma_1(x,y) \right|} \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y)}{\sigma_0(x,y)} \right| \geq \frac{\varepsilon}{3} \right\}$$

$$K_2 := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{\left| \sigma_1(x,y) \right|} \geq \frac{\varepsilon}{6} \right\}$$

$$K_3 := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y)}{\sigma_0(x,y)} \right| \geq \frac{\varepsilon}{3M} \right\}.$$

It follows from (9) that

$$K \subset \bigcup_{i=1}^{3} K_i.$$

Also, defining

$$K_4 := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \frac{w(f,\delta_{mn})}{\left| \sigma_1(x,y) \right|} \geq \sqrt{\frac{\varepsilon}{3}} \right\}$$

$$K_5 := \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;x,y) - f_0(x,y)}{\sigma_0(x,y)} \right| \geq \sqrt{\frac{\varepsilon}{3}} \right\}.$$

We have

$$K_1 \subset K_4 \cup K_5$$

which yields

$$K \subset \bigcup_{i=2}^{5} K_i.$$

Therefore, using (a) and (b), we get

$$P - \lim_{j,k} \frac{1}{jk} \left\| \left\{ (m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in D} \left| \frac{\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)} \right| \geq \varepsilon \right\} \right\| = 0.$$

So, the proof is completed.

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References

[1] F. Altomare and M. Campiti, Korovkin type approximation theory and its applica-tion, Walter de Gruyter Publ., Berlin (1994).

[2] Anastassiou, G. A. and Duman, O., A Baskakov type generalization of statistical Korovkin theory. J. Math. Anal. Appl. 340, 476-486 (2008).

[3] Anastassiou, G. A. and Duman, O., Statistical fuzzy approximation by fuzzy positive linear operators. Compt. Math. Appl. 55, 573-580, (2008).

[4] E. W. Chittenden, Relative uniform convergence of sequences of functions, Transactions of the AMS, 15, 197-201 (1914).

[5] E. W. Chittenden, On the limit functions of sequences of continuous functions converg-ing relative uniformly, Transactions of the AMS, 20, 179-184 (1919).

[6] E. W. Chittenden, Relative uniform convergence and classification of functions, Transactions of the AMS, 23, 1-15, (1922).

[7] F. Dirik and K. Demirci, Korovkin type approximation theorem for functions of two variables in statistical sense, Turk J. Math. 34, 73-83, (2010).

[8] Hamilton, H. J., Transformations of multiple sequences, Duke Math. J. 2, 29-60 (1936).

[9] S. Karakus, K. Demirci, Korovkin-type approximation theorem for double sequences of positive linear operators via statistical A summability, Results. Math.. 63, 1-13, (2013).

[10] K. Demirci, S. Orhan, Statistical relative uniform convergence of positive linear operators, Results. Math.. 69, 359-367, (2016).

[11] P.P. Korovkin, Linear operators and approximation theory, Hindustan Publ. Co., Delhi, (1960)

[12] Moricz, F., Statistical convergence of multiple sequences, Arch. Math., 81, 82-89, (2003).

[13] E. H. Moore, An introduction to a form of general analysis, The New Haven Mathe-matical Colloquium, Yale University Press, New Haven, (1910).

[14] Mursaleen and Osama H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288, 223-231, (2003).

[15] Pringsheim, A., Zur theorie der zweifach unendlichen zahlenfolgen, Math. Ann., 53, 289-321, (1900).

[16] Robison, G. M., Divergent double sequences and series, Amer. Math. Soc. Transl., 28, 50-73, (1926).

[17] B. Yilmaz, K. Demirci, S. Orhan, Relative modular convergence of positive linear operators, Positivity, 20, 565-577, (2016).