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Mulatu Numbers Which Are Concatenation of Two Fibonacci Numbers

Fatih ERDUVAN[1], Merve GÜNEY DUMAN[*2]

Abstract

Let (M_k) be the sequence of Mulatu numbers defined by $M_0 = 4, M_1 = 1, M_k = M_{k-1} + M_{k-2}$ and (F_k) be the Fibonacci sequence given by the recurrence $F_k = F_{k-1} + F_{k-2}$ with the initial conditions $F_0 = 0,\ F_1 = 1$ for $k \geq 2$. In this paper, we showed that all Mulatu numbers, that are concatenations of two Fibonacci numbers are $11, 28$. That is, we solved the equation $M_k = 10^d F_m + F_n$, where d indicates the number of digits of F_n. We found the solutions of this equation as $(k, m, n, d) \in \{(4,2,2,1), (6,3,6,1)\}$. Moreover the solutions of this equation displayed as $M_4 = \overline{F_2 F_2} = 11$ and $M_6 = \overline{F_3 F_6} = 28$. Here the main tools are linear forms in logarithms and Baker Davenport basis reduction method.

Keywords: Mulatu and Fibonacci numbers, linear forms in logarithms, exponential Diophantine equations

1. INTRODUCTION

Let (M_k) be the sequence of Mulatu numbers defined by $M_0 = 4,\ M_1 = 1,\ M_k = M_{k-1} + M_{k-2}$ for $k \geq 2$. The Mulatu numbers are introduced in [1]. Let (F_k) be the Fibonacci sequence given by the recurrence $F_k = F_{k-1} + F_{k-2}$ with the initial conditions $F_0 = 0,\ F_1 = 1$ for $k \geq 2$. Binet formulas of these numbers are

$$M_k = \left(\frac{10-\sqrt{5}}{5}\right)\alpha^k + \left(\frac{10+\sqrt{5}}{5}\right)\beta^k,$$
$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. α and β are the roots of $1 + x - x^2 = 0$. The relation between M_k and F_k with α is given by

$$\alpha^{k-1} \leq M_k \leq 4\alpha^k \quad \text{for all } k \geq 0 \tag{1}$$

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \quad \text{for all } k \geq 1. \tag{2}$$

By induction, the inequalities (1) and (2) can be proved. In [2], let d be the number of digits of F_n, authors gave solutions of the equation

$$F_k = 10^d F_m + F_n$$

as

$$(k, m, n, d) = (7,1,4,1), (7,2,4,1), (8,3,1,1), (8,3,2,1), (10,5,5,1).$$

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In [3], Alan determined that Fibonacci numbers which are concatenations of two Lucas numbers and also Lucas numbers which are concatenations of two Fibonacci numbers. Inspired of these works, we solved the Diophantine equation

$$M_k = 10^d F_m + F_n, \quad (3)$$

where d indicates the number of digits of F_n. If $m = 0$, then we have

$$M_k = F_n$$

from the equality (3). Moreover, the equality

$$M_k = F_{k-3} + F_{k-1} + F_{k+2}$$

can be found in [4]. When we consider the above two equalities together, we get Fibonacci numbers that are sums of the three Fibonacci numbers. But this problem is also solved in [5]. Moreover, we will take $m \geq 2$ since F_1 and F_2 values are the same.

2. PRELIMINARIES

Let γ be an algebraic number of degrees d over $\mathbb{Q}$ with minimal primitive polynomial

$$c_0 x^d + c_1 x^{d-1} + \cdots + c_d = c_0 \sum_{i=1}^{d} (x - \gamma^{(i)}) \in \mathbb{Z}[x],$$

with $\gamma^{(i)}$'s are conjugates of γ and $c_0 > 0$. Then logarithmic height of γ is given

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\left(\max\{|\gamma^{(i)}|, 1\}\right)\right).$$

The following properties are given in [6].

$$h(\gamma_1 \mp \gamma_2) \leq \log 2 + h(\gamma_1) + h(\gamma_2) \quad (4)$$

$$h(\gamma_1 \gamma_2^{\pm 1}) \leq h(\gamma_1) + h(\gamma_2) \quad (5)$$

$$h(\gamma_1^{\,r}) = |r| h(\gamma_1). \quad (6)$$

The following lemma can be found in [7].

Lemma 1. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be positive real algebraic numbers and let $b_1, b_2, \ldots, b_n$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ over $\mathbb{Q}$. Let
$B \geq \max\{|b_1|, |b_2|, \ldots, |b_n|\}$,
$A_i \geq \max\{D \cdot h(\gamma_i), |\log \gamma_i|, (0.16)\}$

for all $i = 1,2, \ldots, n$. If

$$\Gamma := \gamma_1^{\,b_1} \cdot \gamma_2^{\,b_2} \cdots \gamma_n^{\,b_n} - 1 \neq 0$$

then

$$|\Gamma| > \exp(-1.4 \cdot 30^{n+3} \cdot n^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdot A_2 \cdots A_n).$$

In [8], the authors put forward a different version of conclusion of Dujella and Pethő's Lemma in [9]. This lemma is given below.

Lemma 2. Let τ be irrational number, M be a positive integer and $\frac{p}{q}$ be a convergent of the continued fraction of τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Put

$$\varepsilon := \|\mu q\| - M\|\tau q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no positive integer solution (r, s, t) to the inequality

$$0 < |r\tau - s + \mu| < A \cdot B^{-t}$$

subject to the restrictions that

$$r \leq M \text{ and } t \geq \frac{\log\left(Aq/\varepsilon\right)}{\log B}.$$

The following lemma are given in [10].

Lemma 3. Let $\rho, \Gamma \in \mathbb{R}$. If $0 < \rho < 1$ and $|\Gamma| < \rho$, then

$$|\log(1 + \Gamma)| < \frac{\log(1-\rho)}{-\rho} \cdot |\Gamma|$$

and

$|\Gamma| < \frac{-\rho}{e^{-\rho}-1} \cdot |e^{\Gamma} - 1|$.

The following lemma is also useful. We will use it to prove our theorem.

Lemma 4. Suppose that $M_k = 10^d F_m + F_n$, where d indicates the number of digits of F_n. Then we have the following inequalities.

a) $d < \frac{n+3}{4}$
b) $F_n < 10^d < 10F_n$
c) $n + m - 7 < k < n + m + 4$
d) $k - m > 1$
e) $k - n \geq -1$

Proof. a) Since d is the number of digits of F_n, we can write $d = \lfloor \log_{10} F_n \rfloor + 1$. From here, we find

$$\frac{n-2}{5} < \log_{10} \alpha^{n-2} \leq \log_{10} F_n < \lfloor \log_{10} F_n \rfloor + 1 = d.$$

Thus, we obtain $d < \frac{n+3}{4}$.

b) Since $d = \lfloor \log_{10} F_n \rfloor + 1$, we find

$$F_n = 10^{\log_{10} F_n} < 10^d \leq 10^{\log_{10} F_n + 1} < 10F_n.$$

c) By using Lemma 4(b), the inequalities (1) and (2) we obtain

$$\alpha^{k-1} \leq M_k = 10^d F_m + F_n < 10F_n F_m + F_n F_m = 11F_n F_m < \alpha^{n+m+3}$$

and

$$\alpha^{k+3} > 4\alpha^k \geq M_k = 10^d F_m + F_n > F_n F_m > \alpha^{n+m-4}.$$

Thus, we get $n + m - 7 < k < n + m + 4$.

d) Let consider the inequality

$$M_k = 10^d F_m + F_n \geq 10F_m$$

and the equality

$$M_k = 4F_{k+1} - 3F_k$$

given in [1]. If $k \leq m + 1$ then we have

$$10F_m \leq M_k = 4F_{k+1} - 3F_k = 3F_{k-1} + F_{k+1} \leq 3F_m + F_{m+2} \leq 6F_m,$$

which is a contradiction. Thus, $k - m > 1$.

e) Considering the inequality

$$M_k = 10^d F_m + F_n > F_n F_m + F_n \geq 2F_n,$$

it can be seen that $k - n \geq -1$ similar to the proof of Lemma 4(d).

3. MAIN THEOREM

Theorem 5. Let d be the number of digits of F_n, $d \geq 1$, $m \geq 2$ and $n, k \geq 0$. If $M_k = 10^d F_m + F_n$ then

$$(k, F_m, F_n, M_k) \in \{(4,1,1,11), (6,2,8,28)\}.$$

Proof. Assume that the equation (3) is satisfied. If $0 \leq k \leq 109$, we find $(k, M_k, F_m, F_n) \in \{(4,11,1,1), (6,28,2,8)\}$. So suppose that $k \geq 110$. We arrange the equation (3) as

$$\left(\frac{10-\sqrt{5}}{5}\right)\alpha^k - 10^d \cdot \frac{\alpha^m}{\sqrt{5}} = -\left(\frac{10+\sqrt{5}}{5}\right)\beta^k - 10^d \frac{\beta^m}{\sqrt{5}} + F_n.$$

If we multiply both sides of the above equality by $\frac{\sqrt{5}}{10^d \cdot \alpha^m}$ and taking absolute values of this equality, we obtain

$$\left|\frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{10^d} - 1\right| \leq \frac{2\sqrt{5}+1}{10^d \cdot \alpha^{k+m}} + \frac{1}{\alpha^{2m}} + \frac{\sqrt{5}\cdot F_n}{10^d \cdot \alpha^m} \leq \frac{1}{\alpha^m}\left(\frac{2\sqrt{5}+1}{10^d \cdot \alpha^k} + \frac{1}{\alpha^m} + \sqrt{5}\right),$$

i.e.,

$$\left|\frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{10^d} - 1\right| \leq \frac{2.62}{\alpha^m}, \qquad (7)$$

where we use that $k \geq 110$, $m \geq 2$ and $d \geq 1$. Now, we are ready to apply Lemma 1 with $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := 2\sqrt{5}-1$ and $b_1 := k-m$, $b_2 := -d$, $b_3 := 1$. Furthermore, $D = 2$. Put

$$\Gamma_1 := \frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{10^d} - 1.$$

Suppose that $\Gamma_1 = 0$. Then, we get $\alpha^{k-m} = \frac{(1+2\sqrt{5})10^d}{19}$. If we conjugate in $\mathbb{Q}(\sqrt{5})$, then we obtain $\beta^{k-m} = \frac{(1-2\sqrt{5})10^d}{19}$. From this, it can be seen that $F_{k-m} = \frac{4\cdot 10^d}{19}$. This is not possible. Moreover $h(\gamma_1) = \frac{\log\alpha}{2}$, $h(\gamma_2) = \log 10$ and

$$h(\gamma_3) = h(2\sqrt{5}-1) \leq \frac{\log 80}{2}.$$

Hereby, we can choose $\mathrm{A}_1 := \log\alpha$, $\mathrm{A}_2 := \log 100$, $\mathrm{A}_3 := \log 80$. Considering Lemma 4(a), (c), we can write

$$d < \frac{n+3}{4} < k-m+3 < k+3. \tag{8}$$

From here, we can say $B := k+3$. Let $A = (-1.4)\cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. By using Lemma 1 and the inequality (7), we have

$$2.62 \times \alpha^{-m} > |\Gamma_1| > \exp((1+\log 2)\cdot \mathrm{A} \cdot (1+\log(\mathrm{k}+3)) \cdot \log 100 \cdot \log\alpha \cdot \log 80)$$

i.e.,

$$m\log\alpha - \log 2.62 < 9.42 \times 10^{12} \cdot (1+\log(k+3)). \tag{9}$$

Now, from (3), we write

$$\alpha^k\left(\frac{10-\sqrt{5}}{5} - \frac{\alpha^{n-k}}{\sqrt{5}}\right) - F_m \cdot 10^d = -\left(\frac{10+\sqrt{5}}{5}\right)\beta^k - \frac{\beta^n}{\sqrt{5}},$$

i.e.,

$$\alpha^k\left((2\sqrt{5}-1) - \alpha^{n-k}\right) - 10^d \cdot \sqrt{5} \cdot F_m = -(2\sqrt{5}+1)\beta^k - \beta^n. \tag{10}$$

After making necessary calculations, we get

$$\left|1 - \frac{10^d \cdot F_m \cdot \sqrt{5}}{\alpha^k\left((2\sqrt{5}-1)-\alpha^{n-k}\right)}\right| \leq \frac{1}{\alpha^k}\left|\frac{1}{(2\sqrt{5}-1)-\alpha^{n-k}}\right|\left(\frac{2\sqrt{5}+1}{\alpha^k} + \frac{1}{\alpha^n}\right),$$

i.e.,

$$\left|1 - \frac{10^d \cdot \sqrt{5} \cdot F_m}{\alpha^k\left((2\sqrt{5}-1)-\alpha^{n-k}\right)}\right| \leq \frac{0.55}{\alpha^k}, \tag{11}$$

where we kept in view that $n \geq 0$, $k \geq 110$ and $k-n \geq -1$. Put $\gamma_1 := \alpha$, $\gamma_2 := 10$, $\gamma_3 := \frac{\sqrt{5}\cdot F_m}{(2\sqrt{5}-1)-\alpha^{n-k}}$, $b_1 := -k$, $b_2 := d$ and $b_3 := 1$. Moreover, $D = 2$. Let

$$\Gamma_2 := 1 - \frac{10^d \cdot \sqrt{5} \cdot F_m}{\alpha^k\left((2\sqrt{5}-1)-\alpha^{n-k}\right)}.$$

If $\Gamma_2 = 0$, then we can write

$$10^d F_m = \left(2 - \frac{\sqrt{5}}{5}\right)\alpha^k - \frac{\sqrt{5}}{5}\alpha^n. \tag{12}$$

Conjugating in $\mathbb{Q}(\sqrt{5})$, we get

$$10^d F_m = \left(2 + \frac{\sqrt{5}}{5}\right)\beta^k + \frac{\sqrt{5}}{5}\beta^n. \tag{13}$$

Thus, from the equalities (12) and (13), we obtain $2 \cdot 10^d \cdot F_m = M_k - F_n$. This is impossible since $M_k = 10^d F_m + F_n$. As

$$\begin{aligned} h(\gamma_3) &\leq h(F_m) + h(\sqrt{5}) + h(2\sqrt{5}-1) \\ &\quad + (k-n)h(\alpha) + \log 2 \\ &\leq (2m+3)\frac{\log\alpha}{2} + \log 10 + 2\log 2 \\ &= \frac{2\log 40 + 3\log\alpha}{2} + m\log\alpha, \end{aligned}$$

we can choose $\mathrm{A}_1 := \log\alpha$, $\mathrm{A}_2 := \log 100$, $\mathrm{A}_3 := 8.83 + 2m\log\alpha$. By using (8), we obtain $B := k+1$. By using the inequality (11) and Lemma 1, we can say

$$0.55 \times \alpha^{-k} > |\Gamma_2| >$$

$\exp((1+\log 2)\cdot A\cdot(1+\log(k+1))\cdot \log\alpha\cdot\log 100\cdot(8.83+2m\log\alpha)),$

i.e.,

$$k\log\alpha-\log(0.55)<2.15\times 10^{12}\cdot(1+\log(k+1)\cdot(8.83+2m\log\alpha)). \quad (14)$$

Combining the inequalities (9) and (14), we get $k<4.03\times 10^{29}$. Now, let us reduce this bound. Let

$$z_1:=(k-m)\log\alpha-d\log 10+\log(2\sqrt{5}-1)$$

and so $\Gamma_1:=e^{z_1}-1$. From (7), we have

$$|\Gamma_1|:=|e^{z_1}-1|<\frac{2.62}{\alpha^m}<0.62.$$

Taking $\rho:=0.62$, we obtain the inequality

$$|z_1|<-\frac{\log 0.38}{0.62}\cdot\frac{2.62}{\alpha^m}<4.09\times\alpha^{-m}$$

by Lemma 3. Therefore, we can write

$$0<\left|(k-m)\log\alpha-d\log 10+\log(2\sqrt{5}-1)\right|<4.09\times\alpha^{-m}$$

i.e.,

$$0<\left|(k-m)\frac{\log\alpha}{\log 10}-d+\frac{\log(2\sqrt{5}-1)}{\log 10}\right|<1.78\times\alpha^{-m}. \quad (15)$$

Put $\tau:=\frac{\log\alpha}{\log 10}\notin\mathbb{Q}$, $\mu:=\left(\frac{\log(2\sqrt{5}-1)}{\log 10}\right)$, $A:=1.78$, $B:=\alpha$, $t:=m$ and $M:=4.03\times 10^{29}$. We have $k-m>1$ from Lemma 4(d). Moreover, we found $q_{61}>6M$ and then

$$\varepsilon:=\|\mu q_{61}\|-M\|\tau q_{61}\|<0.47.$$

According to Lemma 2, if (15) has a solution, then

$$m\le\frac{\log\left(\frac{Aq_{61}}{\varepsilon}\right)}{\log B}\le 151.55.$$

Thus, $m\le 151$. By using (14), we have $k<2.68\times 10^{16}$. Put

$$z_2:=d\log 10-k\log\alpha+\log\left(\frac{\sqrt{5}\cdot F_m}{(2\sqrt{5}-1)-\alpha^{n-k}}\right)$$

and so $\Gamma_2:=1-e^{z_2}$. Since $k\ge 110$, it is clear that

$$|\Gamma_2|:=|1-e^{z_2}|<0.55\cdot\alpha^{-k}<0.01$$

by the inequality (11). So, choosing $\rho:=0.01$ in Lemma 3, we get

$$\left|d\log 10-k\log\alpha+\log\left(\frac{\sqrt{5}\cdot F_m}{(2\sqrt{5}-1)-\alpha^{n-k}}\right)\right|<\frac{\log\left(\frac{100}{99}\right)}{0.01}\cdot\frac{0.55}{\alpha^k}<0.56\times\alpha^{-k}$$

i.e.,

$$0<\left|d\frac{\log 10}{\log\alpha}-k+\frac{\log\left(\frac{\sqrt{5}\cdot F_m}{(2\sqrt{5}-1)-\alpha^{n-k}}\right)}{\log\alpha}\right|<1.17\times\alpha^{-k}, \quad (16)$$

where $\tau:=\frac{\log 10}{\log\alpha}$ and $\mu:=\frac{\log\left(\frac{\sqrt{5}\cdot F_m}{(2\sqrt{5}-1)-\alpha^{n-k}}\right)}{\log\alpha}$. By using (8), we can take $M:=2.68\times 10^{16}$. Here we found $q_{40}\ge 6M$. Thus, we can say

$$\varepsilon:=\|\mu q_{40}\|-M\|\tau q_{40}\|<0.498$$

for $-1\le k-n\le m+4$ and $3\le m\le 151$. In Lemma 2, let $A:=1.17$, $B:=\alpha$, $t:=k$. Hence, if (16) has a solution, then from Lemma 2, we get

$$k\le\frac{\log\left(\frac{Aq_{40}}{\varepsilon}\right)}{\log B}\le 104.5.$$

Thus $k\le 104$ contradicts our presumption that $k\ge 110$. When we take into account the case $m=2$ for $k\ge 110$ it can be seen that $k\le 91$. This is a contradiction. Thus, the proof of our theorem is finished.

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The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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