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Research Article

Convergence of a Four-Step Iteration Process for $G-$nonexpansive Mappings in Banach Spaces with a Digraph

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Abstract: This review reckons with iterative scheme of Thianwan to approximate a common fixed point for four $G-$nonexpansive mappings (tersely $G-nm$). We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Keywords: Fixed point, digraph, $G-$nonexpansive mappings.

1. Introduction and Preliminaries

Let X be a Banach space, $K \neq \varnothing, K \subseteq X$. *Directed graph* mostly enrolled qua *digraph* is a double: $G = (V(G), E(G))$, that here $V(G)$ is the set of vertices of graph and $E(G)$ is the set of its edges that involves overall the loops, scilicet $(x,x) \in E(G)$ for all $x \in V(G)$. Given that G enjoys no parallel edges. If x, y occur vertices of G, here a *path* in G ranging x from y of length N is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x = x_0$, $y = x_N$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = \overline{1,N}$. Digraph G is alleged to become *transitive* if, for all $x,y,z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, we acquire $(x,z) \in E(G)$ [2]. A mapping $f : K \to K$ is asserted to become

- *$G-$nonexpansive* (tersely $G-nm$) [3] if it yields (i) $(x,y) \in E(G) \Rightarrow (fx, fy) \in E(G)$ (f preserves edges of G), (ii) $(x,y) \in E(G) \Rightarrow \|fx - fy\| \leq \|x - y\|$;
- *semi-compact* [9] if for $\{x_n\}$ in K with $\|x_n - fx_n\| \to 0$ as $n \to \infty$, there appears a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to f^* \in K$.

The mappings $f_i : K \to K$ are supply condition (A'') [1] if there is a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $0 < g(t)$ for all $t \in (0,\infty)$ such that $\|x - f_i x\| \geq g(d(x, F_f))$ for all $i = \overline{1,k}$, $x \in K$, where $d(x, F_f) = \inf\left\{\|x - f^*\| : f^* \in F_f = \cap_{c=1}^k F(f_c) \neq \varnothing\right\}$.

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Let $x_0 \in V(G)$ and $\Upsilon \subseteq V(G)$. We state that [5], (i) Υ is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in \Upsilon$, (ii) Υ dominates x_0 if for each $x \in \Upsilon$, $(x_0, x) \in E(G)$.

Let G be a digraph such that $V(G) = K$. Then, K is alleged to get property P [8] if for each sequence $\{x_n\}$ in $K \rightharpoonup x \in K$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $(x_{n_l}, x) \in E(G)$ for all $l \in N$.

Remark 1.1 *[6] If G is transitive, then Property P is equal to the speciality: if $\{x_n\} \subseteq K$ with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_l}\}$ of $\{x_n\} \rightharpoonup x \subseteq X$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Phuengrattana and Suantai [15] gave on the rate of convergence of Mann, Ishikawa, Noor and $SP-$iterations for continuous functions on an arbitrary interval. Şahin and Başarır [16] presented on the strong and $\Delta-$convergence of $SP-$iteration on $CAT(0)$ space.

Motivated by [11–13] and above results, the iterative scheme is defined as follows:

$$\begin{aligned} t_n &= (1-\beta_n)\,x_n + \beta_n f_1 x_n, \\ y_n &= (1-\xi_n)\,x_n + \xi_n f_2 t_n, \\ s_n &= (1-\varrho_n)\,y_n + \varrho_n f_3 y_n, \\ x_{n+1} &= (1-\theta_n)\,x_n + \theta_n f_4 s_n,\ n \geq 1, \end{aligned} \tag{1}$$

where $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [0,1]$, for all $i = \overline{1,4}$, $f_i : K \to K$ are $G-nm$. We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Lemma 1.2 *[10] Let X be a uniformly convex Banach space. Suggesting that $0 < b \leq \nu_n \leq c < 1$, $n \geq 1$. Let $\{x_n\}, \{y_n\} \subseteq X$ be such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|\nu_n x_n + (1-\nu_n)\,y_n\| = a$, where $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Main Results

$F_f = \cap_{c=1}^{4} F(f_c) \neq \varnothing$. For $x_0 \in K$, let $\{x_n\}$ be the sequence created by (1).

Proposition 2.1 *Let $u_0 \in F_f$ be such that (x_0, u_0) and (u_0, x_0) are in $E(G)$. Then, (x_n, u_0), (u_0, x_n), (x_n, s_n), (s_n, x_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$.*

Proof We shall demonstrate our deductions by induction. Let $(x_0, u_0) \in E(G)$. By virtue of edge-preserving of f_1, we have $(f_1 x_0, u_0) \in E(G)$, and thus $(t_0, u_0) \in E(G)$ from the convexity of $E(G)$. Due to edge-preserving of f_2, we get $(f_2 t_0, u_0) \in E(G)$. By using the convexity of $E(G)$

and (x_0,u_0), $(f_2t_0,u_0)\in E(G)$, we own $(y_0,u_0)\in E(G)$. As f_3 is edge-preserving, we possess $(f_3y_0,u_0)\in E(G)$ and $(s_0,u_0)\in E(G)$ from the convexity of $E(G)$. Owing to edge-preserving of f_4, $(f_4s_0,u_0)\in E(G)$. Again the convexity of $E(G)$ and (x_0,u_0), $(f_4s_0,u_0)\in E(G)$, we acquire $(x_1,u_0)\in E(G)$. Continuing in this fashion for (x_1,u_0) instead of (x_0,u_0), we get (t_1,u_0), (y_1,u_0), (s_1,u_0), $(x_2,u_0)\in E(G)$.

Suppose that $(x_v,u_0)\in E(G)$ for $v\geq 1$. Because of edge-preserving of f_1, we attain $(f_1x_v,u_0)\in E(G)$, and thus $(t_v,u_0)\in E(G)$ from the convexity of $E(G)$. On account of edge-preserving of f_2, we achieve $(f_2t_v,u_0)\in E(G)$. Using the convexity of $E(G)$ and (x_v,u_0), $(f_2t_v,u_0)\in E(G)$, we obtain $(y_v,u_0)\in E(G)$. Because f_3 is edge-preserving, we own $(f_3y_v,u_0)\in E(G)$ and so $(s_v,u_0)\in E(G)$ from the convexity of $E(G)$. In view of edge-preserving of f_4, $(f_4s_v,u_0)\in E(G)$. Repetition the convexity of $E(G)$ and (x_v,u_0), $(f_4s_v,u_0)\in E(G)$, we belong $(x_{v+1},u_0)\in E(G)$. Repeating the procedure on one occasion for $(x_{v+1},u_0)\in E(G)$, we get (t_{v+1},u_0), (y_{v+1},u_0), (s_{v+1},u_0), $(x_{v+2},u_0)\in E(G)$.

Hence, (x_n,u_0), (t_n,u_0), (y_n,u_0), $(s_n,u_0)\in E(G)$ for $n\geq 1$. Utilizing an analog argumentum, we infer that (u_0,x_n), (u_0,t_n), (u_0,y_n), $(u_0,s_n)\in E(G)$ from $(u_0,x_0)\in E(G)$. As the graph G is transitivity, we acquire for $n\geq 1$ (x_n,s_n), (s_n,x_n), (y_n,x_n), (x_n,y_n), (t_n,x_n), (x_n,t_n) and $(x_n,x_{n+1})\in E(G)$. □

Lemma 2.2 *If K is a nonempty closed convex subset of a real uniformly convex Banach space X, $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\}\subseteq[a,b]$, where $0<a<b<1$ and (x_0,u_0), $(u_0,x_0)\in E(G)$ for $x_0\in K$ and $u_0\in F_f$, then*

(i) $\|x_{n+1}-u_0\|\leq\|x_n-u_0\|$ *for* $n\geq 1$, *and hence* $\|x_n-u_0\|\to 0$ *as* $n\to\infty$;

(ii) $\lim_{n\to\infty}\|x_n-f_ix_n\|=0$ *for all* $i=\overline{1,4}$.

Proof (i) By Proposition 2.1, (x_n,u_0), (u_0,x_n), (s_n,x_n), (x_n,s_n), (y_n,x_n), (x_n,y_n), (x_n,t_n), (t_n,x_n), (u_0,s_n), (s_n,u_0), (u_0,y_n), (y_n,u_0), (u_0,t_n), (t_n,u_0), (x_n,x_{n+1}) are in $E(G)$. It follows from (1) that

$$\begin{aligned}\|t_n-u_0\| &= \|-u_0+(-\beta_n+1)x_n+\beta_nf_1x_n\|\\ &\leq (-\beta_n+1)\|-u_0+x_n\|+\beta_n\|f_1x_n-u_0\|\\ &\leq (-\beta_n+1)\|-u_0+x_n\|+\beta_n\|-u_0+x_n\|\\ &= \|-u_0+x_n\|. \end{aligned}\tag{2}$$

Using (1) & (2), we have

$$\begin{aligned} \|y_n - u_0\| &\leq (1-\xi_n)\|x_n - u_0\| + \xi_n \|f_2 t_n - u_0\| \\ &\leq (1-\xi_n)\|x_n - u_0\| + \xi_n \|t_n - u_0\| \\ &\leq \|x_n - u_0\|. \end{aligned} \tag{3}$$

Similarly, along with (3), we get

$$\begin{aligned} \|s_n - u_0\| &\leq (1-\varrho_n)\|y_n - u_0\| + \varrho_n \|f_3 y_n - u_0\| \\ &\leq (1-\varrho_n)\|y_n - u_0\| + \varrho_n \|y_n - u_0\| \\ &\leq \|y_n - u_0\| \\ &\leq \|x_n - u_0\|. \end{aligned} \tag{4}$$

By (4), we possess

$$\begin{aligned} \|-u_0 + x_{n+1}\| &\leq (-\theta_n + 1)\|-u_0 + x_n\| + \theta_n \|-u_0 + f_4 s_n\| \\ &\leq (-\theta_n + 1)\|-u_0 + x_n\| + \theta_n \|s_n - u_0\| \\ &\leq \|x_n - u_0\|. \end{aligned} \tag{5}$$

Hence, $\lim_{n\to\infty}\|x_n - u_0\|$ exists.

(ii) By assumption (i), $\{x_n\}$ is bounded. Let

$$\lim_{n\to\infty}\|x_n - u_0\| = M. \tag{6}$$

If $M = 0$, then, by $G-nm$ of $\{f_1, f_2, f_3, f_4\}$, it is obvious. Next, suppose $M > 0$. We shall show that, for all $i = \overline{1,4}$, $\|x_n - f_i x_n\| \to 0$ as $n \to \infty$.

Getting $\limsup$ on both parts of (2), (3) & (4), we have

$$\limsup_{n\to\infty}\|t_n - u_0\| \leq M, \tag{7}$$

$$\limsup_{n\to\infty}\|y_n - u_0\| \leq M, \tag{8}$$

$$\limsup_{n\to\infty}\|s_n - u_0\| \leq M. \tag{9}$$

It implies by (7), (8) & (9) and the $G-nm$ of $\{f_1, f_2, f_3, f_4\}$ that

$$\begin{aligned} \|f_1 x_n - u_0\| &\leq \|x_n - u_0\| \\ \limsup_{n\to\infty}\|f_1 x_n - u_0\| &\leq M, \end{aligned} \tag{10}$$

$$\begin{aligned} \|f_2 t_n - u_0\| &\leq \|t_n - u_0\| \\ \limsup_{n\to\infty} \|f_2 t_n - u_0\| &\leq M, \end{aligned} \tag{11}$$

$$\begin{aligned} \|f_3 y_n - u_0\| &\leq \|y_n - u_0\| \\ \limsup_{n\to\infty} \|f_3 y_n - u_0\| &\leq M, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|f_4 s_n - u_0\| &\leq \|s_n - u_0\| \\ \limsup_{n\to\infty} \|f_4 s_n - u_0\| &\leq M. \end{aligned} \tag{13}$$

Since $\lim_{n\to\infty} \|x_{n+1} - u_0\| = M$, we get

$$\lim_{n\to\infty} \|(1-\theta_n)(x_n - u_0) + \theta_n (f_4 s_n - u_0)\| = M. \tag{14}$$

By Lemma 1.2, we obtain

$$\|x_n - f_4 s_n\| \to 0 \text{ as } n \to \infty. \tag{15}$$

Now, using the $G - nm$ of $\{f_1, f_2, f_3, f_4\}$, we have

$$\begin{aligned} \|-u_0 + x_n\| &\leq \|f_4 s_n - u_0\| + \|-f_4 s_n + x_n\| \\ &\leq \|x_n - f_4 s_n\| + \|s_n - u_0\| \qquad (16) \\ &\leq \|x_n - f_4 s_n\| + \|(1-\varrho_n)(y_n - u_0) + \varrho_n (f_3 y_n - u_0)\| \\ &\leq \|x_n - f_4 s_n\| + (1-\varrho_n)\|y_n - u_0\| + \varrho_n \|f_3 y_n - u_0\| \\ &\leq \|x_n - f_4 s_n\| + \|y_n - u_0\| \qquad (17) \\ &\leq \|x_n - f_4 s_n\| + \|(1-\xi_n)(x_n - u_0) + \xi_n (f_2 t_n - u_0)\| \\ &\leq \|x_n - f_4 s_n\| + (1-\xi_n)\|x_n - u_0\| + \xi_n \|f_2 t_n - u_0\| \\ &\leq \frac{1}{\xi_n}\|x_n - f_4 s_n\| + \|t_n - u_0\| \\ &\leq \frac{1}{a}\|x_n - f_4 s_n\| + \|t_n - u_0\|. \qquad (18) \end{aligned}$$

Taking $\liminf$ on both sides of (16), (17), (18) and using (15), we obtain

$$M \leq \liminf_{n\to\infty} \|s_n - u_0\|, \tag{19}$$

$$M \leq \liminf_{n\to\infty} \|y_n - u_0\|, \tag{20}$$

$$M \leq \liminf_{n\to\infty} \|t_n - u_0\|, \tag{21}$$

respectively.

By combining (7) & (21), (8) & (20), (9) & (19), we get

$$\lim_{n\to\infty}\|t_n-u_0\|=\lim_{n\to\infty}\|y_n-u_0\|=\lim_{n\to\infty}\|s_n-u_0\|=M, \tag{22}$$

respectively. Namely,

$$\begin{aligned}\lim_{n\to\infty}\|(1-\beta_n)(x_n-u_0)+\beta_n(f_1x_n-u_0)\| &= M,\\ \lim_{n\to\infty}\|(1-\xi_n)(x_n-u_0)+\xi_n(f_2t_n-u_0)\| &= M,\\ \lim_{n\to\infty}\|(1-\varrho_n)(y_n-u_0)+\varrho_n(f_3y_n-u_0)\| &= M,\end{aligned}$$

respectively. It follows from (6), (8), (10), (11) & (12) and Lemma 1.2 that

$$\lim_{n\to\infty}\|x_n-f_1x_n\| = 0, \tag{23}$$

$$\lim_{n\to\infty}\|x_n-f_2t_n\| = 0, \tag{24}$$

$$\lim_{n\to\infty}\|y_n-f_3y_n\| = 0, \text{ resp.} \tag{25}$$

It implies by (23) & (24) that

$$\begin{aligned}\|x_n-f_2x_n\| &\le \|x_n-f_2t_n\|+\|f_2t_n-f_2x_n\|\\ &\le \|x_n-f_2t_n\|+\|t_n-x_n\|\\ &\le \|x_n-f_2t_n\|+\beta_n\|f_1x_n-x_n\|\\ &\le \|x_n-f_2t_n\|+b\|f_1x_n-x_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{26}$$

By (1) & (24), we have

$$\begin{aligned}\|x_n-y_n\| &= \|x_n-[(1-\xi_n)x_n+\xi_nf_2t_n]\|\\ &\le \xi_n\|x_n-f_2t_n\|\\ &\le b\|x_n-f_2t_n\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{27}$$

It follows from (25) & (27), we get

$$\begin{aligned}\|x_n-f_3x_n\| &\le \|-y_n+x_n\|+\|y_n-f_3y_n\|+\|f_3y_n-f_3x_n\|\\ &\le \|-y_n+x_n\|+\|y_n-f_3y_n\|\\ &\quad+\|-x_n+y_n\| \to 0 \text{ as } n\to\infty.\end{aligned} \tag{28}$$

By (1), (25) & (27), we have

$$\begin{aligned}
\|s_n - x_n\| &\leq \|-y_n + s_n\| + \|y_n - x_n\| \\
&= \|[(1-\varrho_n) y_n + \varrho_n f_3 y_n] - y_n\| + \|-x_n + y_n\| \\
&\leq \varrho_n \|y_n - f_3 y_n\| + \|-x_n + y_n\| \\
&\leq b \|y_n - f_3 y_n\| + \|-x_n + y_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (29)
\end{aligned}$$

Using (15) & (29), we obtain

$$\begin{aligned}
\|x_n - f_4 x_n\| &\leq \|x_n - f_4 s_n\| + \|f_4 s_n - f_4 x_n\| \\
&\leq \|x_n - f_4 s_n\| \\
&\quad + \|s_n - x_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (30)
\end{aligned}$$

From (23), (26), (28) & (30), we get

$$\|x_n - f_i x_n\| \to 0 \text{ as } n \to \infty \text{ for all } i = \overline{1,4}. \qquad (31)$$

□

Theorem 2.3 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a,b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that (x_0, u_0), (u_0, x_0) are in $E(G)$ for $x_0 \in K$. Supposing that K hold the property P, $\{f_1, f_2, f_3, f_4\}$ satisfy the condition(A''), F_f is dominated by x_0 and F_f dominates x_0, then $\{x_n\} \longrightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that (x_n, u_0), (u_0, x_n),(s_n, x_n), (x_n, s_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. Due to Lemma 2.2 (ii) and *condition* (A''), we attain that $\lim_{n\to\infty} g(d(x_n, F_f)) = 0$. As g is nondecreasing with $g(0) = 0$, we hold $d(x_n, F_f) \to 0$ as $n \to \infty$. Thus, we can receive a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and $\{u_l^*\} \subset F_f$ such that $\|x_{n_l} - u_l^*\| < 2^{-l}$. Due to the fact that strong convergence implies weak convergence and by Remark 1.1, we hold $(x_{n_l}, u_l^*) \in E(G)$. Using the proof method of [11], we own

$$\|x_{n_{l+1}} - u_l^*\| \leq \|x_{n_l} - u_l^*\| < \frac{1}{2^l},$$

and so

$$\|-u_{l+1}^* + u_l^*\| \leq \|-x_{n_{l+1}} + u_l^*\| + \|-u_{l+1}^* + x_{n_{l+1}}\| \leq 3.2^{-(1+l)}.$$

We deduce that $\{u^*_{l+1}\}$ is a Cauchy sequence. Therefore, we have $u^*_l \to r$. By closed of F_f, $r \in F_f$ in that case $x_{n_l} \to r$. Because of Lemma 2.2 (i), $x_n \to r \in F_f$. □

Theorem 2.4 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a,b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that (x_0,u_0), (u_0,x_0) are in $E(G)$ for $x_0 \in K$. Supposing that K has the property P and one of $\{f_1,f_2,f_3,f_4\}$ is semi-compact, F_f is dominated by x_0 and F_f dominates x_0, then $\{x_n\} \longrightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that (x_n,u_0), (u_0,x_n), (x_n,s_n), (s_n,x_n), (x_n,y_n), (y_n,x_n), (x_n,t_n), (t_n,x_n), (u_0,s_n), (s_n,u_0), (u_0,y_n), (y_n,u_0), (u_0,t_n), (t_n,u_0), (x_n,x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. We have $\lim_{n\to\infty}\|x_n - f_j x_n\| = 0$ from Lemma 2.2 (ii). Assume that f_j is semi-compact for all $j = \overline{1,4}$. Then, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l\to\infty}\|x_{n_l} - \upsilon\| = 0$ for some $\upsilon \in K$. This together with Remark 1.1 implies that $(x_{n_l},\upsilon) \in E(G)$. It follows from the $G-nm$ of $\{f_1,f_2,f_3,f_4\}$ and Lemma 2.2 (ii) that

$$\begin{aligned} \|\upsilon - f_j\upsilon\| &\le \|\upsilon - x_{n_l}\| + \|x_{n_l} - f_j x_{n_l}\| + \|f_j x_{n_l} - f_j \upsilon\| \\ &\to 0 \text{ as } l \to \infty, \end{aligned}$$

for all $j = \overline{1,4}$. Hereat, $\upsilon \in F_f$ so that $\lim_{n\to\infty}\|x_n - \upsilon\|$ exists. Thus, $x_n \to \upsilon$ as $n \to \infty$. □

We indicate an instance which is inspired by Example 4.5 in [7].

Example 2.5 $K = [0,2] \subseteq X = \mathbb{R}$. *Let G be a digraph described by $V(G) = K$ and $(x,y) \in E(G)$ iff $1.20 \ge y \ge x \ge 0.50$. Denote $\{f_1,f_2,f_3,f_4\} : K \to K$ by $f_1 x = 1 + \frac{23}{49}\tan(-1+x)$, $f_2 x = 1 + \frac{29}{45}\tan(-1+x)$, $f_3 x = 1 + \frac{23}{49}\arcsin(-1+x)$, $f_4 x = 1 + \frac{29}{45}\arcsin(-1+x)$ for any $x \in K$ and $i = 1,2,3,4$. It is easy to see that f_1,f_2,f_3,f_4 are $G-nm$, but f_1,f_2,f_3,f_4 are not nonexpansive. Let $\beta_n = \frac{6n+5}{8n+15}$, $\xi_n = \frac{3n+1}{9n+20}$, $\varrho_n = \frac{10n+3}{11n+4}$, $\theta_n = \frac{7n+11}{13n+47}$ for $n \ge 1$. $F_f = \cap_{c=1}^4 F(f_c) = \{1\}$ as in Figure 1.*

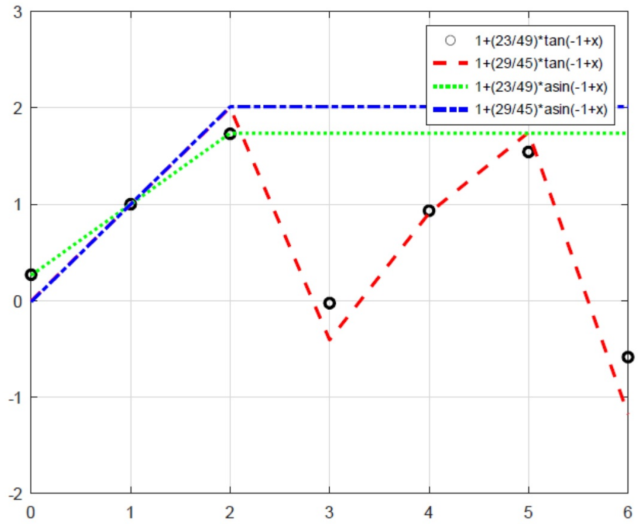

Figure 1: Plot showing $F_f = \cap_{c=1}^{4} F(f_c) = \{1\}$

Table 1 The value of the sequence $\{x_n\}$ *with initial value* $x_0 = 1.20000$, $x_0 = 0.80000$ *and* $n = 20$, *respectively.*

n	x_n	x_n
1	1.20000	0.80000
2	1.15950	0.84047
3	1.12180	0.87822
4	1.09010	0.90994
5	1.06500	0.93499
6	1.04600	0.95395
7	1.03210	0.96788
8	1.02210	0.97787
9	1.01510	0.98492
10	1.01020	0.98981
11	1.00680	0.99317
12	1.00450	0.99545
13	1.00300	0.99699
14	1.00200	0.99802
15	1.00130	0.99870
16	1.00090	0.99915
17	1.00060	0.99945
18	1.00040	0.99964
19	1.00030	0.99977
20	1.00020	0.99985

Remark 2.6 *(i) If* $\xi_n \equiv 0$ *and* $f_1 = f_2 = f_3 = f_4 = f$ *in (1), then Theorem 2.3 generalize the results of Theorem 3.6 in [14] for self-map.*

(ii) If $\xi_n = \varrho_n \equiv 0$ *and* $f_1 = f_2 = f_3 = f_4 = f$ *in (1), we attain convergence of the Mann iteration to some fixed points of* f *on Banach space involving a digraph.*

(iii) If $f_1 = f_2 = f_3 = f_4 = f$ *in (1), then Theorem 2.3 extends the results of [12] without errors for self-map.*

(iv) If $f_1 = f_2$, $f_3 = f_4$ *in (1), then Theorem 2.3 improves the results of [13] without errors for self-map.*

(v) If $\xi_n \equiv 0$ *in (1), then Theorem 2.4 reduces to the results of [4].*

3. Conclusion

In this writting, we reckons with four step iteration scheme to common fixed points of four $G-nm$ described on Banach space involving a digraph. Our findings evolve the equal results of Shahzad (2005) [14], Thianwan (2008) [12], Kızıltunç et al. (2010) [13] and Tripak (2016) [4]. Within the future scope of the idea, reader can show that (1) compare convergence rate Picard, Mann, Ishikawa and $SP-$iteration process for contractions.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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