Communications in Advanced Mathematical Sciences
Vol. 6, No. 2, 78-85, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1232982

A Qualitative Investigation of the Solution of the Difference Equation $\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(\pm 1\pm\Psi_{m-3}\Psi_{m-5})}$

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Abstract
We explore the dynamics of adhering to rational difference formula

$$\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm 1\pm\Psi_{m-3}\Psi_{m-5}\right)} \quad m\in\mathbb{N}_0$$

where the initials Ψ_{-5}, Ψ_{-4}, Ψ_{-3},Ψ_{-2}, Ψ_{-1}, Ψ_0 are arbitrary nonzero real numbers. Specifically, we examine global asymptotically stability. We also give examples and solution diagrams for certain particular instances.

Keywords: Boundedness, Equilibrium point, Global asymptotic stability, Solution of difference equation, Stability.
2010 AMS: 39A10, 39A30.

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Received: 12 January 2023, **Accepted:** 29 May 2023, **Available online:** 30 June 2023
How to cite this article: B. Oğul, D. Şimşek, T. F. Ibrahim, *A Qualitative Investigation of the Solution of the Difference Equation* $\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(\pm 1\pm\Psi_{m-3}\Psi_{m-5})}$, Commun. Adv. Math. Sci., (6)2 (2023) 78-85.

1. Introduction

Because of its employment in discrete-time systems with microprocessors, difference equations are becoming increasingly important in engineering. The study of rational difference equations and their qualitative features has recently sparked a surge of interest. We refer the reader to [1–3] for some literature in this field.

Important rational difference equations were investigated by several authors. As examples:
Aloqeili, [4] has actually gotten the solutions to the difference equation

$$\Psi_{m+1}=\frac{\Psi_{m-1}}{a-\Psi_m\Psi_{m-1}}.$$

Çınar [5], researched adhering to problems with positive first values:

$$\Psi_{m+1}=\frac{Q_{m-1}}{-1+a\Psi_m\Psi_{m-1}}$$

for $m = 0, 1, 2, \ldots$

Gelişken [6] investigated behaviors of

$$\Psi_{m+1} = \frac{A_1 M_{m-(3k-1)}}{B_1 + C_1 M_{m-(3k-1)} \Psi_{m-(2k-1)} M_{m-(k-1)}},$$
$$M_{m+1} = \frac{A_2 \Psi_{m-(3k-1)}}{B_2 + C_2 \Psi_{m-(3k-1)} M_{m-(2k-1)} \Psi_{m-(k-1)}}.$$

Karataş et al. [7] deal with

$$\Psi_{m+1} = \frac{\Psi_{m-5}}{1 + \Psi_{m-2}\Psi_{m-5}}.$$

Oğul et al. [8] deal with

$$\Psi_{m+1} = \frac{\Psi_{m-17}}{\pm 1 \pm \Psi_{m-2}\Psi_{m-5}\Psi_{m-8}\Psi_{m-11}\Psi_{m-14}\Psi_{m-17}}.$$

Şimşek et al. [9] examine the equation

$$\Psi_{m+1} = \frac{\Psi_{m-13}}{1 + \Psi_{m-1}\Psi_{m-3}\Psi_{m-5}\Psi_{m-7}\Psi_{m-9}\Psi_{m-11}}.$$

Yalçınkaya et al. [10] have studied

$$\Psi_{m+1} = \frac{a\Psi_{m-k}}{b + c_m^p}.$$

For more related works we refer to [11–18].

Our objective in this study is to check out actions of the solution of adhering to nonlinear difference formula

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm 1 \pm \Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0$$

where the initials are arbitrary real numbers. Additionally, we obtain these types of solutions.

2. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(1+\Psi_{m-3}\Psi_{m-5})}$

In this part we give the solutions of

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(1 + \Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0 \tag{2.1}$$

where the initials are real numbers.

Theorem 2.1. *Let* $\{\Psi_m\}_{m=-5}^{\infty}$ *be a solution of* (2.1). *Then for* $m \in \mathbb{N}_0$

$$\Psi_{4m+1} = \frac{DF^{m+1}}{B^{m+1}} \prod_{i=0}^{m} \left(\frac{1+(i)BD}{1+(i+1)DF} \right), \qquad \Psi_{4m+2} = \frac{CE^{m+1}}{A^{m+1}} \prod_{i=0}^{m} \left(\frac{1+(i)CA}{1+(i+1)CE} \right),$$
$$\Psi_{4m+3} = \frac{B^{m+2}}{F^{m+1}} \prod_{i=0}^{m} \left(\frac{1+(i+1)DF}{1+(i+1)BD} \right), \qquad \Psi_{4m+4} = \frac{A^{m+2}}{E^{m+1}} \prod_{i=0}^{m} \left(\frac{1+(i+1)CE}{1+(i+1)CA} \right),$$

where, $\Psi_{-5} = F, \quad \Psi_{-4} = E, \Psi_{-3} = D, \quad \Psi_{-2} = C, \quad \Psi_{-1} = B, \quad \Psi_0 = A.$

Proof. Assume $m > 0$ and this our supposition remains true for $m-1$.
That is,

$$\Psi_{4m-3} = \frac{DF^m}{B^m} \prod_{i=0}^{m-1} \left(\frac{1+(i)BD}{1+(i+1)DF} \right), \quad \Psi_{4m-2} = \frac{CE^m}{A^m} \prod_{i=0}^{m-1} \left(\frac{1+(i)CA}{1+(i+1)CE} \right),$$
$$\Psi_{4m-1} = \frac{B^{m+1}}{F^m} \prod_{i=0}^{m-1} \left(\frac{1+(i+1)DF}{1+(i+1)BD} \right), \quad \Psi_{4m} = \frac{A^{m+1}}{E^m} \prod_{i=0}^{m-1} \left(\frac{1+(i+1)CE}{1+(i+1)CA} \right), \quad \Psi_{4m-5} = \frac{B^m}{F^{m-1}} \prod_{i=0}^{m-2} \left(\frac{1+(i+1)DF}{1+(i+1)BD} \right).$$

At the present time, using the main (2.1), one has

$$\Psi_{4m+1} = \frac{\Psi_{4m-3}\Psi_{4m-5}}{\Psi_{4m-1}\left(1+\Psi_{4m-3}\Psi_{4m-5}\right)}$$
$$= \frac{\frac{DF^m}{B^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)BD}{1+(i+1)DF}\right)\frac{B^m}{F^{m-1}}\prod_{i=0}^{m-2}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)}{\frac{B^{m+1}}{F^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)+\frac{B^{m+1}}{F^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)\frac{DF^m}{B^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)BD}{1+(i+1)DF}\right)\frac{B^m}{F^{m-1}}\prod_{i=0}^{m-2}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)}.$$

Hence, we have

$$\Psi_{4m+1} = \frac{DF^{m+1}}{B^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i)BD}{1+(i+1)DF}\right).$$

Similarly, it is easily obtained in other relationships. □

Theorem 2.2. (2.1) *has one equilibrium* $\overline{\Psi} = 0$ *and this equilibrium isn't locally asymptotically stable.*

Proof. We may express the equilibrium points of (2.1) as

$$\overline{\Psi} = \frac{\overline{\Psi}^2}{\overline{\Psi}(1+\overline{\Psi}^2)},$$

$$\overline{\Psi}^2\left(1+\overline{\Psi}^2\right) = \overline{\Psi}^2.$$

After that

$$\overline{\Psi}^4 = 0.$$

As a result, the equilibrium of (2.1) is $\overline{\Psi} = 0$.
Assume that $f : (0,\infty)^4 \to (0,\infty)$ is the function defined by

$$f(\tau,\kappa,\rho) = \frac{\tau\rho}{\kappa(1+\tau\rho)}.$$

As a result, it follows that

$$f_\tau(\tau,\kappa,\rho) = \frac{\rho}{\kappa(1+\tau\rho)^2}, \qquad f_\kappa(\tau,\kappa,\rho) = -\frac{\tau\rho}{\kappa^2(1+\tau\rho)}, \qquad f_\rho(\tau,\kappa,\rho) = \frac{\tau}{\kappa(1+\tau\rho)^2}.$$

We see that

$$f_\tau(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1, \qquad f_\kappa(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1, \qquad f_\rho(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1.$$

□

We confirm our results with the following numerical examples.

Example 2.3. *Assume that*

$$\Psi_{-5} = 0.3, \qquad \Psi_{-4} = 0.32, \qquad \Psi_{-3} = 0.34, \qquad \Psi_{-2} = 0.36, \qquad \Psi_{-1} = 0.38, \qquad \Psi_0 = 0.4.$$

See Figure 2.1.

Example 2.4. *Assume that*

$$\Psi_{-5} = 0.35, \qquad \Psi_{-4} = 0.32, \qquad \Psi_{-3} = 0.34, \qquad \Psi_{-2} = 0.38, \qquad \Psi_{-1} = 0.42, \qquad \Psi_0 = 0.43.$$

See Figure 2.2.

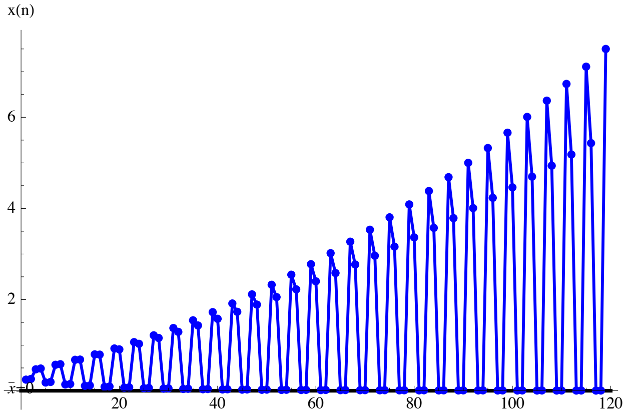

Figure 2.1

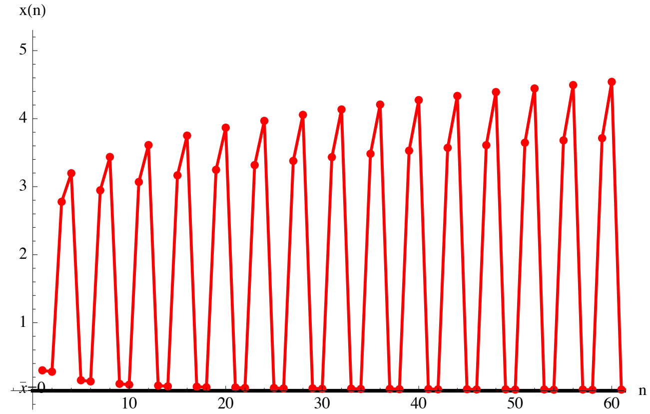

Figure 2.2

3. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(1-\Psi_{m-3}\Psi_{m-5})}$

We deal with the difference equation

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(1-\Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0. \tag{3.1}$$

Theorem 3.1. *Let $\{\Psi_m\}_{m=-7}^{\infty}$ represent a solution of (3.1). In that case for $m \in \mathbb{N}_0$*

$$\Psi_{4m+1} = \frac{DF^{m+1}}{B^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i)BD}{-1+(i+1)DF} \right), \qquad \Psi_{4m+2} = \frac{CE^{m+1}}{A^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i)CA}{-1+(i+1)CE} \right),$$

$$\Psi_{4m+3} = \frac{B^{m+2}}{F^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i+1)DF}{-1+(i+1)BD} \right), \qquad \Psi_{4m+4} = \frac{A^{m+2}}{E^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i+1)CE}{-1+(i+1)CA} \right),$$

where, $\Psi_{-5} = F, \quad \Psi_{-4} = E, \quad \Psi_{-3} = D, \quad \Psi_{-2} = C, \quad \Psi_{-1} = B, \quad \Psi_0 = A.$

Proof. The proof is similar to the proof of Theorem 2.1 and therefore it will be omitted. □

Theorem 3.2. *The unique equilibrium $\overline{\Psi} = 0$ in (3.1) isn't locally asymptotically stable.*

Proof. For confirming outcomes of this section, we take into consideration mathematical instances which stand for various kind of solutions to (3.1). □

Example 3.3. *Figure 3.1 depicts the actions taken when*

$$\Psi_{-5} = 3, \quad \Psi_{-4} = 3.9, \quad \Psi_{-3} = 3.1, \quad \Psi_{-2} = 2.8, \quad \Psi_{-1} = 2.5, \quad \Psi_0 = 3.5.$$

Example 3.4. *Figure 3.2 depicts the actions taken when*

$$\Psi_{-5} = 5.1, \quad \Psi_{-4} = 4.9, \quad \Psi_{-3} = 4.3, \quad \Psi_{-2} = 5.3, \quad \Psi_{-1} = 4.5, \quad \Psi_0 = 4.6.$$

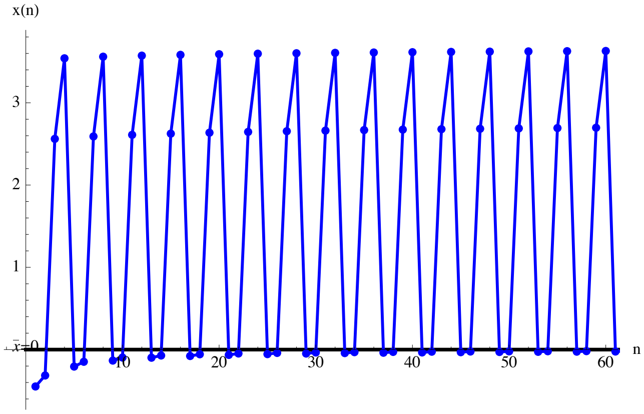

Figure 3.1

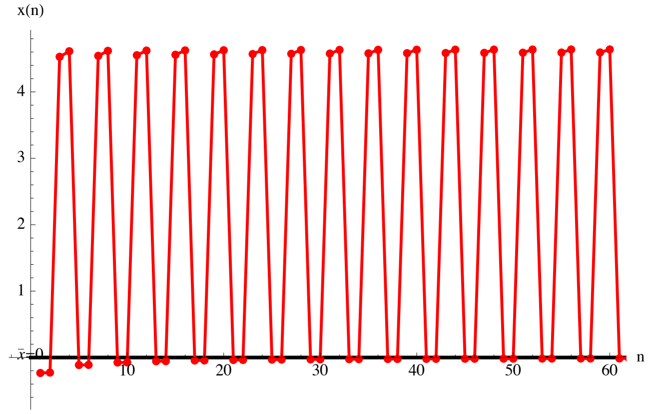

Figure 3.2

4. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(-1+\Psi_{m-3}\Psi_{m-5})}$

In this part, we study

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(-1+\Psi_{m-3}\Psi_{m-5})}, \quad m \in \mathbb{N}_0. \tag{4.1}$$

Theorem 4.1. *Let* $\{\Psi_m\}_{m=-5}^{\infty}$ *represent a solution of* (4.1). *In that case for,* $m = 0, 1, 2, \ldots$

$$\Psi_{8m+1} = \frac{-DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^{m+1}}, \quad \Psi_{8m+2} = \frac{-CE^{2m+1}(1+AC)^m}{A^{2m+1}(1+CE)^{m+1}}, \quad \Psi_{8m+3} = \frac{B^{2m+2}(1+DF)^{m+1}}{F^{2m+1}(1+BD)^{m+1}},$$

$$\Psi_{8m+4} = \frac{A^{2m+2}(1+CE)^{m+1}}{E^{2m+1}(1+AC)^{m+1}}, \quad \Psi_{8m+5} = \frac{DF^{2m+2}(1+BD)^{m+1}}{B^{2m+2}(1+DF)^{m+1}}, \quad \Psi_{8m+6} = \frac{CE^{2m+2}(1+AC)^{m+1}}{A^{2m+2}(1+CE)^{m+1}},$$

$$\Psi_{8m+7} = \frac{B^{2m+3}(1+DF)^{m+1}}{F^{2m+2}(1+BD)^{m+1}}, \quad \Psi_{8m+8} = \frac{A^{2m+3}(1+CE)^{m+1}}{E^{2m+2}(1+AC)^{m+1}}.$$

Proof. Assume that $m > 0$ and our supposition hold for $m-1$.

$$\Psi_{8m-7} = \frac{-DF^{2m}(1+BD)^{m-1}}{B^{2m}(1+DF)^m}, \quad \Psi_{8m-6} = \frac{-CE^{2m}(1+AC)^{m-1}}{A^{2m}(1+CE)^m}, \quad \Psi_{8m-5} = \frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m},$$

$$\Psi_{8m-4} = \frac{A^{2m+1}(1+CE)^m}{E^{2m}(1+AC)^m}, \quad \Psi_{8m-3} = \frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}, \quad \Psi_{8m-2} = \frac{CE^{2m+1}(1+AC)^m}{A^{2m+1}(1+CE)^m},$$

$$\Psi_{8m-1} = \frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m}, \quad \Psi_{8m} = \frac{A^{2m+2}(1+CE)^m}{E^{2m+1}(1+AC)^m}.$$

Now, it follows from (4.1) that

$$\Psi_{8m+1} = \frac{\Psi_{8m-3}\Psi_{8m-5}}{\Psi_{8m-1}(-1+\Psi_{8m-3}\Psi_{8m-5})}$$
$$= \frac{\frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}\frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m}}{-\frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m} + \frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m}\frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}\frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m}}.$$

Then, we have

$$\Psi_{8m+1} = \frac{-DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^{m+1}}.$$

The other relations can be provided in the same way. □

Theorem 4.2. (4.1) *contains three equilibriums,* 0, $\pm\sqrt{2}$ *and they aren't locally asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 2.2 and therefore it will be omitted. □

Example 4.3. *Figure 4.1 depicts the actions taken when*

$$\Psi_{-5} = 4.3, \quad \Psi_{-4} = 4.7, \quad \Psi_{-3} = 4.9, \quad \Psi_{-2} = 3.8, \quad \Psi_{-1} = 3.6, \quad \Psi_0 = 3.3.$$

Example 4.4. *Figure 4.2 depicts the actions taken when*

$$\Psi_{-5} = 4, \quad \Psi_{-4} = 4.5, \quad \Psi_{-3} = 5.3, \quad \Psi_{-2} = 4.7, \quad \Psi_{-1} = 5.1, \quad \Psi_0 = 5.5.$$

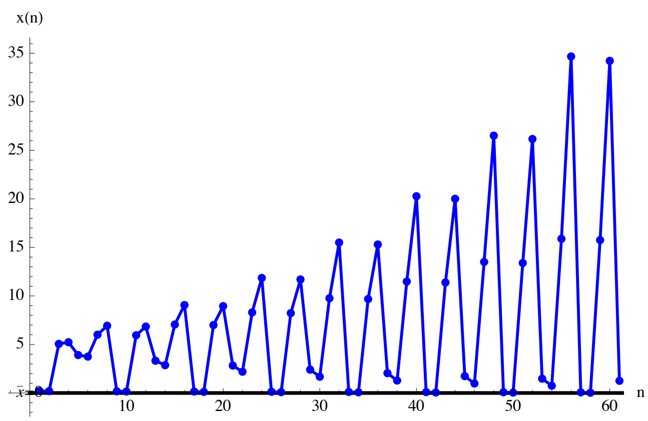

Figure 4.1

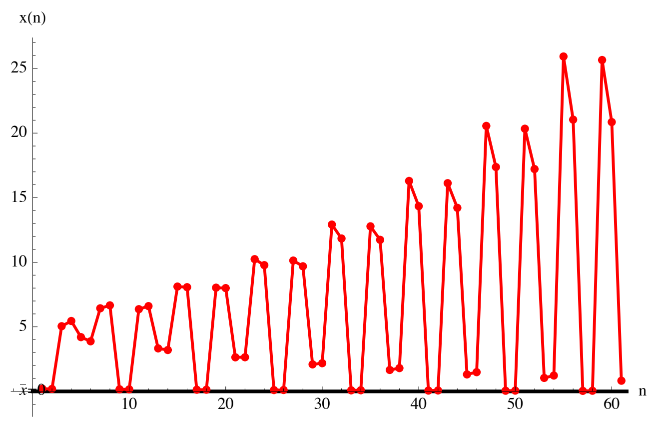

Figure 4.2

5. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(-1-\Psi_{m-3}\Psi_{m-5})}$

In this section, we find the solutions of

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(-1 - \Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0. \tag{5.1}$$

Theorem 5.1. *Assume that, $\{\Psi_m\}_{m=-5}^{\infty}$ represent a solution of (5.1).*

$$\Psi_{8m+1} = \frac{DF^{2m+1}(-1+BD)^m}{B^{2m+1}(-1+DF)^{m+1}}, \quad \Psi_{8m+2} = \frac{CE^{2m+1}(-1+AC)^m}{A^{2m+1}(-1+CE)^{m+1}}, \quad \Psi_{8m+3} = \frac{B^{2m+2}(-1+DF)^{m+1}}{F^{2m+1}(-1+BD)^{m+1}},$$
$$\Psi_{8m+4} = \frac{A^{2m+2}(-1+CE)^{m+1}}{E^{2m+1}(-1+AC)^{m+1}}, \quad \Psi_{8m+5} = \frac{DF^{2m+2}(-1+BD)^{m+1}}{B^{2m+2}(-1+DF)^{m+1}}, \quad \Psi_{8m+6} = \frac{CE^{2m+2}(-1+AC)^{m+1}}{A^{2m+2}(-1+CE)^{m+1}},$$
$$\Psi_{8m+7} = \frac{B^{2m+3}(-1+DF)^{m+1}}{F^{2m+2}(-1+BD)^{m+1}}, \quad \Psi_{8m+8} = \frac{A^{2m+3}(-1+CE)^{m+1}}{E^{2m+2}(-1+AC)^{m+1}}.$$

Proof. The proof is similar to the proof of Theorem 4.1 and therefore it will be omitted. □

Theorem 5.2. (5.1) *contains three equilibriums,* $0, \pm\sqrt{-2}$ *and these aren't locally asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 2.2 and therefore it will be omitted. □

Example 5.3. *See Figure 5.1 for the initials*

$$\Psi_{-5} = 2.85, \quad \Psi_{-4} = 2.8, \quad \Psi_{-3} = 2.75, \quad \Psi_{-2} = 2.7, \quad \Psi_{-1} = 2.6, \quad \Psi_0 = 2.55.$$

Example 5.4. *We consider*

$$\Psi_{-5} = 2, \quad \Psi_{-4} = 2.8, \quad \Psi_{-3} = 2.4, \quad \Psi_{-2} = 2.7, \quad \Psi_{-1} = 2.3, \quad \Psi_0 = 2.5.$$

See Figure 5.2.

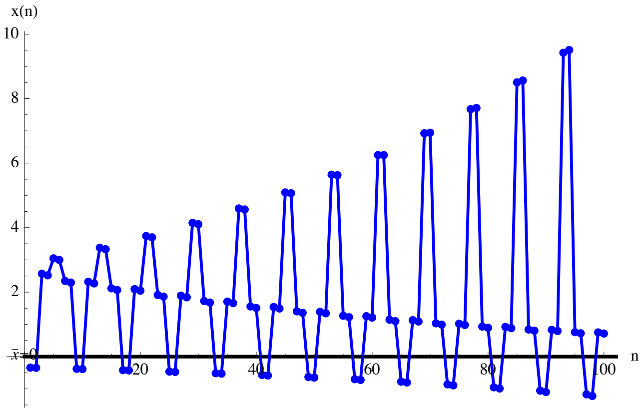

Figure 5.1

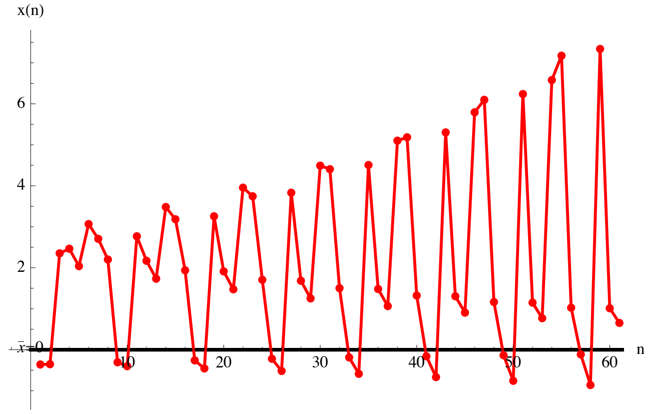

Figure 5.2

6. Conclusion

We explore the behavior of the following difference equation

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm 1 \pm \Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0$$

with positive real integers as initials. Local stability is discussed. Furthermore, we obtain the solution to several exceptional circumstances. Finally, a few numerical examples are shown.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

Availability of data and materials: Not applicable.

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