Helices on a surface in Euclidean 3- space

Ufuk Öztürk [1]*, Hasan Hilmi Hacısalihoğlu[2]

[1] Çankırı Karatekin University, Faculty of Science, Department of Mathematics,
18100, Çankırı, Turkey, +90 (376) 218 9537,
ozturkufuk06@gmail.com, uuzturk@asu.edu
[2] Bilecik Şeyh Edebali University, Faculty of Arts and Sciences, Department of Mathematics,
11210, Bilecik, Turkey,
[2] Current Address: Taşpınar Mahallesi 2856. Sokak No:14, Ahlatlıbel, Gölbaşı, Ankara,
+90 535 492 9030, h.hacisali@science.ankara.edu.tr
* Corresponding author

Received: 28th August 2016
Accepted: 4th December 2016
DOI: http://dx.doi.org/10.18466/cbujos.302650

Abstract
In this paper, we define the strip slant helices according to the frame of the strip and introduce some characterizations for strip slant helices using the curvatures of the strip. We also determine the axis of the strip slant helices. Moreover, we investigate some characterizations for the strip slant helices when the curve of the strip is a geodesic curve or an asymptotic curve or a principal curve.
Keywords —Geodesic curvature, geodesic torsion, helix, normal curvature, strip

1 Introduction

Some of the classical results of differential geometry topics in Riemannian geometry have been treated by the researchers. Several authors introduced different types of helices and give some characterizations of these special curves for a long time. The helix is generally known as a curve in DNA double and α-form. Also we can see the helix curve in the field of computer aided design and computer graphics. In differential geometry; it is well-known that a general helix (or a curve of constant slope) is a curve whose tangent's makes a constant angle with a fixed direction, which is called the axis of the helix. The ratio of the curvature and the torsion of such curve is a constant, which is the necessary and sufficient condition for a curve to be a general helix [1].

In [2] izumiya Izumiya and Takeuchi introduced a slant helix as a curve in the Euclidean 3-space having a property that its principal normal vector makes a constant angle with a constant direction (see also [3]) and in [4] Kula et al. consider the tangent spherical indicatrix (the normal and binormal indicatrix, respectively) and characterize slant helices by certain differential equations verified for each one of these indicatrices. Moreover, in [5] Ali and López generalize the definition of slant helices in the Euclidean four-dimensional space $\mathbb{E}^4$, and present different characterizations of them. Recently, in [6] Ali and Turgut give some characterizations of slant helices in the n-dimensional Euclidean space. Moreover, they introduce the type-2 harmonic curvatures of a regular curve.

In differential geometry of surfaces, a *strip* or *curve-surface pair* is a natural moving frame constructed along the curve α on a surface and it is the analog of the Frenet-Serret frame. On the strip in Euclidean space have studied in [7,8]. In [7] Hacısalihoglu studied a relation between the Serret-Frenet formulae of a curve α in a hypersurface M and the curvatures of M in

Euclidean space $\mathbb{E}^n$. In [8] Sabuncuoglu and Hacısalihoglu calculated the higher curvature of a strip in $\mathbb{E}^n$.

In this paper, we define the strip slant helices according to the frame of the strip and characterize the strip slant helices using the curvatures of the strip. We also determine the axis of the strip slant helices. Moreover, we investigate some characterizations for the strip slant helices when the curve of the strip is a geodesic curve or an asymptotic curve or a principal curve.

2 Basic Concept

We now recall some basic concepts on classical differential geometry of space curves and the definition of the strip in Euclidean 3-space. Let

$$\begin{aligned} \alpha : I \subset \mathbb{R} &\longrightarrow \mathbb{E}^3 \\ s &\longrightarrow \alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s)) \end{aligned}$$

be a curve parameterized by arc length. There exist Frenet frame $\{T, N, B\}$ at each point of α where $T(s) = \alpha'(s)$ is the unit tangent vector, $N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ is the principal normal vector and $B(s) = T(s) \times N(s)$ is the binormal vector field. Differentiating the Frenet frame yields the classic Frenet equations:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of α, respectively.

Definition 1 ([8]) *Let M and α be a surface and a unite velocity curve on M in $\mathbb{E}^3$, respectively. The locus of the surface elements of M, which are the part of the tangent plane of α at a neighborhood of every point of α, are called a strip or curve-surface pair along the curve α which is showed by (α, M).*

Let α be a regular unit speed curve in $\mathbb{E}^3$ with the Frenet frame $\{T, N, B\}$ lying fully on a regular surface M and ζ be a unit normal vector field of the surface M at the point $\alpha(s)$. Then, we have

$$\zeta \times \xi = \eta, \tag{2.2}$$

is the binormal vector field of the strip where $\xi = T$. Thus, we obtain the system of orthonormal vector fields $\{\xi(s), \eta(s), \zeta(s)\}$ is called ***the strip three-bundle*** and we have the following Frenet–Serret type formulae:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix}. \tag{2.3}$$

Here, $k_n(s) = \langle \xi'(s), \zeta(s) \rangle = \kappa \sin\theta$ is the normal curvature, $k_g(s) = \langle \xi'(s), \eta(s) \rangle = \kappa \cos\theta$ is the geodesic curvature, $\tau_g(s) = \langle \eta'(s), \zeta(s) \rangle = \tau - \theta'$ is the geodesic torsion and θ is the angle between the vectors η and N, [7,8].

3 Strip slant helices in Euclidean 3-space

In this section, we consider a regular unit speed curve α on a regular surface M in $\mathbb{E}^3$and introduce strip slant helices according to the frame $\{\xi, \eta, \zeta\}$ of (α, M). We give a classisication of such curves in the Euclidean 3-space $\mathbb{E}^3$. Throughout this section let $\mathbb{R}_0$ denotes $\mathbb{R}\backslash\{0\}$.

Definition 2 *The strip (α, M) in $\mathbb{E}^3$ is called $\xi-$strip slant helix if there exists a non-zero fixed direction $U \in \mathbb{E}^3$ such that*

$$\langle \xi, U \rangle = constant$$

holds. The fixed direction U is called the axis of the strip slant helix.

Definition 3 *The strip (α, M) in $\mathbb{E}^3$ is called $\eta-$strip slant helix if there exists a non-zero fixed direction $V \in \mathbb{E}^3$ such that*

$$\langle \eta, V \rangle = constant$$

holds. The fixed direction V is called the axis of the strip slant helix.

Definition 4 *The strip (α, M) in $\mathbb{E}^3$ is called $\zeta-$strip slant helix if there exists a non-zero fixed direction $W \in \mathbb{E}^3$ such that*

$$\langle \zeta, W \rangle = constant$$

holds. The fixed direction W is called the axis of the strip slant helix.

Let us first characterize ξ-type slant helices.

Case 1 ($\xi-$strip slant helices) If (α, M) is $\xi-$strip slant helix parameterized by the arclength s in $\mathbb{E}^3$, then according to Definition 2, there exists a non-zero constant vector field $U \in \mathbb{E}^3$ such as

$$g(\xi, U) = c, \quad c \in \mathbb{R}_0. \tag{2.4}$$

ith respect to the Frenet frame $\{\xi, \eta, \zeta\}$ of (α, M), the fixed direction U can be decomposed as

$$U = c\xi + u_2\eta + u_3\zeta, \tag{2.5}$$

where u_2 and u_3 are differentiable functions of the curvatures. Differentiating the equation (2.5) with respect to s and using equations (2.3), we obtain the following system of differential equations

$$\begin{cases} u_3k_n + u_2k_g = 0, \\ u_2' - u_3\tau_g + ck_g = 0, \\ u_3' + u_2\tau_g + ck_n = 0. \end{cases} \tag{2.6}$$

From the first and the second equations of (2.6) we get

$$\begin{cases} u_2 = -ce^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right), \\ u_3 = c\frac{k_g}{k_n} e^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right), \end{cases} \tag{2.7}$$

where $c \in \mathbb{R}_0$.

Substituting (2.7) in the third equation of (2.6) we obtain that the curvature functions of (α, M) satisfy the relation

$$a\left[\left(\frac{k_g}{k_n} \right)' - \tau_g \left(\frac{k_g}{k_n} \right)^2 + \tau_g \right] + \frac{k_g^2}{k_n} + k_n = 0, \tag{2.8}$$

where $a = e^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right)$.

Conversely, assume that (2.8) holds. Consider the vector U given by

$$U = c\xi - ce^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right) \eta + c\frac{k_g}{k_n} e^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right) \zeta,$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using the equations (2.3), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$g(\xi, U) = c, \quad c \in \mathbb{R}_0.$$

According to Definition 2, (α, M) is a $\xi-$strip slant helix with the axis U.

Therefore, we can give the following theorem and corollary.

Theorem 1 *Let (α, M) be a strip in $\mathbb{E}^3$ with the curvatures k_g, k_n and τ_g. Then (α, M) is a $\xi-$strip slant helix if and only if its curvature functions k_g, k_n and τ_g satisfy the relation*

$$a\left[\left(\frac{k_g}{k_n} \right)' - \tau_g \left(\frac{k_g}{k_n} \right)^2 + \tau_g \right] + \frac{k_g^2}{k_n} + k_n = 0, \tag{2.9}$$

where $a = e^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right)$.

Corollary 1 *The axis of the $\xi-$strip slant helix (α, M) in $\mathbb{E}^3$ is given by*

$$U = c\xi - ce^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right) \eta + c\frac{k_g}{k_n} e^{-\int \frac{\tau_g k_g}{k_n} ds} \left(\int k_g e^{\int \frac{\tau_g k_g}{k_n} ds} ds \right) \zeta,$$

where $c \in \mathbb{R}_0$.

Substituting $c = 0$ in relation (2.6), we get

$$u_2 = a_1 e^{-\int \frac{\tau_g k_g}{k_n} ds}, \quad u_3 = -a_1 \frac{k_g}{k_n} e^{-\int \frac{\tau_g k_g}{k_n} ds},$$

where $a_1 \in \mathbb{R}_0$.

CBU J. of Sci., Volume 13, Issue 1, 2017, p 113-123

Therefore, we obtain the next corollary

Corollary 2 *Let (α, M) be a $\xi-$strip slant helix with the axis U in $\mathbb{E}^3$. If its tangent vector ξ is orthogonal to the axis U, then the axis U is given by*

$$U = a_1 e^{-\int \frac{\tau_g k_g}{k_n} ds}\eta - a_1 \frac{k_g}{k_n} e^{-\int \frac{\tau_g k_g}{k_n} ds}\zeta,$$

where $a_1 \in \mathbb{R}_0$.

Now, we consider the following subcases when the curvatures k_g, k_n and τ_g are zero, respectively.

Case 1.1 Let the curve α be a geodesic curve (i. e. $k_g = 0$). In this case, from (2.6) we have

$$\begin{cases} u_3 k_n = 0, \\ u_2' - u_3\tau_g = 0, \\ u_3' + u_2\tau_g + ck_n = 0. \end{cases} \tag{2.10}$$

From the first equation of (2.10) we get $k_n = 0$ or $u_3 = 0$.

(i) If $k_n = 0$ for all s, then $\kappa = 0$ which means that the curve α is a straight line.

(ii) If $u_3 = 0$ for all s, then from (2.10) we get $u_2 = -c\frac{k_n}{\tau_g} \in \mathbb{R}_0$. Also, since $k_g = \kappa\cos\theta = 0$, we find that $\theta = \mp\frac{\pi}{2}$ so by using the definition of k_n and τ_g, we have $k_n = \mp\kappa$ and $\tau_g = \tau$. Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mp\kappa \\ 0 & 0 & \tau \\ \pm\kappa & -\tau & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix}.$$

and the axis of (α, M) always lies in the plane $sp\{\xi, \eta\}$ and is given by

$$U = c\xi \pm c\tfrac{\kappa}{\tau}\eta, \tag{2.11}$$

where $\frac{\kappa}{\tau} = constant$ and $c \in \mathbb{R}_0$.

Therefore, we can give the following corollary.

Corollary 3 Let (α, M) be a $\xi-$strip slant helix with the axis U is given by (2.11). Then the curve α is a geodesic curve on M if only if the curve α is a general helix.

Case 1.2 Let the curve α be an asymptotic curve (i. e. $k_n = 0$). In this case, from (2.6) we have

$$\begin{cases} u_2 k_g = 0, \\ u_2' - u_3\tau_g + ck_g = 0, \\ u_3' + u_2\tau_g = 0. \end{cases} \tag{2.12}$$

From the first equation of (2.12) we get $k_g = 0$ or $u_2 = 0$.

(i) If $k_g = 0$ for all s, then $\kappa = 0$ which means that the curve α of (α, M) is a straight line.

(ii) If $u_2 = 0$ for all s, then from (2.12) we get $u_3 = c\frac{k_g}{\tau_g} \in \mathbb{R}_0$. Also, since $k_n = \kappa\sin\theta = 0$, we find that $\theta = k\pi \ \ (k = 0, 1)$ so by using the definition of k_g and τ_g, we have $k_g = \mp\kappa$ and $\tau_g = \tau$. Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & \mp\kappa & 0 \\ \pm\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix},$$

and the axis of (α, M) always lies in the plane $sp\{\xi, \zeta\}$ and is given by

$$U = c\xi \mp c\tfrac{\kappa}{\tau}\zeta, \tag{2.13}$$

where $\frac{\kappa}{\tau} = constant$ and $c \in \mathbb{R}_0$.

Therefore, we can give the following corollary.

Corollary 4 *Let (α, M) be a $\xi-$strip slant helix with the axis U is given by (2.13). Then the curve α is an asymptotic curve on M if only if the curve α is a general helix.*

Case 1.3 Let the curve α be a principal curve (i. e.

$\tau_g = 0$), then from (2.6) we have

$$\begin{cases} u_3 k_n + u_2 k_g = 0, \\ u_2' + c k_g = 0, \\ u_3' + c k_n = 0. \end{cases} \tag{2.14}$$

and we get

$$\begin{array}{l} u_2 = -c \int k_g ds, \\ u_3 = -c \int k_n ds. \end{array} \tag{2.15}$$

Substituting (2.15) in the first equation of (2.14), we obtain that the curvature functions of (α, M) satisfy the relation as follows

$$k_g \int k_g ds + k_n \int k_n ds = 0.$$

Therefore, the axis of (α, M) is given by

$$U = c\xi - c\left(\int k_g ds\right)\eta - c\left(\int k_n ds\right)\zeta, \tag{2.16}$$

where $c \in \mathbb{R}_0$.

Therefore, we obtain the next corollary.

Corollary 5 *Let (α, M) be a $\xi-$strip slant helix in $\mathbb{E}^3$. If the curve α is a line of principal curvature of M, then the axis U of the $\xi-$strip slant helix (α, M) is*

$$U = c\xi - c\left(\int k_g ds\right)\eta - c\left(\int k_n ds\right)\zeta,$$

where $c \in \mathbb{R}_0$ and

$$k_g\left(\int k_g ds\right) + k_n\left(\int k_n ds\right) = 0.$$

Next, let us consider η-type slant helices.

Case 2 ($\eta-$strip slant helices) If (α, M) is an $\eta-$strip slant helix parameterized by the arclength s in $\mathbb{E}^3$, then according to Definition 3 there exists a non-zero constant vector field $U \in \mathbb{E}^3$ such that

$$g(\eta, V) = c, \quad c \in \mathbb{R}_0. \tag{2.17}$$

With respect to the Frenet frame $\{\xi, \eta, \zeta\}$ of (α, M), the fixed direction V can be decomposed as

$$V = v_1 \xi + c\eta + v_3 \zeta, \tag{2.18}$$

where v_1 and v_3 are differentiable functions of the curvatures. Differentiating the equation (2.18) with respect to s and using equations (2.3), we obtain the following system of differential equations

$$\begin{cases} v_1' - v_3 k_n - c k_g = 0, \\ v_1 k_g - v_3 \tau_g = 0, \\ v_3' + v_1 k_n + c\tau_g = 0. \end{cases} \tag{2.19}$$

From the second and the third equations of (2.19) we get

$$\begin{cases} v_1 = -c\frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right), \\ v_3 = -c e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right), \end{cases} \tag{2.20}$$

where $c \in \mathbb{R}_0$.

Substituting (2.20) in the first equation of (2.19), we obtain the relation

$$a\left[\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - k_n\right] + \frac{\tau_g^2}{k_g} + k_g = 0, \tag{2.21}$$

where $a = e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)$.

Conversely, assume that (2.21) holds. Consider the vector V given by

$$V = -c\frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)\xi + c\eta - c e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)\zeta,$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using the equations (2.3) and (2.19), we find $V' = 0$. Hence V is a fixed direction. It can be easily checked that

$$g(\eta, V) = c, \quad c \in \mathbb{R}_0.$$

According to Definition 3, (α, M) is an $\eta-$strip slant helix with the axis V.

Theorem 2 *Let (α, M) be a strip in $\mathbb{E}^3$ with the curvatures k_g, k_n and τ_g. Then (α, M) is an $\eta-$strip slant helix if and only if its curvature functions k_g, k_n and τ_g satisfy the relation*

$$a\left[\left(\frac{\tau_g}{k_g}\right)' - k_n\left(\frac{\tau_g}{k_g}\right)^2 - k_n\right] + \frac{\tau_g^2}{k_g} + k_g = 0,$$

where $a = e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)$.

Corollary 6 *The axis of $\eta-$strip slant helix (α, M) in $\mathbb{E}^3$ is given by*

$$\begin{aligned} V = & -c\frac{\tau_g}{k_g}e^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)\xi \\ & + c\eta - ce^{-\int \frac{\tau_g k_n}{k_g} ds}\left(\int \tau_g e^{\int \frac{\tau_g k_n}{k_g} ds} ds\right)\zeta, \end{aligned}$$

where $c \in \mathbb{R}_0$.

Putting $c = 0$ in relation (2.19), we get

$$\begin{cases} v_1' - v_3 k_n = 0, \\ v_1 k_g - v_3 \tau_g = 0, \\ v_3' + v_1 k_n = 0. \end{cases}$$

and

$$\begin{aligned} v_1 &= a_2 \frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds} \\ v_3 &= a_2 e^{-\int \frac{\tau_g k_n}{k_g} ds}, \end{aligned}$$

where $a_2 \in \mathbb{R}_0$.

Therefore, we obtain the next corollary.

Corollary 7 *Let (α, M) be a $\eta-$strip slant helix with the axis V in $\mathbb{E}^3$. If its binormal vector η is orthogonal to the axis V, then the axis V is given by*

$$V = a_2 \frac{\tau_g}{k_g} e^{-\int \frac{\tau_g k_n}{k_g} ds} \xi + a_2 e^{-\int \frac{\tau_g k_n}{k_g} ds} \zeta,$$

where $a_2 \in \mathbb{R}_0$.

Now, we consider the following special cases when the curvatures k_g, k_n and τ_g are zero, respectively.

Case 2.1 Let the curve α is a geodesic curve (i. e. $k_g = 0$), then from (2.19) we have

$$\begin{cases} v_1' - v_3 k_n = 0, \\ v_3 \tau_g = 0, \\ v_3' + v_1 k_n + c\tau_g = 0. \end{cases} \tag{2.22}$$

From the second equation of (2.22) we get $\tau_g = 0$ or $v_3 = 0$.

(i) If $\tau_g = 0$ for all s, then from the first and the second equations of (2.22) we get

$$\frac{d}{ds}\left(\frac{1}{k_n}\frac{dv_1}{ds}\right) + v_1 k_n = 0.$$

Putting $p(s) = \frac{1}{k_n(s)}$, the above equation can be rewritten as

$$\frac{d}{ds}\left(p(s)\frac{dv_1}{ds}\right) + \frac{v_1}{p(s)} = 0.$$

By changing the variables in the above equation by $t(s) = \int \frac{1}{p(s)} ds$, we find

$$\frac{d^2 v_1}{dt^2} + v_1 = 0.$$

The solution of the previous differential equation is given by

$$v_1(t) = C_1 \cos(t) + C_2 \sin(t),$$

and since $t(s) = \int k_n(s) ds$, we get

$$v_1 = C_1 \cos\left(\int k_n ds\right) + C_2 \sin\left(\int k_n ds\right).$$

Also, the first equation of (2.22) we have

$$v_3 = -C_1 \sin\left(\int k_n ds\right) + C_2 \cos\left(\int k_n ds\right).$$

On the other hand, since $k_g = \kappa\cos\theta = 0$, we find that $\theta = \mp\frac{\pi}{2}$ so by using definition k_n and τ_g we have $k_n = \mp\kappa$ and $\tau_g = \tau = 0$. Therefore, the axis of (α, M) is given by

$$V = v_1\xi + c\eta + v_3\zeta, \tag{2.23}$$

where

$$\begin{aligned} v_1 &= C_1\cos\left(\int\kappa ds\right) \mp C_2\sin\left(\int\kappa ds\right), \\ v_3 &= -C_1\sin\left(\int\kappa ds\right) \mp C_2\cos\left(\int\kappa ds\right). \end{aligned}$$

(ii) If $v_3 = 0$ for all s, then from (2.22) we get $v_1 = -c\frac{\tau_g}{k_n} \in \mathbb{R}_0$. Also, since $k_g = \kappa\cos\theta = 0$, we find that $\theta = \mp\frac{\pi}{2}$ so by using the definition of k_n and τ_g, we have $k_n = \mp\kappa$ and $\tau_g = \tau$. Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mp\kappa \\ 0 & 0 & \tau \\ \pm\kappa & -\tau & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix},$$

and the axis of (α, M) always lies on the plane $sp\{\xi, \eta\}$ and is given by

$$V = \pm c\tfrac{\tau}{\kappa}\xi + c\eta, \tag{2.24}$$

where $\frac{\tau}{\kappa} = constant$ and $c \in \mathbb{R}_0$.

Therefore, we can give the following corollaries.

Corollary 8 *Let (α, M) be an $\eta-$strip slant helix with the axis V is given by (2.23). If the curve α of (α, M) is a geodesic curve on M, then the position vector of the curve α always lies in the plane $sp\{\xi, \zeta\}$.*

Corollary 9 *Let (α, M) be an $\eta-$strip slant helix with the axis V is given by (2.24). If the curve α of (α, M) is a geodesic curve on M, then the curve α is a general helix.*

Also, we can give the following corollary which gives the relationship between $\xi-$strip slant helices and $\eta-$strip slant helices.

Corollary 10 *Let the curve α be a geodesic curve on M. Then, (α, M) is a $\xi-$strip slant helix if and only if (α, M) is an $\eta-$strip slant helix with the axis (2.24).*

Case 2.2 Let the curve α is an asymptotic curve (i. e. $k_n = 0$), then from (2.19) we have

$$\left\{\begin{aligned} v_1' - ck_g &= 0, \\ v_1k_g - v_3\tau_g &= 0, \\ v_3' + c\tau_g &= 0. \end{aligned}\right. \tag{2.25}$$

From the first and the third equation of (2.25), we get

$$\begin{aligned} v_1 &= c\int k_g ds, \\ v_3 &= -c\int \tau_g ds. \end{aligned} \tag{2.26}$$

Substituting (2.26) in the second equation of (2.25), we obtain that the curvature functions of (α, M) satisfy the relation as follows

$$k_g\int k_g ds + \tau_g\int\tau_g ds = 0.$$

Also, since $k_n = \kappa\sin\theta = 0$, we find that $\theta = k\pi$ $(k = 0, 1)$ so by using the definition of k_g and τ_g, we have $k_g = \pm\kappa$ and $\tau_g = \tau$. Therefore, the axis of (α, M) is given by

$$V = \pm c\left(\int\kappa ds\right)\xi + c\eta - c\left(\int\tau ds\right)\zeta, \tag{2.27}$$

where $c \in \mathbb{R}_0$. Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & \pm\kappa & 0 \\ \mp\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix}.$$

Case 2.3 Let the curve α is a principal curve (i. e. $\tau_g = 0$), then from (2.19) we have

CBU J. of Sci., Volume 13, Issue 1, 2017, p 113-123

$$\begin{cases} v_1' - v_3 k_n - c k_g = 0, \\ v_1 k_g = 0, \\ v_3' + v_1 k_n = 0. \end{cases} \tag{2.28}$$

From the second equation of (2.28) we get $k_g = 0$ or $u_1 = 0$.

(i) If $k_g = 0$ for all s, then from the first and third equation of (2.28) we have

$$\frac{d}{ds}\left(\frac{1}{k_n}\frac{dv_1}{ds}\right) + v_1 k_n = 0.$$

The solution of the previous differential equation is given by

$$v_1 = C_1 \cos\left(\int k_n ds\right) + C_2 \sin\left(\int k_n ds\right).$$

Also, the first equation of (2.28) we have

$$v_3 = -C_1 \sin\left(\int k_n ds\right) + C_2 \cos\left(\int k_n ds\right).$$

On the other hand, since $k_g = \kappa \cos\theta = 0$, we find that $\theta = \mp\frac{\pi}{2}$ so by using the definition of k_n and τ_g , we have $k_n = \mp\kappa$ and $\tau = 0$. Therefore, the axis of (α, M) is given by

$$V = v_1\xi + c\eta + v_3\zeta,$$

where

$$\begin{aligned} v_1 &= C_1 \cos\left(\int \kappa ds\right) \mp C_2 \sin\left(\int \kappa ds\right), \\ v_3 &= -C_1 \sin\left(\int \kappa ds\right) \mp C_2 \cos\left(\int \kappa ds\right). \end{aligned}$$

(ii) If $v_1 = 0$ for all s, then from (2.28) we get $v_3 = -c\frac{k_g}{k_n} \in \mathbb{R}_0$ so by the definition of k_n and k_g we get $\theta = \theta_0 = constant$. Thus, we obtain $k_n = \kappa \sin\theta_0$, $k_g = \kappa\cos\theta_0$ and $\tau = 0$. Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \kappa \begin{bmatrix} 0 & \cos\theta_0 & \sin\theta_0 \\ -\cos\theta_0 & 0 & 0 \\ -\sin\theta_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix},$$

and the axis of (α, M) always lies in the plane $sp\{\eta, \zeta\}$ and is given by

$$V = c\eta - c\kappa(\tan\theta_0)\zeta, \tag{2.29}$$

where $c \in \mathbb{R}_0$ and $\theta_0 = constant$.

Therefore, we can give the following corollary which gives the relationship between $\xi-$strip slant helices and $\eta-$strip slant helices.

Corollary 11 *Let the curve α be a principal curve on M . Then, (α, M) is a $\xi-$strip slant helix if and only if (α, M) is an $\eta-$strip slant helix with the axis (2.29).*

Finally, let us characterize ζ-type slant helices.

Case 3 ($\zeta-$strip slant helices) If (α, M) is a $\zeta-$strip slant helix parameterized by the arclength s in $\mathbb{E}^3$, then according to Definition 4, there exists a non-zero constant vector field $U \in \mathbb{E}^3$ such that

$$g(\zeta, W) = c, \quad c \in \mathbb{R}_0. \tag{2.30}$$

With respect to the Frenet frame $\{\xi, \eta, \zeta\}$ of (α, M), the fixed direction U can be decomposed as

$$W = w_1\xi + w_2\eta + c\zeta, \tag{2.31}$$

where w_1 and w_2 are differentiable functions of the curvatures. Differentiating the equation (2.21) with respect to s and using the equations (2.3), we obtain the following system of differential equations

$$\begin{cases} w_1' - w_2 k_g - c k_n = 0, \\ w_2' + w_1 k_g - c\tau_g = 0, \\ w_1 k_n + u_2 \tau_g = 0. \end{cases} \tag{2.32}$$

From the first and the third equations of(2.32) we get

$$\begin{cases} w_1 = ce^{-\int \frac{k_n k_g}{\tau_g} ds} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right), \\ w_2 = -c\frac{k_n}{\tau_g} e^{-\int \frac{k_n k_g}{\tau_g} ds} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right), \end{cases} \quad (2.33)$$

where $c \in \mathbb{R}_0$. Substituting (2.33) in the second equation of (2.32), we obtain that the curvature functions of (α, M) satisfy the relation

$$a \left[\left(\frac{k_n}{\tau_g} \right)' - k_g^2 \left(\frac{k_n}{\tau_g} \right)^2 - k_g \right] + \frac{k_n^2}{\tau_g} + \tau_g = 0, \quad (2.34)$$

where $a = e^{-\int \frac{k_n k_g}{\tau_g} ds} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right)$.

Conversely, assume that (2.34) holds. Consider the vector W given by

$$W = ce^{-\int \frac{k_n k_g}{\tau_g} ds} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right) \xi \\ - c\frac{k_n}{\tau_g} e^{-\int \frac{k_n k_g}{\tau_g} ds} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right) \eta + c\zeta,$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using the equations (2.3) and (2.32), we find $W' = 0$. Hence W is a fixed direction. It can be easily checked that

$$g(\zeta, W) = c, \quad c \in \mathbb{R}_0.$$

According to Definition 4, (α, M) is a $\zeta-$strip slant helix with the axis W.

Therefore, we can give the following theorem and corollary.

Theorem 3 *Let (α, M) be a strip in $\mathbb{E}^3$ with the curvatures k_g, k_n and τ_g. Then (α, M) is a $\zeta-$strip slant helix if and only if its curvature functions k_g, k_n and τ_g satisfy the relation*

$$a \left[\left(\frac{k_n}{\tau_g} \right)' - k_g^2 \left(\frac{k_n}{\tau_g} \right)^2 - k_g \right] + \frac{k_n^2}{\tau_g} + \tau_g = 0,$$

where $\tau_g \neq 0$ for all s and

$$a = \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right) e^{-\int \frac{k_n k_g}{\tau_g} ds}.$$

Corollary 12 *The axis of a $\zeta-$strip slant helix (α, M) in $\mathbb{E}^3$ is given by*

$$W = c \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right) e^{-\int \frac{k_n k_g}{\tau_g} ds} \xi \\ - c\frac{k_n}{\tau_g} \left(\int k_n e^{\int \frac{k_n k_g}{\tau_g} ds} ds \right) e^{-\int \frac{k_n k_g}{\tau_g} ds} \eta + c\zeta,$$

where $c \in \mathbb{R}_0$.

Putting $c = 0$ in relation (2.32), we get

$$\begin{cases} w_1' - w_2 k_g = 0, \\ w_2' + w_1 k_g = 0, \\ w_1 k_n + w_2 \tau_g = 0. \end{cases}$$

and

$$w_1 = a_3 e^{-\int \frac{k_n k_g}{\tau_g} ds}, \\ w_2 = -a_3 \frac{k_n}{2\tau_g} e^{-\int \frac{k_n k_g}{\tau_g} ds},$$

where $a_3 \in \mathbb{R}_0$.

Therefore, we obtain the next corollary.

Corollary 13 *Let (α, M) be a $\zeta-$strip slant helix with the axis W in $\mathbb{E}^3$. If its normal vector ζ is orthogonal to the axis W, then the axis W is given by*

$$W = a_3 e^{-\int \frac{k_n k_g}{\tau_g} ds} \xi - a_3 \frac{k_n}{2\tau_g} e^{-\int \frac{k_n k_g}{\tau_g} ds} \eta,$$

where $a_3 \in \mathbb{R}_0$.

Now, we consider the following special cases when the curvatures k_g, k_n and τ_g of the strip are zero, respectively.

Case 3.1 Let the curve α is a geodesic curve (i. e. $k_g = 0$), then we find that $\theta = \mp\frac{\pi}{2}$ so by using the definition of k_n and τ_g, we have $k_n = \mp\kappa$ and

$\tau_g = \tau$. Also, from (2.32) we have

$$\begin{cases} w_1' - ck_n = 0, \\ w_2' - c\tau_g = 0, \\ w_1 k_n + w_2 \tau_g = 0. \end{cases} \tag{2.35}$$

From the first and second equations of (2.35) we get

$$\begin{array}{l} w_1 = c\int k_n ds = \mp c\int \kappa ds, \\ w_2 = c\int \tau_g ds = c\int \tau ds. \end{array} \tag{2.36}$$

Substituting (2.36) in the third equation of (2.35), we obtain that the curvature functions of (α, M) satisfy the relation as follows

$$k_g \int k_g ds + \tau_g \int \tau_g ds = 0,$$

or

$$\kappa \int \kappa ds + \tau \int \tau ds = 0.$$

Thus, we have the frame $\{\xi, \eta, \zeta\}$ of the strip (α, M) as follows:

$$\begin{bmatrix} \xi'(s) \\ \eta'(s) \\ \zeta'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mp\kappa \\ 0 & 0 & \tau \\ \pm\kappa & -\tau & 0 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{bmatrix},$$

and the axis of (α, M) always lies in the plane $sp\{\eta, \zeta\}$ and is given by

$$W = w_1\xi + w_2\eta + c\zeta,$$

where $c \in \mathbb{R}_0$ and

$$\begin{array}{l} w_1 = \mp c\int \kappa ds, \\ w_2 = c\int \tau ds. \end{array}$$

Case 3.2 Let the curve α is an asymptotic curve (i. e. $k_n = 0$), then from (2.32) we have

$$\begin{cases} w_1' - w_2 k_g = 0, \\ w_2' + w_1 k_g - c\tau_g = 0, \\ w_2 \tau_g = 0. \end{cases} \tag{2.37}$$

From the third equation of (2.37) we get $\tau_g = 0$ or $u_2 = 0$.

(i) If $\tau_g = 0$ for all s, then $k_g = \pm\kappa$ and $\tau = 0$. So from (2.37) we get

$$\begin{array}{l} w_1 = r\cos\upsilon, \\ w_2 = r\sin\upsilon, \end{array}$$

where $r \in \mathbb{R}^+$ and $\upsilon = \mp\int \kappa ds$. Therefore, the axis of (α, M) is given by

$$W = r\cos(\upsilon)\,\xi + r\sin(\upsilon)\,\eta + c\zeta \tag{2.38}$$

where $c \in \mathbb{R}_0$.

(ii) If $w_2 = 0$ for all s, then from (2.37) we have

$$w_1 = c\frac{\tau_g}{k_g} = constant.$$

Also, by using the definition of k_g and τ_g, we have $k_g = \mp\kappa$ and $\tau_g = \tau$ since $k_n = 0$. Then

$$w_1 = \mp c\frac{\tau}{\kappa},$$

and the axis of (α, M) is

$$U = \mp c\frac{\tau}{\kappa}\xi + c\zeta, \tag{2.39}$$

where $\frac{\tau}{\kappa} = constant$ and $c \in \mathbb{R}_0$.

Therefore, we can give the following corollaries.

Corollary 14 *Let (α, M) be a $\zeta-$strip slant helix with the axis W is given by (2.39). If the curve α of (α, M) is an asymptotic curve on M, then the curve α is a general helix.*

Also, we can give the following corollary which gives the relationship between $\xi-$strip slant helices and $\zeta-$strip slant helices.

Corollary 15 *Let the curve α be an asymptotic curve on*

M. Then, (α, M) is a $\xi-$strip slant helix if and only if (α, M) is a $\zeta-$strip slant helix with the axis (2.39).

Case 3.3 Let the curve α is a principal curve (i. e. $\tau_g = 0$), then from (2.32) we have

$$\begin{cases} w_1' - w_2 k_g - c k_n = 0, \\ w_2' + w_1 k_g = 0, \\ w_1 k_n = 0. \end{cases} \tag{2.40}$$

From the third equation of (2.40) we get $k_n = 0$ or $w_1 = 0$.

(i) If $k_n = 0$ for all s, then $k_g = \pm\kappa$ and $\tau = 0$. So from (2.40) we get

$$w_1 = R\cos\varphi,$$
$$w_2 = R\sin\varphi,$$

where $R \in \mathbb{R}^+$ and $\varphi = \mp\int \kappa ds$. Therefore, the axis of (α, M) is given by

$$W = R\cos(\varphi)\,\xi + R\sin(\varphi)\,\eta + c\zeta$$

where $c \in \mathbb{R}_0$.

(ii) If $w_1 = 0$ for all s, then from (2.40) we have

$$w_2 = -c\tfrac{k_n}{k_g} = constant.$$

Also, by using the definition of τ_g we have $\theta = \int \tau ds$. Thus, we obtain $k_n = \kappa\sin\left(\int \tau ds\right)$, $k_g = \kappa\cos\left(\int \tau ds\right)$ and the axis of (α, M) is

$$W = -c\tfrac{k_n}{k_g}\eta + c\zeta, \tag{2.41}$$

where $c \in \mathbb{R}_0$.

Therefore, we can give the following corollary which gives the relationship between $\eta-$strip slant helices and $\zeta-$strip slant helices.

Corollary 16 *Let the curve α of (α, M) be an asymptotic curve on M. Then, (α, M) is an $\eta-$strip slant helix with (2.24) if and only if (α, M) is a $\zeta-$strip slant helix with the axis (2.41).*

4 References

[1] Struik, D. J. Lectures on classical differential geometry. Addison Wesley, Dover, 1988; 240 pp.

[2] Izumiya, S.; Tkeuchi, N. New special curves and developable surfaces. Turk J. Math. 2004; 28, 153–163.

[3] Kula, L.; Yayli, Y. On slant helix and its spherical indicatrix. Appl. Math. Comput. 2005; 169(1), 600—607.

[4] Kula, L.; Ekmekci, N.; Yayli, Y.; Ilarslan, K. Characterizations of slant helices in Euclidean 3-space. Turk. J. Math. 2010; 34, 261–273.

[5] Ali, A. T.; López, R. Slant helices in Euclidean 4-space E^4. J. Egyptian Math. Soc. 2010; 18(2), 223–230.

[6] Ali, A. T.; Turgut, M. Some characterizations of slant helices in the Euclidean space $\mathbb{E}^n$. Hacet. J. Math. Stat. 2010; 39(3), 327–336.

[7] Hacısalihoglu, H. H. On the relations between the higher curvatures of a curve and a strip. Communications de la Faculté des Sciences de L'Université d' Ankara Serie A1. 1982; 31, 5–14.

[8] Sabuncuoglu, A.; Hacisalihoglu, H. H. Higher curvatures of a strip. Comm. Fac. Sci. Univ. Ankara Sér. A1 Math. 1975; 24, 25—33.