An exponential finite difference method based on Padé approximation

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Abstract
This paper reports a new technique of forming improved exponential finite difference solution of the one dimensional Burgers' equation. The technique is called explicit exponential finite difference method based on Padé approximation. The main purpose of the paper is to improve the exponential finite difference method and define an alternative method for the solution of the Burgers' equation. The advantage of the present method is reduced the computation cost to other exponential methods for solving the Burgers' equation. Accuracy of the present method is demonstrated by solving test problems and comparing numerical results with exact solution for different values of Reynolds' number.
Keywords — Burgers' equation, Exponential finite difference method, Explicit exponential finite difference method, Finite difference methods, Padé approximation.

1 Introduction

In this paper, we consider the one-dimensional nonlinear Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}, \ (x,t) \in \Omega \qquad (1.1)$$

where

$$\Omega = (0,1) \times (0,\mathrm{T}]$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < 1$$

and boundary conditions

$$u(0,t) = g_1(t) \ \text{ and } \ u(1,t) = g_2(t), \quad 0 \le t \le T$$

where Re is the Reynolds number and f, g_1 and g_2 are the prescribed functions of the variables. The Burgers' equation was initially given by Bateman [1] and later rediscovered by Burgers as a model of turbulence [2]. Burgers' equation has been found to describe various kind of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in a viscous fluid [3].

The Burgers' equation is an important simple model for the understanding of physical flows. In literature, many numerical methods have been proposed and implemented for approximating solution of the Burgers' equation. Ali *et al.* obtained numerical solution of the equation by a B-spline finite element method [6]. Bahadr used fully implicit finite-difference method for the numerical solution of equation [3]. Kutluay *et al.* applied explicit and exact-explicit finite difference methods for obtained numerical solution of the Burgers' equation [7]. Wei and Gu used conjugate filter approach for solving the equation [8]. Aksan and Özdeş developed variational method constructed on the method of discretization for the numerical solution of Burgers' equation [9]. The least-squares quadratic B-spline finite element method applied to the equation by Kutluay *et al* . [10]. Bahadr and Sağlam used a mixed finite difference and boundary element approach to for solution of the equation [11]. The Galerkin finite element method constructed on the method of discretized in time was applied to solve the one-dimensional nonlinear Burgers' equation by Aksan [12]. Gülsu and Öziş proposed restrictive Taylor approximation classical explicit finite difference method for the equation [13]. Kadalbajoo and Awasthi defined a solution based on Crank-Nicolson finite difference method for the equation [14]. Gülsu used restrictive Padé approximation classical implicit finite difference method for the Burgers' equation [15]. Liao applied a fourth-order compact finite difference method to the equation [16]. Sari and Gürarslan defined a sixth-order compact finite difference method for numerical solution of the one-dimensional Burgers' equation [17]. A compact predictor-corrector finite difference scheme applied to the equation by Zhang and Wang [18]. Mittal and Jain proposed modified cubic B-splines collocation method for the numerical solutions of Burgers' equation [19]. Soliman obtained numerical solutions of the Burgers' equation by the Galerkins' method using cubic B-splines finite elements [20].

The explicit exponential finite difference method was defined by Bhattacharya for the solution of heat equation [21]. Bhattacharya [22] and $h=\Delta x$ is the spatial mesh size, $k=\Delta t$ is the time step, $r_1=\frac{(\mathrm{Re})^{-1}\Delta t}{(\Delta x)^2}$ and $r_2=\frac{\mathrm{Re}\,\Delta x}{2}$. Explicit exponential finite difference method for Eq. (1.1) takes the following form

$$U_i^{n+1}=U_i^n\exp\left\{r_1\left[-r_2\left(U_{i+1}^n-U_{i-1}^n\right)+\frac{\left(U_{i-1}^n-2U_i^n+U_{i+1}^n\right)}{U_i^n}\right]\right\} \tag{2.1}$$

which is valid for values of i lying in the interval $1\le i\le N-1$ [22].

Handschuh and Keith [23] used explicit exponential finite difference method for the solution of Burgers' equation. Bahadr obtained the numerical solution of KdV equation by using the exponential finite-difference technique [24]. Implicit, fully implicit and Crank-Nicolson exponential finite difference methods applied to the Burgers' equation by İnan and Bahadr [25, 27]. Also, İnan and Bahadr [26, 28] solved the Burgers' equation linearized by Hopf-Cole transformation with three different exponential finite difference methods.

It is the purpose of this paper to advance another form of exponential finite difference method for the numerical solution of the Burgers' equation. This method can be defined explicit exponential finite difference method based on Padé approximation. In this paper, we use Padé approximation to approximate the exponential functions on explicit exponential finite difference method. So firstly, we define explicit exponential finite difference method and then we remind Padé approximation.

To examine the ability of this method for solution of the equation, two problems are considered. It is clearly seen from solution of the problems that numerical method is reasonably in good agreement with the exact solution.

2 Explicit Exponential Finite Difference Method(EEFDM)

The solution domain is discretized into cells described by the nodes set (x_i,t_n) in which $x_i=ih\ (i=0,1,2,\ldots,N)$ and $t_n=nk\ (n=0,1,2,\ldots)$,

Where U_i^n denotes the exponential finite difference approximation to the exact solution $u(x,t)$. Eq. (2.1) is system of difference equations.

3 Padé Approximation

The $[L/M]$ Padé approximation to $A(x)$ is shown by

$$R_{L,M} = [L/M] = \frac{P_L(x)}{Q_M(x)} \qquad (3.1)$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial at most M. The formal power series

$$A(x) - \frac{P_L(x)}{Q_M(x)} = O\left(x^{L+M+1}\right), \qquad (3.2)$$

$$A(x) = \sum_{i=1}^{\infty} a_i x^i \qquad (3.3)$$

determine the coefficient of $P_L(x)$ and $Q_M(x)$. Since we can obviously multiply the numerator and denominator by a constant and leave $[L/M](x,t)$ unchanged, we impose the normalization condition

$$Q_M(0) = 1.0. \qquad (3.4)$$

We write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + \ldots + p_L x^L \\ Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \ldots + q_M x^M . \end{aligned} \qquad (3.5)$$

Then we can write Eq. (3.2) as

$$\begin{aligned} a_{L+1} + a_L q_1 + \ldots + a_{L-M+1} q_M &= 0 \\ a_{L+2} + a_{L+1} q_1 + \ldots + a_{L-M+2} q_M &= 0 \\ &\vdots \\ a_{L+M} + a_{L+M-1} q_1 + \ldots + a_L q_M &= 0 \end{aligned} \qquad (3.9)$$

and

$$\begin{aligned} a_0 &= p_0 \\ a_1 + a_0 q_1 &= p_1 \\ a_2 + a_1 q_1 + a_0 q_2 &= p_2 \\ &\vdots \\ a_L + a_{L-1} q_1 + \ldots + a_0 q_L &= p_L \end{aligned} \qquad (3.10)$$

Since the q ' s are known from Eq. (3.9), Eq. (3.10) can be solved easily.

$$[L/M] = \frac{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{vmatrix}}{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}} \qquad (3.11)$$

If Eq. (3.9) and Eq. (3.10) can be solved directly as Eq. (3.11) [30].

4 Explicit Exponential Finite Difference Method Based on Padè Approximation (EEFDM-Padé Technique)

Eq. (2.1) can be written as

$$U_i^{n+1} = U_i^n \exp\{R\} \qquad (4.1)$$

and R is defined following form

$$R = r_1\left[-r_2\left(U_{i+1}^n - U_{i-1}^n\right) + \frac{\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)}{U_i^n} \right] \qquad (4.2)$$

and if Padé approximation is applied for $\exp\{R\}$, explicit exponential finite difference method based on Padé approximation obtained as following form

$$U_i^{n+1} = U_i^n \left\{ \frac{1 + \frac{R}{2}}{1 - \frac{R}{2}} \right\} \qquad (4.3)$$

or

$$U_i^{n+1} = U_i^n \left\{ \frac{1 + \frac{r_1}{2}\left[-r_2\left(U_{i+1}^n - U_{i-1}^n\right) + \frac{\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)}{U_i^n} \right]}{1 - \frac{r_1}{2}\left[-r_2\left(U_{i+1}^n - U_{i-1}^n\right) + \frac{\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)}{U_i^n} \right]} \right\}. \qquad (4.4)$$

5 Numerical Results

We obtain numerical solution of the Burgers' equation by EEFDM-Padé technique for two standard problems. The accuracy of the proposed method is measured in terms of the following error norms defined by

$$L_2 = \|u-U\|_2 = \left(h\sum_{i=0}^{N} |u_i - U_i|^2 \right)^{\frac{1}{2}}, \tag{5.1}$$

$$L_\infty = \|u-U\|_\infty = \max_{0\le i\le N} |u_i - U_i|, \tag{5.2}$$

$$E = \left[\frac{\sum_{i=0}^{N} |u_i - U_i|^2}{\sum_{i=0}^{N} |u_i|^2} \right]^{\frac{1}{2}}. \tag{5.3}$$

From comparisons of the numerical results with the exact solutions it is deduced that the proposed method gives highly accurate solutions. The rates of convergence of the method, computed using

$$rate = \frac{\log\left(E^h / E^{h/2}\right)}{\log(2)} \tag{5.4}$$

where E^h and $E^{h/2}$ are the errors defined in Eq. (5.3) with the grid size h and $h/2$, respectively.

Problem 1.

We first solve the Burgers' equation Eq. (1.1) and the initial condition

$$u(x,0) = \sin(\pi x),\ 0 < x < 1 \tag{5.5}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0,\ 0 \le t \le T \tag{5.6}$$

and the exact solution given by

$$u(x,t) = \left(\frac{2\pi}{\text{Re}}\right) \frac{\sum_{n=1}^{\infty} A_n \exp\left(-n^2\pi^2 \text{Re}^{-1} t\right) n \sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-n^2\pi^2 \text{Re}^{-1} t\right) \cos(n\pi x)} \tag{5.7}$$

with

$$A_0 = \int_0^1 \exp\left\{-\frac{\text{Re}}{2\pi}\left[1-\cos(\pi x)\right]\right\} dx \tag{5.8}$$

$$A_n = 2\int_0^1 \exp\left\{-\frac{\text{Re}}{2\pi}\left[1-\cos(\pi x)\right]\right\} \cos(n\pi x) dx,\ n = 1,2,3,\ldots \tag{5.9}$$

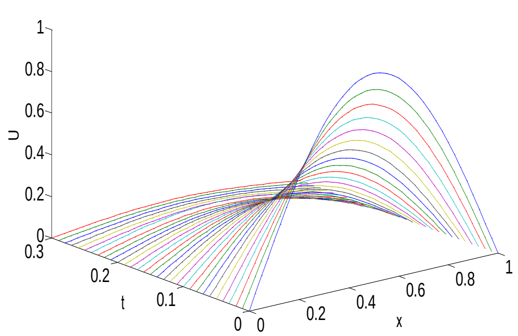
The results for Problem 1 are displayed in Table 1-3 and Fig. 1. The numerical solutions obtained by the present method and the exact solution for different values of Re Reynolds numbers are shown in Table 1-3. It is observed from Table 1-2 that the values of L_2 and L_∞ decrease with decrease of h . The obtained solutions for Problem 1 by the EEFDM-Padé technique are compared with other methods [10, 13, 15, 25, 29] in Table 3. All comparisons show that the present method offers better results than the others. In order to show, how the numerical solutions of the Problem 1 obtained with the present method we give the graphs Fig. 1. Fig. 1a and Fig. 1b display numerical solutions for $\text{Re} = 1$, $N = 40$, $k = 10^{-4}$ and $\text{Re} = 100$, $N = 100$, $k = 10^{-4}$, respectively.

Table 1. Comparison of the solutions with the exact solution at t=0.1 for Re=1 and $k=10^{-5}$ using various mesh sizes.

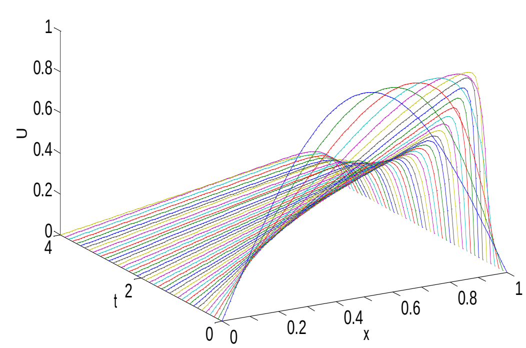
x	N=20	N=40	N=80	N=100	Exact
0.1	0.109727	0.109585	0.109550	0.109545	0.109538
0.2	0.210164	0.209885	0.209815	0.209807	0.209792
0.3	0.292437	0.292031	0.291930	0.291918	0.291896
0.4	0.348604	0.348094	0.347966	0.347951	0.347924
0.5	0.372348	0.371770	0.371626	0.371608	0.371577
0.6	0.359836	0.359243	0.359095	0.359077	0.359046
0.7	0.310625	0.310085	0.309950	0.309934	0.309905
0.8	0.228370	0.227956	0.227852	0.227840	0.227817
0.9	0.120987	0.120762	0.120706	0.120699	0.120687
L_2	0.000553	0.000138	0.000035	0.000022	
L_∞	0.000791	0.000198	0.000050	0.000032	

Table 2. Comparison of the solutions with the exact solution at t=1 for Re=100 and $k=10^{-5}$ using various mesh sizes.

x	N=20	N=40	N=80	N=100	Exact
0.1	0.075420	0.075391	0.075384	0.075383	0.075382
0.2	0.150723	0.150664	0.150649	0.150648	0.150645
0.3	0.225787	0.225695	0.225672	0.225670	0.225666
0.4	0.300480	0.300351	0.300318	0.300314	0.300309
0.5	0.374651	0.374477	0.374433	0.374428	0.374420
0.6	0.448112	0.447892	0.447834	0.447827	0.447816
0.7	0.520123	0.520370	0.520293	0.520283	0.520268
0.8	0.582551	0.591616	0.591510	0.591498	0.591476
0.9	0.546459	0.661216	0.660513	0.660350	0.660019
L_2	0.180470	0.024741	0.004384	0.002715	
L_∞	0.798098	0.152326	0.025042	0.016036	



Şekil 1a. Numerical solutions of Problem 1 at different times for Re=1.



Şekil 1b. Numerical solutions of Problem 1 at different times for Re=100.

Table 3. Comparison of the results for Re=10, N=80 and k=10⁻⁴.

x	t	RHC [13]	RPA [15]	[10]	[29]	I-EFDM [25]	Present Method	Exact
0.25	0.4	0.317062	0.308776	0.31215	0.30415	0.308936	0.308902	0.308894
	0.6	0.248472	0.240654	0.24360	0.23629	0.240775	0.240750	0.240739
	0.8	0.202953	0.195579	0.19815	0.19150	0.195709	0.195691	0.195676
	1.0	0.169527	0.162513	0.16473	0.15861	0.162599	0.162584	0.162565
0.5	0.4	0.583408	0.569527	0.57293	0.56711	0.569727	0.569695	0.569632
	0.6	0.461714	0.447117	0.40588	0.44360	0.447307	0.447275	0.447206
	0.8	0.373800	0.359161	0.36286	0.35486	0.359343	0.359313	0.359236
	1.0	0.306184	0.291843	0.29532	0.28710	0.292026	0.291996	0.291916
0.75	0.4	0.638847	0.625341	0.63038	0.61874	0.625659	0.625695	0.625438
	0.6	0.506429	0.487089	0.49268	0.47855	0.487513	0.487480	0.487215
	0.8	0.393565	0.373827	0.37912	0.36467	0.374203	0.374150	0.373922
	1.0	0.305862	0.029726	0.03038	0.27860	0.287714	0.287658	0.287474

(5.13)

Problem 2.

The initial condition for the current problem is

$$u(x,0)=4x(1-x),\ 0<x<1 \tag{5.10}$$

and the boundary conditions

$$u(0,t)=u(1,t)=0,\ 0\le t\le T \tag{5.11}$$

with the exact solution also given by Eq. (5.7) but with following coefficients.

$$A_0=\int_0^1 \exp\left\{-\frac{\mathrm{Re}}{3}\left[x^2(3-2x)\right]\right\}dx \tag{5.12}$$

$$A_n=2\int_0^1 \exp\left\{-\frac{\mathrm{Re}}{3}\left[x^2(3-2x)\right]\right\}\cos(n\pi x)\,dx,\ n=1,2,3,\ldots$$

In Table 4-6, we compare the numerical results of Problem 2 obtained from new method with the exact solutions for $\mathrm{Re}=1$, $\mathrm{Re}=10$ and $\mathrm{Re}=100$. It is observed from Table 4-5 that the values of L_2 and L_∞ small enough. In Table 6, we compare the numerical results of our method with the methods proposed in [10, 13, 15, 25, 29] for Problem 2. The comparisons showed that the present method offer better results than the others. *t* is clearly seen from all tables that the obtained numerical results with the method present in this paper are in good agreement with the exact solution. Numerical solutions of Problem 2 at different times for $\mathrm{Re}=1$, $N=40$ and $k=10^{-4}$ are displayed in Figure 2a. The computed solutions of the Problem 2 at different times by the method are showed for $\mathrm{Re}=100$, $N=100$ and $k=10^{-4}$ in Figure 2b.

Table 4. Comparison of the solutions with the exact solution at t=0.1 for Re=1 and $k=10^{-5}$ using various mesh sizes.

x	N=20	N=40	N=80	N=100	Exact
0.1	0.113086	0.112941	0.112904	0.112900	0.112892
0.2	0.216634	0.216347	0.216276	0.216267	0.216252
0.3	0.301521	0.301104	0.301000	0.300988	0.300966
0.4	0.359562	0.359038	0.358907	0.358891	0.358863
0.5	0.384216	0.383621	0.383472	0.383454	0.383422
0.6	0.371474	0.370862	0.370709	0.370691	0.370658
0.7	0.320812	0.320253	0.320113	0.320096	0.320066
0.8	0.235945	0.235515	0.235408	0.235395	0.235371
0.9	0.125031	0.124797	0.124738	0.124731	0.124718
L_2	0.000571	0.000143	0.000036	0.000023	
L_∞	0.000817	0.000205	0.000052	0.000033	

Table 5. Comparison of the solutions with the exact solution at t=1 for Re=100 and $k=10^{-5}$ using various mesh sizes.

x	N=20	N=40	N=80	N=100	Exact
0.1	0.078149	0.078104	0.078092	0.078091	0.078088
0.2	0.156043	0.155961	0.155940	0.155937	0.155934
0.3	0.233440	0.233334	0.233306	0.233303	0.233298
0.4	0.310117	0.309995	0.309963	0.309959	0.309953
0.5	0.385861	0.385722	0.385686	0.385682	0.385676
0.6	0.460432	0.460289	0.460248	0.460243	0.460236
0.7	0.532942	0.533444	0.533392	0.533386	0.533376
0.8	0.594410	0.604875	0.604804	0.604795	0.604781
0.9	0.551459	0.674162	0.673574	0.673427	0.673123
l_2	0.189173	0.026018	0.004569	0.002826	
L_∞	0.836100	0.160805	0.026134	0.002826	

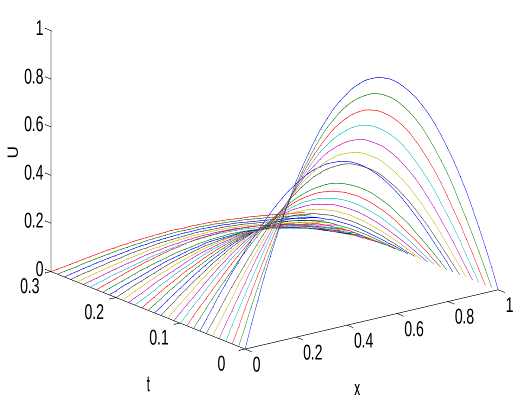


Figure 2a. Numerical solutions of Problem 2 at different times for Re=1.

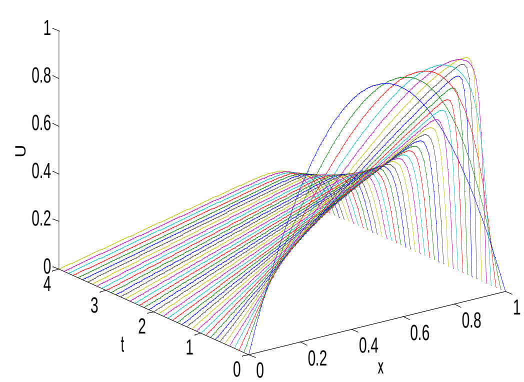


Figure 2b. Numerical solutions of Problem 2 at different times for Re=100.

Table 6. Comparison of the results for Re=10, N=80 and k=10^{-4}.

		k=10^{-5}		k=10^{-4}				
x	t	RHC [13]	RPA [15]	[10]	[29]	I-EFDM [25]	Present Method	Exact
0.25	0.4	0.306529	0.317399	0.32091	0.31247	0.317567	0.317530	0.317523
	0.6	0.236051	0.246058	0.24910	0.24148	0.246175	0.246148	0.246138
	0.8	0.190181	0.199437	0.20211	0.19524	0.199589	0.199570	0.199555
	1.0	0.156646	0.165529	0.16782	0.16153	0.165633	0.165618	0.165599
0.5	0.4	0.565994	0.584429	0.58788	0.58176	0.584627	0.584596	0.584537
	0.6	0.438926	0.457888	0.46174	0.45414	0.458077	0.458045	0.457976
	0.8	0.348328	0.367320	0.37111	0.36283	0.367507	0.367475	0.367398
	1.0	0.280038	0.298271	0.30183	0.29336	0.298455	0.298424	0.298343
0.75	0.4	0.626990	0.645527	0.65054	0.63858	0.645850	0.645887	0.645616
	0.6	0.477908	0.502564	0.50825	0.49362	0.502969	0.502954	0.502676
	0.8	0.360630	0.385232	0.39068	0.37570	0.385613	0.385574	0.385336
	1.0	0.272623	0.295779	0.30057	0.28663	0.296092	0.296048	0.295857

Rate of convergence at $\mathrm{Re}=1$ and $t=0.5$ for the Problem 1 and Problem 2 are shown in Table 7. From the table, we observe that the proposed method is first order accurate in space. From this table, it can be seen that errors approach to zero as the mesh refines, which shows that the scheme is consistent.

Table 7. Rate of convergence for Re=1 at t=0.5.

N	Problem 1	Problem 2
2	-	-
4	2.450709506	2.385884554
8	2.117029032	2.107812354
16	2.006633901	2.006790233
32	2.001942713	2.001512488
64	2.000968722	1.999595552

6 Conclusion

An explicit exponential finite difference method based on Padé approximation is presented for the nonlinear Burgers' equation. Numerical results are obtained for the nonlinear Burgers' equation with various initial and boundary conditions, which manifest high accuracy and efficiency of the present method. The proposed method are seen to be good alternative to existing methods for such problems.

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