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CLASSIFICATION OF FINITE SIMPLICIAL ALGEBRAS

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ABSTRACT

We get a classification of 1-truncated simplicial algebras under certain conditions by using GAP

Keywords: Simplicial algebra, Moore complex, Crossed modules.

1. INTRODUCTION

Simplicial commutative algebras occupy a place somewhere between homological algebra, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time. Their own internal structure has however been studied relatively little [1, 2].

The concept of a crossed module originates in the work [16] of Whitehead in algebraic topology. There the crossed modules were crossed modules of groups. Areas in which crossed modules have been applied include the theory of group presentations, algebraic K-theory, and homological algebra. The commutative algebra version of crossed modules has been used, in essence rather than in name, by Lichtenbaum and Schlessinger [12] also the work of Gerstenhaber [11] essentially involves the notion of crossed modules in commutative algebras. Some categorical results and Koszul complex link are also given by Porter in [14].

A share package XMod, [15], for the GAP computational discrete algebra system was described by C.D. Wensley et al. The 2-dimensional part of this programme contains functions for computing crossed modules and cat1-groups and their morphisms. Arvasi and Odabas describe a package XModAlg [6] for GAP4 which constructs crossed modules of k-algebras and cat1-algebras over k, and their morphisms (see [7]).

By a similar way, we give a GAP implementation for classification of finite simplicial algebras. For this, we added some new functions which do not exist in XModAlg package. One of our main results is the GAP implementation of the equivalent categories crossed modules of algebras and that of simplicial algebras with Moore complex of length 1.

2. SIMPLICIAL ALGEBRAS

In this section we recall a few well-known definitions and facts about simplicial algebras and homology modules. For more details regarding this, we refer to the book *Homologie des algèbres commutatives* by M.André [1]. Let **k** be a fixed commutative ring with $1 \neq 0$. $E_n (n \in \mathrm{N})$

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A simplicial algebra **E** is a collection of **k**-algebras $E_n (n \in \mathbb{N})$ together with, for each $n \geq 0$, **k**-algebra homomorphisms

$$\begin{array}{llll} d_i^n: & E_n & \longrightarrow & E_{n-1} \quad 0 \leq i \leq n \neq 0, \\ s_j^n: & E_n & \longrightarrow & E_{n+1} \quad 0 \leq j \leq n, \end{array}$$

which are called face operators and degeneracies respectively. These homomorphisms are required to satisfy the following axioms:

$$\begin{array}{llcll} 1. & d_i^{n-1}d_j^n & = & d_{j-1}^{n-1}d_i^n & \text{for } 0 \leq i < j \leq n, \\ 2. & s_i^{n+1}s_j^n & = & s_{j+1}^{n+1}s_i^n & \text{for } 0 \leq i \leq j \leq n, \\ 3. & d_i^{n+1}s_j^n & = & s_{j-1}^{n-1}d_i^n & \text{for } 0 \leq i < j \leq n, \\ 4. & d_i^{n+1}s_j^n & = & id & \text{for } i = j \text{ or } i = j+1, \\ 5. & d_i^{n+1}s_j^n & = & s_j^{n-1}d_{i-1}^n & \text{for } 0 \leq j < i-1 \leq n. \end{array}$$

A homomorphism of simplicial algebras $\boldsymbol{f}: \boldsymbol{E} \to \boldsymbol{F}$ is a set of **k**-algebra homomorphisms $f_n: E_n \to F_n$ commuting with all the face operators. We have thus defined the category of simplicial algebras, which we will denote by **SimpAlg**.

2.1. The Moore Complex of a Simplicial Algebra

Recall that given a simplicial algebra **E**, *the Moore complex* $(\mathbf{NE}, \partial)$ *of* **E** is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n: NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The n^{th} homotopy module $\pi_n(\mathbf{E})$ of **E** is the n^{th} homology of the Moore complex of **E**, i.e.,

$$\begin{array}{lll} \pi_n(\mathbf{E}) & \cong & H_n(\mathbf{NE}, \partial) \\ & = & \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1} (\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}). \end{array}$$

By a *k-truncated simplicial algebra*, we mean a simplicial algebra $\mathbf{tr_k E}$ obtained by forgetting dimensions of order > *k* in a simplicial algebra **E**. We denote the category of k-truncated simplicial algebras by $\mathbf{Tr_k SimpAlg}$. Recall from [9] some facts about the skeleton functor. In the category of algebras, **Alg**, there is a truncation functor

$$\mathbf{tr_k}: \mathbf{SimpAlg} \longrightarrow \mathbf{Tr_k SimpAlg}$$

which admits a right adjoint

$$\mathbf{cosk_k}: \mathbf{Tr_k SimpAlg} \longrightarrow \mathbf{SimpAlg}$$

called the *k-coskeleton functor*, and a left adjoint

$$\mathbf{sk_k}: \mathbf{Tr_k SimpAlg} \longrightarrow \mathbf{SimpAlg}$$

called the *k-skeleton functor*.

3. CROSSED MODULES

J.H.C.Whitehead [16] described crossed modules in various contexts especially in his investigations into the algebraic structure of relative homotopy groups. In this section, we introduce the definition and elementary theory of crossed modules of commutative algebras given by T.Porter, [14]. More details about this may be found in [3, 4, 5].

Let *R* be a **k**-algebra with identity. A **pre-crossed module of commutative algebras** is an *R*-algebra *C*, together with a commutative action of *R* on *C* and an *R*-algebra morphism

$$\partial : C \longrightarrow R,$$

such that for all $c \in C, r \in R$

$$CM1) \quad \partial(r \cdot c) \quad = \quad r\,\partial c.$$

This is a crossed R-module if in addition, for all $c, c' \in C$,

$$CM2) \quad \partial c \cdot c' \quad = \quad cc'.$$

The last condition is called *the Peiffer identity*. We denote such a crossed module by (C, R, ∂). Clearly any crossed module is a pre-crossed module.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of k-algebra morphisms,

$$\theta : C \longrightarrow C', \psi : R \longrightarrow R',$$

such that

$$\theta(r \cdot c) = \psi(r) \cdot \theta(c) \text{ and } \partial'\theta(c) = \psi\,\partial(c).$$

In this case, we shall say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity. We therefore can define the category of crossed modules denoting it as **XMod**.

Examples

1. Let I be any ideal of a **k**-algebra R. Consider an inclusion map

$$inc.: I \longrightarrow R.$$

Then $(I, R, inc.)$ is a crossed module. Conversely given any crossed R-module $\partial : C \to R$, one can easily verify that $\partial C = I$ is an ideal in R.

2. Let M be any R-module. It can be considered as an R-algebra with zero multiplication, and then $\mathbf{0}: M \to R$ is a crossed R-module by $\mathbf{0}(c) \cdot c' = 0c' = 0 = cc'$, for all $c, c' \in C$.

Conversely, given any crossed modüle $\partial : C \to R$, then $Ker\,\partial$ is an $R/\,\partial C - \text{module}$.

3. Assume given a simplicial algebra **E** and a simplicial ideal **I**. The inclusion

$$inc.: \mathbf{I} \hookrightarrow \mathbf{E},$$

induces a map

$$\partial : \pi_0(\mathbf{I}) \longrightarrow \pi_0(\mathbf{E}),$$

and **E** acting on **I** by multiplication induces an action of $\pi_0(\mathbf{E})$ on $\pi_0(\mathbf{I})$. Then $(\pi_0(\mathbf{I}), \pi_0(\mathbf{E}), \partial)$ is a crossed module.

Proposition 1 *If (C,R,∂) is a crossed R-module, then*
i) $Ker\, \partial$ is a central ideal of C,
ii) both C/C^2 and $Ker\, \partial$ have natural $R/\,\partial C$-module structure (See [13]).

Theorem 2 *The category of crossed modules is equivalent to the category of simplicial algebras with Moore complex of length* 1. *(See Arvasi and Porter, [3])*

Proof. Let **E** be a simplicial algebra with Moore complex of length 1. Put $M = NE_1, N = NE_0$ and $\partial_1 = d_1$ (restricted to *M*). Then NE_0 acts on NE_1 by multiplication via s_0. Since the Moore complex is of length 1, we have $\partial_2 NE_2 = Kerd_0 Kerd_1 = 0$ and the generators of this ideal are of the form $x(s_0 d_1 y - y)$ with $x, y \in NE_1$. It then follows that for all $x, x' \in M$,

$$\begin{aligned} \partial_1(x) \cdot x' &= d_1(x) \cdot x' & \\ &= s_0 d_1(x) x' & \text{by the action,} \\ &= xx' & \text{since } \partial_2 NE_2 = 0. \end{aligned}$$

Thus $\partial_1 : M \to N$ is a crossed module. This yields a functor

$$\mathbf{N_1} : \mathbf{SimpAlg}_{\leq 1} \longrightarrow \mathbf{XMod}.$$

Conversely, let $\partial_1 : M \to N$ be a crossed module. By using the action of *N* on *M*, one forms the semidirect product $M \rtimes N$ together with homomorphisms

$$d_0(m,n) = n, d_1(m,n) = \partial_1 m + n, s_0(n) = (0,n).$$

Define $E_0 = N$ and $E_1 = M \rtimes N$. Then we have a 1-truncated simplicial algebra $\mathbf{E_{\leq 1}}$.

There is a functor $t_{1]}$ from the category of 1-truncated simplicial algebras to that of simplicial algebras. This enables us to define a functor

$$\mathbf{XMod} \longrightarrow \mathbf{SimpAlg}_{\leq 1},$$

given by sending $\{M, N, \partial\}$ to $\mathbf{E} = t_{1]}\mathbf{E_{\leq 1}}$. **E** is a simplicial algebra whose Moore complex is of length 1. The correspondence gives rise to an equivalence of categories.

4. APPLICATIONS

GAP [10] is an open-source system for discrete computational algebra. The system consists of a library of implementations of mathematical structures: groups, vector spaces, modules, algebras, graphs, codes, designs, etc.; plus databases of groups of small order, character tables, etc. The system has world wide usage in the area of education and scientific research. GAP is free software and user contributions to the system are supported. These contributions are organized in a form of GAP packages and are distributed together with the system. Contributors can submit additional packages for inclusion after a reviewing process.

The Small Groups library provides access to descriptions of the groups of small order up to isomorphism. There is no equivalent library of small algebras in GAP. For (commutative) algebras, we will concentrate on group rings of abelian groups over finite fields, because these algebras are conveniently implemented in GAP. We recall some basic properties of group algebras as follows.

Group Algebras : Let $\mathbf{k}$ be a field and G a multiplicative group, finite or infinite. It is well known that the group algebra $\mathbf{k}G$ is an associative $\mathbf{k}$-algebra with a set $\{e_g : g \in G\}$ as a basis and with multiplication defined distributively using the group multiplication in G.

For $\sigma: G \to H$ a group homomorphism, let the map $\mathbf{k}\sigma$ be the group algebra homomorphism where:

$$\begin{aligned} \boldsymbol{k}\sigma: \ \boldsymbol{k}G &\to \boldsymbol{k}H \\ e_g &\mapsto e_{\sigma(g)}. \end{aligned}$$

In particular $\mathbf{k}\mathrm{id}_G = \mathrm{id}_{\mathbf{k}G}$, and if $\sigma': H \to J$ is a second group homomorphism, then $\mathbf{k}(\sigma * \sigma') = \mathbf{k}\sigma * \mathbf{k}\sigma'$. These facts are summarized in the following proposition (see [8]).

Proposition 3 $\mathbf{k}(.)$: **Gr** $\to$ **Alg** *is a functor.*

The group algebra functor provides a canonical construction for $\mathbf{k}$-algebra from any given group. Conversely, there are at least two canonical ways of extracting a group from a given $\mathbf{k}$-algebra. One is to forget the multiplication and take the additive (abelian) group of the algebra; this gives the forgetful functor **Alg** $\to$ **Ab**. Alternatively, the subset of the algebra consisting of elements which are invertible under multiplication forms a subgroup (with the operation of multiplication) called the group of units of the algebra; this gives a functor $\mathbf{u}(.)$: **Alg** $\to$ **Gr**. In general, the group of units of a non-commutative algebra need not be abelian.

Proposition 4 *The group algebra functor* $\mathbf{k}(.)$: **Gr** $\to$ **Alg** *is left adjoint to the unit group functor* $\mathbf{u}(.)$: **Alg** $\to$ **Gr**.

Proof. Let G be a group and A a $\mathbf{k}$-algebra, and suppose $f: G \to u(A)$. Define a map $\omega_{G,A}: \mathbf{Gr}(G, u(A)) \to \mathbf{Alg}(\mathbf{k}G, A)$ by

$$\omega_{G,A}(f)(\mathbf{e}_g) := \mathbf{e}_{f(g)}$$

(this defines $\omega_{G,A}$ completely, since $\{\mathbf{e}_g : g \in G\}$ is a basis for $\mathbf{k}G$ and, for every $g \in G$, $\omega_{G,A}(f)(\mathbf{e}_g) \in \mathbf{k}u(A) \subseteq A$).

Suppose $\psi: \mathbf{k}G \to A$. Then ψ is completely determined by $\{\psi(\mathbf{e}_g) : g \in G\}$, and for each $g \in G$

$$1_A = \psi(\mathbf{e}_g \mathbf{e}_{g^{-1}}) = \psi(\mathbf{e}_g)\psi(\mathbf{e}_{g^{-1}})$$

so $\psi(\mathbf{e}_g) \in u(A)$. Define the map $\varpi_{G,A}: \mathbf{Alg}(\mathbf{k}G, A) \to \mathbf{Gr}(G, u(A))$ by

$$\varpi_{G,A}(\psi)g = \psi(\mathbf{e}_g).$$

Now $[\varpi_{G,A}\omega_{G,A}(f)](g)$ and $[\omega_{G,A}\varpi_{G,A}(\psi)](g) = \psi(g)$, so $\omega_{G,A}$ is a bijection and

$$\mathbf{Gr}(G, u(A)) \cong \mathbf{Alg}(\mathbf{k}G, A)$$

as required. It is easy to see that ω and ϖ are natural in G and A. The remaining cases are proved similarly (see [8]).

Remark : The functor **k**(.) does not extend to actions and semidirect products. In particular, an action of a group R on a group S does not extend naturally to an action of **k**R on **k**S.

We have developed functions for GAP4 which construct simplicial algebras. Functions to construct simplicial algebras **SimplicialAlgebra**, **SimplicialAlgebraTr1ByFaceDegenere**, **SimplicialAlgebraTr1ByEndomorphisms**, **SimplicialAlgebraTr1**, **IsSimplicialAlgebraTr1** and **IsSimplicialAlgebra**. Attributes of a simplicial algebra constructed in this way include **Source**, **Range**, **Face**, **Degenere**, **Size** and **Name**.

In the following GAP session, we construct a simplicial algebra by using the group algebra GF_2C_6 and GF_2C_3. Also we show usage of the attributes listed above.

```
gap> K := GF(2);
GF(2)
gap> G := SmallGroup(6,2);
<pc group of size 6 with 2 generators>
gap> StructureDescription(G);
"C6"
gap> H := SmallGroup(3,1);
<pc group of size 3 with 1 generators>
gap> StructureDescription(H);
"C3"
gap> KG := GroupRing(K,G);
<algebra-with-one over GF(2), with 2 generators>
gap> IsAlgebra(KG);
true
gap> KH := GroupRing(K,H);
<algebra-with-one over GF(2), with 1 generators>
gap> IsAlgebra(KH);
true
gap> f := AllHomsOfAlgebras(KG,KH);;
gap> g := AllHomsOfAlgebras(KH,KG);;
gap> SA := SimplicialAlgebraTr1ByFaceDegenere(f[6],f[6],g[6]);
[AlgebraWithOne( GF(2), [ (Z(2)^0)*f1, (Z(2)^0)*f2 ] ) ->
AlgebraWithOne( GF(2), [ (Z(2)^0)*f1 ] )]
gap> IsSimplicialAlgebra(SA);
true
gap> IsSimplicialAlgebraTr1(SA);
true
gap> Face(SA);
[ [ (Z(2)^0)*f1*f2 ] -> [ (Z(2)^0)*f1 ], [ (Z(2)^0)*f1*f2 ] ->
[ (Z(2)^0)*f1 ] ]
gap> Degenere(SA);
[ [ (Z(2)^0)*f1 ] -> [ (Z(2)^0)*f2 ] ]
gap> Size(SA);
[ 64, 8 ]
gap> Display(SA);
1-Truncated Simplicial Algebra [..=>..] :-
: source algebra has generators:
[ (Z(2)^0)*<identity> of ..., (Z(2)^0)*f1, (Z(2)^0)*f2 ]
: range algebra has generators:
[ (Z(2)^0)*<identity> of ..., (Z(2)^0)*f1 ]
```

```
: face operator d10 maps source generators to:
[ (Z(2)^0)*<identity> of ..., (Z(2)^0)*<identity> of ...,
(Z(2)^0)*f1 ]
: face operator d11 maps source generators to:
[ (Z(2)^0)*<identity> of ..., (Z(2)^0)*<identity> of ...,
(Z(2)^0)*f1 ]
: degenere operator s00 maps range generators to:
[ (Z(2)^0)*<identity> of ..., (Z(2)^0)*f2 ]
```

We have developed the function **AllSimplicialAlgebrasTr1** which constructs all 1-truncated simplicial algebras. On the other hand, the function **IsMooreComplex1** is used to verify the Moore complex of length 1.

In the following GAP session, we construct all 248 1-truncated simplicial algebras by using the group algebra GF_3Kl_4. 25 of all 248 1-truncated simplicial algebras are Moore complex of length 1.

```
gap> allSA := AllSimplicialAlgebrasTr1(GF(3),Group((1,2),(3,4)));;
gap> Length(allSA);
248
gap> MC1 := Filtered(allSA, SA -> IsMooreComplex1(SA));
gap> Length(MC1);
25
```

By using the natural equivalence of categories of crossed modules and the category of simplicial algebras with Moore complex of length 1, we have developed the functions **XModAlgebraBySimplicialAlgebra** and **SimplicialAlgebraByXModAlgebra** which constructs crossed modules and simplicial algebras from the given simplicial algebras and crossed modules, respectively.

In the following GAP session, we get a crossed module from a simplicial algebras with Moore complex of length 1.

```
gap> SA1 := allSA[1];
[AlgebraWithOne( GF(3), [ (Z(3)^0)*(1,2), (Z(3)^0)*(3,4) ] ) ->
Algebra( GF(3),
[ <zero> of ..., <zero> of ... ] )]
gap> IsMooreComplex1(SA1);
false
gap> XModAlgebraBySimplicialAlgebra(SA1);
SA must be 1-truncated simplicial algebra.
fail
gap> SA2 := allSA[17];
[AlgebraWithOne( GF(3), [ (Z(3)^0)*(1,2), (Z(3)^0)*(3,4) ] ) ->
Algebra( GF(3), [ (Z(3)^0)*(), (Z(3)^0)*(1,2) ] )]
gap> IsMooreComplex1(SA2);
true
gap> Display(SA2);
SimplicialAlgebraTr1 [..=>..] :-
: source algebra has generators:
[ (Z(3)^0)*(), (Z(3)^0)*(1,2), (Z(3)^0)*(3,4) ]
: range algebra has generators:
[ (Z(3)^0)*(), (Z(3)^0)*(1,2) ]
: face operator d10 maps source generators to:
[ (Z(3)^0)*(), (Z(3)^0)*(1,2), (Z(3)^0)*() ]
: face operator d11 maps source generators to:
```

```
[ (Z(3)^0)*(), (Z(3)^0)*(1,2), (Z(3))*() ]
: degenere operator s00 maps range generators to:
[ (Z(3)^0)*(), (Z(3)^0)*(1,2) ]
gap> Size(SA2);
[ 81, 9 ]
gap> CM := XModAlgebraBySimplicialAlgebra(SA2);
[Algebra( GF(3), [ (Z(3)^0)*()+(Z(3)^0)*(3,4),
(Z(3)^0)*(1,2)+(Z(3)^0)*(1,2)(3,4)] ) ->
Algebra( GF(3), [ (Z(3)^0)*(), (Z(3)^0)*(1,2) ] )]
gap> IsXModAlgebra(CM);
true
gap> Display(CM);
Crossed module [..->..] :-
: Source algebra has generators:
[ (Z(3)^0)*()+(Z(3)^0)*(3,4), (Z(3)^0)*(1,2)+(Z(3)^0)*(1,2)(3,4) ]
: Range algebra has generators:
[ (Z(3)^0)*(), (Z(3)^0)*(1,2) ]
: Boundary homomorphism maps source generators to:
[ (Z(3))*(), (Z(3))*(1,2) ]
gap> SA3 := SimplicialAlgebraByXModAlgebra(CM);
[AlgebraWithOne( GF(3), [ (Z(3)^0)*(1,2), (Z(3)^0)*(3,4) ] ) ->
Algebra( GF(3), [ (Z(3)^0)*(), (Z(3)^0)*(1,2) ] )]
gap> IsSimplicialAlgebra(SA3);
true
```

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