Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 1, 272–279.
https://doi.org/10.31197/atnaa.1235476
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

# New existence result under weak topology for fractional differential equations

Ahmed Hallaci[a], Aref Jeribi[b], Bilel Krichen[b], Bilel Mefteh[b]

[a] *Department of exact sciences, Ecole Normale Supérieure de Ouargla, Ouargla, Algeria.*

[b] *Department of Mathematics, Faculy of Sciences of Sfax, Sfax University, Sfax, Tunisia.*

**Abstract**

This paper deals with the existence of integrable solutions for an initial value problem involving Riemann-Liouville-type fractional derivatives. To this end, we transform the posed problem to a sum of two integral operators, then we apply a variant of Krasnoselskii's fixed point theorem under weak topology to conclude the existence of integrable solutions. Lastly, an example to demonstrate the effectiveness of our main result is presented.

*Keywords:* Fractional differential equations Weak topology Existence result Fixed point theorems.
*2010 MSC:* 65L05, 26A33, 47H10, 47H30

## 1. Introduction

Recently, many researchers have been interested in developing the theory of fractional calculus, and this has led to the emergence of many approaches and concepts for fractional derivative and fractional integral. These concepts have been used extensively in the field of differential equations and have been applied a lot in many scientific fields such as physics, chemistry, biology, engineering, viscoelasticity, signal processing and electrochemistry, for more details, see [1, 2, 3, 4, 5]. In the present paper, we prove the existence of weak solutions for the following fractional differential equation with initial conditions

$$\begin{cases} D^{\alpha}u(t) = f(t, L_1u(t)) + g(t, L_2u(t)), t \in I := [0, T], \\ D^{\alpha-i}u|_{t=0} = 0, i = 1, 2, \end{cases} \tag{1}$$

*Email addresses:* hallaci.ahmed@ens-ouargla.dz, hal.ahmedguelm2@gmail.com (Ahmed Hallaci), Aref.Jeribi@fss.rnu.tn (Aref Jeribi), bilel.krichen@fss.usf.tn (Bilel Krichen), bilel.mefteh@gmail.com (Bilel Mefteh)

*Received January 16, 2023; Accepted: March 23, 2023; Online: March 30, 2023.*

where $u$ belongs to $L^1(I,E)$, the space of Lebesgue integrable functions on $I$ with values in a finite dimentional Banach space $(E, \|.\|)$, provided with the norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\| \, dt,$$

$D^\alpha$ is the left Riemann Liouville derivative of order $1 < \alpha \leq 2$. Here, $f(.,.)$ and $g(.,.)$ are nonlinear functions, $L_i : L^1(I,E) \to L^1(I,E), i = 1,2$, are continuous linear maps.

Fractional differential equation has been studied in many research papers (see for example the papers [6, 7, 8, 9] and the references therein). All papers deal with the existence, uniqueness and stability of strong solutions using fixed point theorems in Banach spaces. Beside, to the best of our knowledge, the use of fixed point theorems under weak topology for fractional differential equations is still not sufficiently generalized. However, fixed point theory under weak topology has been used in some papers and monographs for integral equations and fractional differential equations to prove the existence of integrable solutions [10, 11, 12] and the book [13].

In [10], Hallaci et al. studied the following fractional differential equation

$$\begin{cases} D^\alpha u(t) = h(t) f(t, u(t)) + g(t, Hu(t)), t \in I := [0, T], \\ \lim\limits_{t \to 0+} t^{2-\alpha} u(t) = \lim\limits_{t \to 0+} t^{2-\alpha} u'(t) = 0, \end{cases}$$

we have come to prove the existence of the integrable solutions for this problem by using Krasnoselskii type fixed point theorem with the De Blasi measure of weak noncompactness.

In [11] Latrach and Taoudi established a new variant of Krasnoselskii type fixed point theorem under weak topology and use it to investigate the existence of integrable solutions for the following integral equation

$$u(t) = g(t, u(t)) + \lambda \int_\Omega k(t,s) f(s, u(s)) ds.$$

Taoudi et al.[12] prove the existence of integrable solutions for a generalized mixed type operator equation

$$u(t) = g(t, u(t)) + (BN_f UAu)(t),$$

In [13], Jeribi and Krichen studied the existence of solutions for the following variants of Hammerstein's integral equation:

$$u(t) = g(t, L_2 u(t)) + \lambda \int_0^t k(t,s) f(s, L_1 u(s)) ds,$$

where $u \in L^1([0,T])$. For the last two equations, the authors investigated the existence of solutions using fixed point theorem involving sum of two operators under weak topology.

By combining with the fixed point theory under weak topology with De Blasi measure of weak noncompactness and the theory of fractional differential equations, we give sufficient conditions on the functions $f$ and $g$ to prove that IVP (1) has at least one integrable solution. To this purpose, we give some preliminary concepts and Lemmas about fractional calculus theory and weak topology. Then, we transform IVP (1) into Volterra type integral equation employing some useful definitions and lemmas of fractional integral and derivative. After that, we present our main result which based on a variant of fixed point theorem developed in [13].

## 2. Preliminaries

This section is intended to present the notations used in our paper and some ancillary facts that will be required in our considerations. Furthermore, we provide definitions of the key concepts applied in our study as well as point out some key properties of the concepts used in our reasonings.

**Definition 2.1.** *([1, 2]) The Riemann-Liouville fractional integral of the function $u$ of order $\alpha \geq 0$ is defined by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t} \frac{u(\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$

*where $\Gamma(.)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt$.*

**Definition 2.2.** *([1, 2]) The Riemann-Liouville fractional derivative of the function $u$ of order $\alpha \in (n-1, n]$ is defined by*

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{0}^{t} \frac{u(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau.$$

For $\alpha > 0$ be a real number, we have the following lemma.

**Lemma 2.3.** *[2] For $\alpha \in ]n-1, n]$ and $t \in I$, we have the properties*
(1) $D^{\alpha}I^{\alpha}u(t) = u(t)$,
(2) *if $\alpha < \beta$ for an integer $\beta$, then $D^{\alpha}I^{\beta}u(t) = I^{\beta-\alpha}u(t)$,*
(3) *for $q > 0$, the Laplace transform of the Riemann-Liouville fractional derivative $D^{\alpha}u(t)$ and the power function $t \mapsto t^q$ are given respectively by:*
*(i)* $L\{D^{\alpha}u(t), z\} = z^{\alpha}U(z) - \sum_{i=0}^{n-1} z^i \left[D^{\alpha-i-1}u(t)\right]_{t=0}$,
*(ii)* $L\{t^q, z\} = \Gamma(q+1)z^{-(q+1)}$,
*where $U(z)$ denotes the Laplace transforme of $u(t)$.*

**Definition 2.4.** *Let $E$ be a Banach space. We denote by $B(E)$ the collection of all nonempty bounded subsets of $E$ and $\mathcal{W}(E)$ is the subset of $B(E)$ consisting of all weakly compact subsets of $E$. Denote by $B_r$ the closed ball in $E$ centered at 0 with radius $r$. The notion of weak non-compactness measure has been introduced by De Blasi [14] by the map $w : B(E) \to [0, \infty)$ defined as follows*

$$\omega(M) = \inf\{r > 0 : \exists W \in \mathcal{W}(E) \text{ with } M \subseteq W + B_r\},$$

*for each $M \in B(E)$.*

The following useful properties of the function $\omega(.)$ are presented in [14, 15].

**Lemma 2.5.** *Let $M_1, M_2 \in B(E)$, then we have*
- $M_1 \subseteq M_2$ *implies that* $\omega(M_1) \leq \omega(M_2)$.
- $\omega(M_1) = 0$ *if and only if* $\overline{M_1}^{\omega} \in \mathcal{W}(E)$ *where* $\overline{M_1}^{\omega}$ *is the weak closure of* $M_1$.
- $\omega\left(\overline{M_1}^{\omega}\right) = \omega(M_1)$.
- $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
- $\omega(\delta M_1) = |\delta|\,\omega(M_1)$ *for all* $\delta \in \mathbb{R}$.
- $\omega(co(M_1)) = \omega(M_1)$.
- $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
- *If* $(M_n)_{n\geq 1}$ *is a sequence of nonempty, weakly closed subsets of $E$ with $M_1$ bounded and* $M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$ *with* $\lim_{n\to}\omega(M_n) = 0$, *then* $\bigcap_{n=1}^{\infty} M_n \neq \phi$ *and* $\omega\left(\bigcap_{n=1}^{\infty} M_n\right) = 0$.

In $L^1$ space, the measure $\omega(.)$ possesses the following form (see[15]).

**Definition 2.6.** *Let $\Omega$ be compact subset of $\mathbb{R}^n$ and let $M$ be a bounded subset of $L^1(\Omega, E)$ where $E$ is a finite-dimensional Banach space. Then $\omega(.)$ possesses the following form*

$$\omega(M) = \lim_{\epsilon \to 0} \sup \left\{ \sup_{\Psi \in M} \left\{ \int_D \|\Psi(t)\| \, dt : meas(D) \leq \epsilon \right\} \right\},$$

*for any nonempty subset $D$ of $\Omega$, where $meas(.)$ denotes the Lebesgue measure.*

**Definition 2.7.** *Let $\Omega \subset \mathbb{R}^n$ and let $E, F$ be two Banach spaces. A function $f : \Omega \times E \to F$ is said to be weak Carathéodory, if:*
*$(i)$ For any $u \in E$, the map $t \longmapsto f(t,u)$ is measurable from $\Omega$ to $F$, and*
*$(ii)$ For almost all $t \in \Omega$, the map $u \longmapsto f(t,u)$ is weakly sequentially continuous from $E$ to $F$.*

Let $m(\Omega, E)$ be the set of all measurable functions $u : \Omega \to E$. If $f$ is a weak Carathéodory function, then $f$ defines a mapping $N_f : m(\Omega, E) \to m(\Omega, F)$ by $N_f u(t) := f(t, u(t))$, for all $t \in \Omega$. This mapping is called the Nemytskii's operator associated to $f$.

The following Lemma for the weak sequentially continuity of Nemytskii operator is needed.

**Lemma 2.8** ([16])**.** *(Weak sequentially continuity of Nemytskii's operator)*
*Let $\Omega \subset \mathbb{R}^n$ and let $E$ be reflexive Banach space, and $p, q \geq 1$ and let $L : L^p(\Omega, E) \to L^p(\Omega, E)$ be a continuous linear map. Let $f : \Omega \times E \to E$ be a weak Carathéodory map satisfying*

$$\|f(t,u)\| \leq A(t)\, h(\|u\|),$$

*where $A \in L^q(\Omega, \mathbb{R}_+)$ and $h \in L^\infty_{loc}(\mathbb{R}_+)$. Then, if either $q > 1$ or $p = q = 1$, the map $N_f \circ L : L^p(\Omega, E) \to L^q(\Omega, E)$ is weakly sequentially continuous.*

**Lemma 2.9** ([17])**.** *Let $(S, \sum, \pi)$ be a positive measure space. If a set $K$ in $L^1(S, \sum, \pi)$ is weakly sequentially compact, then, for any nonempty subset $E$ of $S$, we have*

$$\lim_{\pi(E) \to 0} \int_E f(s)\, \pi(ds) = 0$$

*uniformly for $f \in K$. If $\pi(S) < \infty$, then conversely this condition is sufficient for a bounded set $K$ to be weakly sequentially compact.*

The following variant of Krasnoselskii fixed point theorem is of fundamental importance in our considerations.

**Theorem 2.10** ([13])**.** *. Let $M$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$. Suppose that $A : M \to E$ and $B : E \to E$ are two weakly sequentially continuous mappings such that:*
*$(i)$ $A(M)$ is relatively weakly compact,*
*$(ii)$ $B$ is a contraction, and*
*$(iii)$ $(x = Bx + Ay, y \in M) \Rightarrow x \in M$.*
*Then, $A + B$ has, at least, a fixed point in $M$.*

## 3. Main Result

In this section, we are interested by showing the existence of integrable solutions for problem (1) by applying theorem 2.10. Before this, we present the following lemma:

**Lemma 3.1.** *Let $1 < \alpha \leq 2$. The unique solution of linear IVP*

$$\begin{cases} D^{\alpha}u(t) = y(t), t \in I := [0, T], \\ D^{\alpha-i}u|_{t=0} = 0, i = 1, 2, \end{cases} \tag{2}$$

*is given by*

$$u(t) = I^{\alpha}y(t), t \in I. \tag{3}$$

*Proof.* We take $\left[D^{\alpha-i}u\left(t\right)\right]_{t=0} = b_i, i = 1, 2$. Applying Laplace transform on both side of first equation of problem (2) and using property three of Lemma 2.3, we get

$$z^{\alpha}U(z) - \sum_{i=0}^{1} z^{i}[D^{\alpha-i-1}u(t)]_{t=0} = Y\left(z\right),$$

where $U\left(z\right)$ and $Y\left(z\right)$ denote the Laplace transformes of $u\left(t\right)$ and $y\left(t\right)$ respectively.

In other words, we can write

$$U(z) = z^{-\alpha}Y(z) + \sum_{i=0}^{1} b_{i+1}z^{i-\alpha}.$$

By applying the inverse Laplace transform with taking into account the notion of convolution product, we find

$$\begin{aligned} u(t) &= \frac{1}{\Gamma\left(\alpha\right)}\int_0^t (t-s)^{\alpha-1}y\left(s\right)ds + \sum_{i=0}^{1}\frac{b_{i+1}}{\Gamma\left(\alpha-i\right)}t^{\alpha-i-1} \\ &= \frac{1}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}y\left(s\right)ds + \sum_{i=1}^{2}\frac{b_i}{\Gamma\left(\alpha-i+1\right)}t^{\alpha-i}, \end{aligned}$$

since $b_i = 0$, $i = 1, 2$ then

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds = I^{\alpha}y(t).$$

Conversely, the expression of $u(t)$ given by (3) satisfies the two equations of problem (2) from properties one and two of lemma 2.3. This completes the proof. □

Clearly, from lemma 3.1, IVP (1) is equivalent to the following operator equation:

$$u = \mathcal{A}u + \mathcal{B}u \tag{4}$$

where $\mathcal{A}$ (resp. $\mathcal{B}$) are two operators defined from $L^1(I, E)$ into itself by:

$$\mathcal{A} := \mathcal{I}\mathcal{N}_f L_1, \tag{5}$$

$$(resp.\mathcal{B} := \mathcal{J}\mathcal{N}_g L_2), \tag{6}$$

which represent the product operator of linear map $L_1$ (resp. $L_2$) and Nemytskii's operator associated to $f(.,.)$ (resp. $g(.,.)$) and the Volterra type linear integral operator $\mathcal{I}$ (resp. $\mathcal{J}$) defined from $L^1\left(I, E\right)$ into itself by:

$$\mathcal{I}z\left(t\right) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z\left(s\right)ds$$

$$\left(resp.\mathcal{J}z(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds\right).$$

Now, we present the main result of this paper, consider the following hypotheses:

$(H1)$ The function $g$ is a measurable, $g(.,0) \in L^1(I,E)$ and $g$ is Lipschitzian with respect to the second variable, i.e., there exists $\lambda > 0$ such that

$$\|g(t,u) - g(t,v)\| \leq \lambda \|u - v\|, \text{ for all } t \in I \text{and} u, v \in E.$$

$(H2)$ The functions $f, g$ satisfy the weak Carathéodory conditions and there exist functions $A_i \in L^1(I, \mathbb{R}_+)$ and nondecreasing functions $h_i \in L^\infty_{loc}(\mathbb{R}_+), i = 1, 2$, such that

$$\|f(t,u)\| \leq A_1(t)\, h_1(\|u\|) \text{ and } \|g(t,u)\| \leq A_2(t)\, h_2(\|u\|).$$

**Theorem 3.2.** *Assume that $(H1) - (H2)$ hold. Then IVP (1) has at least one integrable solution on $I$, provided*

$$\frac{\lambda T^{\alpha+1}}{\Gamma(\alpha)} \|L_2\|_{\mathcal{L}} < 1. \tag{7}$$

*Proof.* Consider operator equation defined by (4). Choosing $R \geq R_0$ where

$$R_0 = \min\left\{ \frac{\|A_1\|_1 \|h_1\|_\infty + \|g(.,0)\|_1}{\frac{\Gamma(\alpha)}{T^\alpha} - \lambda \|L_2\|_{\mathcal{L}}}, \frac{T^\alpha}{\Gamma(\alpha)} \left(\|A_1\|_1 \|h_1\|_\infty + \|A_2\|_1 \|h_2\|_\infty\right) \right\}.$$

Clearly, $R_0 > 0$ according to the relationship (7), we define the bounded, closed, convex set $B_R = \{u \in L^1(I,E) : \|u\|_1 \leq R\}$, we will show that operators $\mathcal{A}$ and $\mathcal{B}$ defined by (5), (6) satisfy all hypotheses of Theorem 2.10.

**Claim 1**: We show that $\mathcal{A} := \mathcal{I}\mathcal{N}_f L_1$ and $\mathcal{B} := \mathcal{J}\mathcal{N}_g L_2$ are weakly sequentially continuous on $L^1(I,E)$. To this end, taking into account Lemma 2.8, hypothesis $(H2)$, we prove that $\mathcal{N}_f L_1$ and $\mathcal{N}_g L_2$ are weakly sequentially continuous on $L^1(I,E)$. Beside, $\mathcal{I}$ and $\mathcal{J}$ are linear continuous operators from $L^1(I,E)$ into itself, then $\mathcal{A}$ and $\mathcal{B}$ are weakly sequentially continuous on $L^1(I,E)$.

**Claim 2**: We prove that $\mathcal{B}$ is a contraction mapping, let $u, v \in L^1(I,E)$. Using $(H2)$ and Hölder inequality, then for all $t \in I$, we have

$$\begin{aligned} \|\mathcal{B}u(t) - \mathcal{B}v(t)\| &\leq \int_0^t (t-s)^{\alpha-1} \|g(s, L_2u(s)) - g(s, L_2v(s))\| \, ds \\ &\leq \frac{\lambda T^\alpha}{\Gamma(\alpha)} \|L_2\|_{\mathcal{L}} \|u - v\|_1, \end{aligned} \tag{8}$$

applying $L^1(I,E)-$norm on both sides of inequatity (8), we get

$$\|\mathcal{B}u - \mathcal{B}v\|_1 \leq \frac{\lambda T^{\alpha+1}}{\Gamma(\alpha)} \|L_2\|_{\mathcal{L}} \|u - v\|_1.$$

So, $\mathcal{B}$ is a contraction mapping on $L^1(I,E)$ with constant $\frac{\lambda T^{\alpha+1}}{\Gamma(\alpha)} \|L_2\|_{\mathcal{L}}$.

**Claim 3**: We show that $v = \mathcal{A}u + \mathcal{B}v \in B_R$, for all $u \in B_R$, indeed

$$\begin{aligned} \|v(t)\| &= \|\mathcal{A}u(t) + \mathcal{B}v(t)\| \\ &\leq \|\mathcal{I}\mathcal{N}_f L_1 u(t)\| + \|\mathcal{J}\mathcal{N}_g L_2 v(t)\|, \end{aligned}$$

using respectively $(H1), (H2)$, we find

$$\|v(t)\| \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|A_1\|_1 \|h_1\|_\infty + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(\lambda \|L_2\|_{\mathcal{L}} \|v\|_1 + \|g(.,0)\|_1\right), \tag{9}$$

$$\|v(t)\| \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|A_1\|_1 \|h_1\|_\infty + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|A_2\|_1 \|h_2\|_\infty, \tag{10}$$

here, $\|L_2\|_{\mathcal{L}}$ denotes the standard norm of linear operator spaces.
Applying $L^1(I,E)-$norm on both sides of inequatities (9) and ( 10), we get

$$\|v\|_1 \leq \frac{T^{\alpha}}{\Gamma(\alpha)} \|A_1\|_1 \|h_1\|_{\infty} + \frac{T^{\alpha}}{\Gamma(\alpha)} \left(\lambda \|L_2\|_{\mathcal{L}} \|v\|_1 + \|g(.,0)\|_1\right).$$

and

$$\|v\|_1 \leq \frac{T^{\alpha}}{\Gamma(\alpha)} \|A_1\|_1 \|h_1\|_{\infty} + \frac{T^{\alpha}}{\Gamma(\alpha)} \|A_2\|_1 \|h_2\|_{\infty}.$$

This means that, for $R \geq R_0 : v = \mathcal{A}u + \mathcal{B}v \in B_R$ for all $u \in B_R$.

**Claim 4**: We show that $\mathcal{A}B_R$ is relatively weakly compact for all $R \geq R_0$ using De Blasi measure of weak noncompactness in $L^1(I,E)$. Indeed, let $S$ be a bounded subset of $B_R$ and let $\epsilon$ be a positive real number. For any nonempty subset $J$ of $I$, and for all $u \in S$, we have

$$\begin{aligned}
\int_J \|\mathcal{N}_f L_1 u(t)\| \, dt &\leq \int_J \|A_1(t) h_1(\|L_1 u(t)\|)\| \, dt \\
&\leq \|h_1\|_{\infty} \int_J \|A_1(t)\| \, dt \\
&\leq \|h_1\|_{\infty} \|A_1\|_1, \text{ on } J.
\end{aligned}$$

Then, by using Lemma 2.9, we get

$$\lim_{\epsilon \to 0} \sup \left\{ \|h_1\|_{\infty} \int_J \|A_1(t)\| \, dt : meas(J) \leq \epsilon \right\} = 0.$$

So, we have $\omega(\mathcal{N}_f L_1 S) = 0$ which means that $\mathcal{N}_f L_1 S$ is relatively weakly compact. Moreover, $\mathcal{I}$ is bounded due to the boundness of Riemann-Liouville integral operator on $L^1(I)$, then we conclude that $\omega(\mathcal{A}S) = 0$, and consequently, $\mathcal{A}S$ is relatively weakly compact. Finally, by applying Theorem 2.10, we deduce that the operator $\mathcal{A} + \mathcal{B}$ has, at least, one fixed point on $I$, which is the solution of IVP (1). $\square$

## 4. An example

To illustrate the application of the obtained result, we consider the following example:

$$\begin{cases} D^{\alpha}u(t) = t\sin[L_1 u(t)]^2 + t^2 e^{-\left[1+e^{-L_2 u(t)}\right]}, t \in I := [0, \frac{\pi}{2}], \\ \lim\limits_{t\to 0+} t^{2-\alpha}u(t) = \lim\limits_{t\to 0+} t^{2-\alpha}u'(t) = 0, \end{cases} \tag{11}$$

We took: $T = \frac{\pi}{2}, (E, \|.\|) = (\mathbb{R}, |.|), \alpha = 1.5, f(t,u(t)) = t\sin[u(t)]^2, g(t,u(t)) = t^2 e^{-\left[1+e^{-u(t)}\right]}; L_1, L_2 : L^1(I,\mathbb{R}) \longrightarrow L^1(I,\mathbb{R}), L_1 u(t) = \int_0^{\frac{\pi}{2}} u(t)\, dt, L_2 u(t) = m(t) u(t)$ with $m : \mathcal{C}(I,\mathbb{R}) \longrightarrow \mathcal{C}(I,\mathbb{R}), m^* = \max_{t\in I} m(t)$. It's clear that:

$$|f(t,u(t))| \leq A_1(t) h_1(|u(t)|), |g(t,u(t))| \leq A_2(t) h_2(|u(t)|),$$

where $A_1(t) = t, A_2(t) = \frac{t^2}{e}, \|A_1\|_1 = \frac{\pi^2}{8}, \|A_2\|_1 = \frac{\pi^3}{24e}$ and $h_1, h_2$ are positive nondecreasing functions defined on $\mathbb{R}_+$ by $h_1(u) = \sin u, h_2(u) = e^{-e^{-u}}$, and

$$\|h_1\|_{\infty} = \|h_2\|_{\infty} = 1, \|L_1\|_{\mathcal{L}} = \frac{\pi}{2}, \|L_2\|_{\mathcal{L}} = m^*.$$

Moreover, since $z + e^{-z} \geq 1$ for all real $z$, then one gets

$$|g(t,u) - g(t,u)| \leq \left(\frac{\pi}{2e}\right)^2 |u - v|, \lambda = \left(\frac{\pi}{2e}\right)^2,$$

$$g(t,0) = \left(\frac{t}{e}\right)^2, \|g(t,0)\|_1 = \frac{\pi^3}{24e^2}.$$

On the other hand, for $m(t) = \frac{1}{2}cos(t)$, we have: $\frac{\lambda T^{\alpha+1}}{\Gamma(\alpha)} \|L_2\|_{\mathcal{L}} = 0.5826 < 1$ and $R_0 = \min\{4.97376, 3.79639\} = 3.79639$. In conclusion, problem (11) has at least one integrable solution on $[0, \frac{\pi}{2}]$.

## 5. Conclusions

In this paper, we presented a study of the existence of integrable solutions to an initial value problem of fractional differential equations involving Riemann-Liouville fractional derivative using the fixed-point theorems of the sum of two operators under weak topology. This type of fixed point theorems is developed extensively in the book [13]. Beside this theorem, we used the De Blasi measure of weak non-compactness and the notion of weak Carathéodory functions. Finally, we mention that the field of differential equations of fractional orders has not been previously studied using this kind of fixed point theorems.
**Acknowledgement.** We extend our thanks in advance to the anonymous reviewers for accepting to review our paper and for providing us with useful comments on this paper.

## References

[1] R. (Ed.). Hilfer, Applications of fractional calculus in physics, World scientific, 2000.
[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, elsevier, 204(2006).
[3] I. Podlubny, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Math. Sci. Eng. 198(1999).
[4] S. Das, I. Pan, Fractional order signal processing: introductory concepts and applications, Springer Science and Business Media, 2011.
[5] I. Petráš, Fractional-order nonlinear systems: modeling, analysis and simulation, Springer Science and Business Media, 2011.
[6] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications, 338(2) 1340-1350.
[7] R. P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, Computers and Mathematics with Applications, 59(3) (2010) 1095-1100.
[8] K. Nouri, M. Nazari, B. Keramati, Existence results of solutions for some fractional neutral functional integro-differential equations with infinite delay, doi:10.20944/preprints201706.0090.v1, 2017.
[9] B. Ahmad, S. K. Ntouyas, Existence and uniqueness of solutions for caputo-hadamard sequential fractional order neutral functional differential equations, Electronic Journal of Differential Equations, 36(2017) 1-11.
[10] A. Hallaci, A. Jeribi, B. Krichen, B. Mefteh, Existence Results for Fractional Differential Equations Under Weak Topology Features, Pan-American Journal of Mathematics, 1 (2022) 14.
[11] K. Latrach, M. A. Taoudi, Existence results for a generalized nonlinear Hammerstein equation on L1 spaces, Nonlinear Analysis, 66(2007) 2325-2333.
[12] M. A. Taoudi, N. Salhi, B. Ghribi, Integrable solutions of a mixed type operator equation, Applied Mathematics and Computation, 216(4) (2010) 1150-1157.
[13] A. Jeribi, B. Krichen, Nonlinear Functional Analysis in Banach Spaces and Banach Algebras: FixedPoint Theory Under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications, (Vol.12), CRC Press, 2015.
[14] F. S. De Blasi, On a property of the unit sphere in a Banach space. Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, (1977) 259-262.
[15] J. Appell, Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili, 1984.
[16] C. S. Barroso, E. V. Teixeira, A topological and geometric approach to fixed points results for sum of operators and applications, Nonlinear Analysis: Theory, Methods and Applications, 60(4) (2005) 625-650.
[17] N. Dunford, J. T. Schwartz, W. G. Bade, R. G. Bartle, Linear Operators: Spectral Theory, (Vol. 7), Interscience Publishers, 1958.