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On the Semi-Analytical Solutions for the Kudryashov-Sinelshchikov Dynamical Equation Arising in Mixtures of Liquid and Gas Bubbles Without Neglecting of Heat Transfer and Viscosity

Emre Aydın[1], İnci Çılıngır Süngü[2,*]

[1]Department of Mathematics , Institute of Graduate Education, Ondokuz Mayıs University, 55270, Samsun, Turkey.
[2]Department of Mathematics, Faculty of Education, Ondokuz Mayıs University, 55270, Samsun, Turkey.

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Abstract. In this study, Banach contraction method (BCM), Daftardar-Jafari method (DJM) and modified variational iteration method (MVIM) are proposed for the semi-analytical solutions of the Kudryashov-Sinelshchikov (K-S) dynamical equation. It has been shown that the analytical and semi-analytical solutions for the K-S dynamical equation with initial value problems by using semi-analytical methods can be obtained. In addition, the effectiveness and usefulness of the semi-analytical methods used are supported by tables and 3D figures. As the number of iteration or terms increases, how the semi-analytical solutions behave over time and converge to the exact solution is shown in detail with 2D figures. Also, it is shown comparison of semi-analytical solutions with exact solutions and error analysis with the help of tables. It has been discussed the methods are compared with each other and whether they are suitable for the K-S dynamical equation.

2020 AMS Classification: 35G25, 35C06, 47H10, 47H09

Keywords: Kudryashov-Sinelshchikov (K-S) dynamical equation , Banach contraction method (BCM), Daftardar-Jafari method (DJM), modified variational iteration method (MVIM).

1. Introduction

Nonlinear evolution equations (NLEE) are employed to depict numerous natural patterns and occurrences. Furthermore, nonlinear evolution equations play a significant role in mathematical physics. Lately, numerous domains of application associated with nonlinear evolution equations have been explored. In the year 2010, Kudryashov and Sinelshchikov [23] derived a broader and practical nonlinear evolution equation that effectively describes pressure waves in a combination of liquid and gas bubbles while considering the fluid's viscosity and heat transfer, without neglecting these factors. That is,

$$\Theta_t + \alpha\Theta_x + \Theta_{xxx} - \beta\left(\Theta_x\Theta_{xx}\right)_x - \gamma\Theta_x\Theta_{xx} = 0. \tag{1.1}$$

In this context, the variable Θ represents heat transfer and viscosity, denotes density. The real parameters α, β, γ are used to determine the wave pressure in the mixture of liquid and gas bubbles, taking into account the effects of heat transfer and viscosity without neglecting them. Also, Kudryashov-Sinelshchikov dynamical equation describes the motions of plasma waves, water waves with surface tension and capillary gravity water waves. In recent years, studies on the Kudryashov-Sinelshchikov (K-S) equations have increased. It is located Bernoulli sub-equation function method

*Corresponding Author
Email addresses: emr_aydn_55@outlook.com (E. Aydın), incicilingir@gmail.com (İ. Çilingir Süngü)

on the K-S equation in [10], the improved the (G'/G)-expansion method in [21], first integral method in [31], improved F-expansion method in [18], modification of truncated expansion method in [29], travelling and solitary wave solutions in [30], polynomial expansion method in [25], integral bifurcation method in [12], trial solution algorithm in [22], the multiple (G'/G)- expansion method in [19], F-expansion method in [33], sine-cosine method and ansatz method in [17], RB sub-ODE method in [32]. In this study, semi-analytical methods are used to demonstrate the practicality, compatibility and effectiveness of Daftardar-Jafari method (DJM), Banach contraction method (BCM) and modified variational iteration method (MVIM) on the K-S dynamical equation which has been solved semi-analytically by any researcher until now. It has been shown that analytical and semi-analytical solutions are obtained for the K-S dynamical equation. The convergence of semi-analytical solutions to the exact solution is visualized in detail with 2D and 3D figures. In addition, for some x and t values, comparison of semi-analytical solutions with exact solutions and error analysis are given with the help of tables. Many researchers have papers on Banach contraction method that is an iterative method based on fixed point theorem in [8, 9, 13, 16, 24], Daftardar-Jafari method that is an iterative method based on fixed point theorem and containing a correction term in [4–7, 11, 14, 27, 28] and modified variational iteration method that is an iterative method based on variation and containing a correction term in [1–3, 15, 20, 26] with differential equations such as Jeffery-Homel flow problem, Boussinesq equation, the SIR epidemic model, Fornberg-Whitham equation.

2. Material and Methods

This section will provide an introduction to the fundamental frameworks of DJM, BCM and MVIM. Equation (1.1) is written in operator form as:

$$L_t\Theta + \widetilde{N}\Theta = 0, \tag{2.1}$$

where $L_t = \frac{d}{dt}, \tilde{N}\Theta = \alpha\Theta_x + \Theta_{xxx} - \beta(\Theta_x\Theta_{xx})_x - \gamma\Theta_x\Theta_{xx}$. The inverse operator of L_t is defined by $L_t^{-1} = \int_0^t(\cdot)d\xi$. Equation (2.1) can be rewritten as:

$$\Theta = \varphi(x) + N\Theta, \tag{2.2}$$

where N is defined as $N\Theta = L_t^{-1}(-\widetilde{N\Theta})$ and N is a nonlinear operator in a Banach space $B \to B, \varphi(x) = \Theta(x, 0)$.

2.1. Banach Contraction Method (BCM).

Definition 2.1 ([13]). Consider two metric spaces (B_1, d) and (B_2, d), and let F be a mapping from (B_1, d) to (B_2, d). F is defined as Lipschitz if there exists a real number$L \geq 0$ such that for any $x, y \in (B_1, d)$, the inequality $d(Fx, Fy) \leq d(x, y)$ holds. If the Lipschitz constant L is less than 1 ($L < 1$), then F is referred to as a contraction.

Theorem 2.2 ([13]). *Let F be a contraction mapping, with Lipschitz constant L, of a Banach space (B, d) into itself. Then, F has a unique fixed point Θ in (B, d). Moreover, if x_0 is an arbitrary point in (B, d) and x_r is defined by $x_(r + 1) = F(x_r), r = 0, 1, ...$ Then, $\lim_{r\to 0} x_r = 0$ and $d(x_r, \Theta) \leq \frac{L^r}{1-r}d(x_1, x_0)$.*

Theorem 2.3 ([13]). *Consider a mapping F from a Banach space (B, d) to itself, such that each iteration F^n is a contraction mapping of (B, d) for some positive integer n. Then, according to the Banach fixed-point theorem, F has a unique fixed point in (B, d).*

The successive approximations for Equation (2.2) are defined as

$$\begin{aligned} \theta_0 &= \psi \\ \theta_1 &= \psi + N(\theta_0) \\ \theta_2 &= \psi + N(\theta_1) \\ &\vdots \\ \theta_r &= \psi + N(\theta_{r-1}), r = 1, 2, ... \end{aligned} \tag{2.3}$$

The BCM relies on Banach's contraction principle in the newly derived approaches. In accordance with Theorem (2.2), the mapping N of the rth sequence mentioned in (2.3) converges to the exact solution. That is,

$$\lim_{r\to\infty} \theta_r = \Theta(x, t). \tag{2.4}$$

2.2. **Daftardar-Jafari Method (DJM).** Let's assume that the solutions of Equation (2.2) can be represented as an infinite series of Θ, denoted as:

$$\Theta = \sum_{i=1}^{\infty} \theta_i.$$

In this case, the nonlinear operator N can be decomposed as illustrated below [7]:

$$N(\sum_{i=1}^{\infty} \theta_i) = N(\theta_0) + \sum_{i=1}^{\infty}(N(\sum_{j=0}^{i} \theta_j) - N(\sum_{j=0}^{i-1} \theta_j)).$$

When (2.3) and (2.4) are substituted in Equation (2.2) and after simplification, the following equation is obtained:

$$\sum_{i=1}^{\infty} \theta_i = \psi + N(\theta_0) + \sum_{i=1}^{\infty}(N(\sum_{j=0}^{i} \theta_j) - N(\sum_{j=0}^{i-1} \theta_j)).$$

Thus, the iteration relation for DJM is determined in the form:

$$\begin{aligned}
\theta_0 &= \phi \\
\theta_1 &= N(\theta_0) \\
\theta_2 &= N(\theta_0 + \theta_1) - N(\theta_0) \\
\theta_3 &= N(\theta_0 + \theta_1 + \theta_2) - N(\theta_0 + \theta_1) \\
&\vdots \\
\theta_{r+1} &= N(\theta_0 + \theta_1 + \theta_2 + ... + \theta_r) - N(\theta_0 + \theta_1 + \theta_2 + ... + \theta_{r-1}) \\
\Rightarrow \theta_1 + \theta_2 + ... + \theta_{r+1} &= N(\theta_0 + \theta_1 + \theta_2 + ... + \theta_r).
\end{aligned}$$

Finally, the r-term approximate for Equation (2.2) is $\theta_r(x,t) = \theta_0 + \theta_1 + \theta_2 + ... + \theta_{r-1}$ and $\lim_{r\to\infty} \theta_r(x,t) = \Theta(x,t)$ under proper conditions.

2.3. **Modified Variational Iteration Method (MVIM).** Equation (1.1) is written in operator form as:

$$L_t\Theta + R_x\Theta + N\Theta = 0, \tag{2.5}$$

where $L_t = \frac{\mathrm{d}}{\mathrm{d}t}, R_x = \frac{\mathrm{d}^3}{\mathrm{d}x^3}, N\Theta = \alpha\Theta_x - \beta(\Theta_x\Theta_{xx})_x - \gamma\Theta_x\Theta_{xx}$. The inverse operator of L_t is defined by $L_t^{-1} = \int_0^t (\cdot)d\xi$. The correction functional of Equation (2.5) is:

$$\theta_{r+1}(x,t) = \theta_0(x,t) + \int_0^t \lambda[R_x\theta_r(x,\xi) + N\theta_r(x,\xi)]d\xi, \tag{2.6}$$

where λ is a Lagrange multiplier. For $L_t = \frac{\partial^m}{\partial t^m}$, The Lagrange multiplier can be expressed in the following form [14]:

$$\lambda(x,t) = \frac{(-1)^m}{(m-1)!}(t-x)^{(m-1)}, m \geq 1. \tag{2.7}$$

Consider the initial approximate function θ_0 be taken as $\Theta(x,0)$. In order to eliminate unnecessary terms, Equation (2.6) is reorganized into the following form:

$$\theta_{r+1} = \theta_0 + \int_0^t \lambda[R_x\theta_{r-1} + G_{r-1}]d\xi + \int_0^t \lambda[R_x(\theta_r - \theta_{r-1}) + G_r - G_{r-1}]d\xi$$

$$\Rightarrow \theta_{r+1} = \theta_r + \int_0^t \lambda[R_x(\theta_r - \theta_{r-1}) + G_r - G_{r-1}]d\xi, \tag{2.8}$$

where polynomials G_r obtain from the equality $N\theta_r = G_r + o(t^{r+1})$. Equation (2.8) is referred to as the modified correction functional. Since $L_t = \frac{\partial}{\partial t}$ in Equation (1.1), we get $\lambda(x,t) = -1$ by rearranging the Equation (2.7). The simplified modified correction functional for the Kudryashov-Sinelshchikov dynamical equation is:

$$\theta_{r+1} = \theta_r - \int_0^t [R_x(\theta_r - \theta_{r-1}) + G_r - G_{r-1}]d\xi$$

$$\theta_0 = \Theta(x,0), \theta_{-1} = 0, G_{-1} = 0, r \geq 1.$$

Eventually, the components $\theta_0, \theta_1, \theta_2, \theta_3, ...$ are identified via iteration and the semi-analytical solutions are fully determined. It happens $\theta_r \to \Theta(x,t)$ for $r \to \infty$ under necessary conditions at the end of this process.

3. Numerical Experiments and Simulations

In this section, semi-analytical solutions of Kudryashov-Sinelshchikov dynamical equation are obtained with BCM, DJM and MVIM.

Example 3.1. Let $\alpha = 1, \beta = 1, \gamma = -1$ and $\Theta(x,0) = 2 - e^{-x}$ in Equation (1.1) for all semi-analytical methods.

$$\begin{cases} \Theta_t + \alpha\Theta_x + \Theta_{xxx} - \beta(\Theta_x\Theta_{xx}) - \gamma\Theta_x\Theta_{xx} = 0 \\ \Theta(x,0) = 2 - e^{-x}. \end{cases} \tag{3.1}$$

The semi-analytical solutions with 5 iterations obtained by BCM for the Kudryashov-Sinelshchikov dynamical equation with initial value problems are:

$$\begin{gathered} \theta_0 = 2 - e^{-x}, \\ \theta_1 = 2 - e^{-x} - e^{-x}t, \\ \theta_2 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2, \\ \theta_3 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3, \\ \theta_4 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3 - \frac{1}{24}e^{-x}t^4, \\ \theta_5 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3 - \frac{1}{24}e^{-x}t^4 - \frac{1}{120}e^{-x}t^5, \\ \vdots \end{gathered}$$

Continuing in the same way, it is seen that rth sequence of the iteration converges to the exact solution of the Kudryashov-Sinelshchikov dynamical equation. That is,

$$\lim_{r\to\infty} \theta_r = \Theta(x,t) = 2 - e^{-x+t}.$$

The approximate parts with 5 iterations obtained by DJM for the Kudryashov-Sinelshchikov dynamical equation with initial value problems are

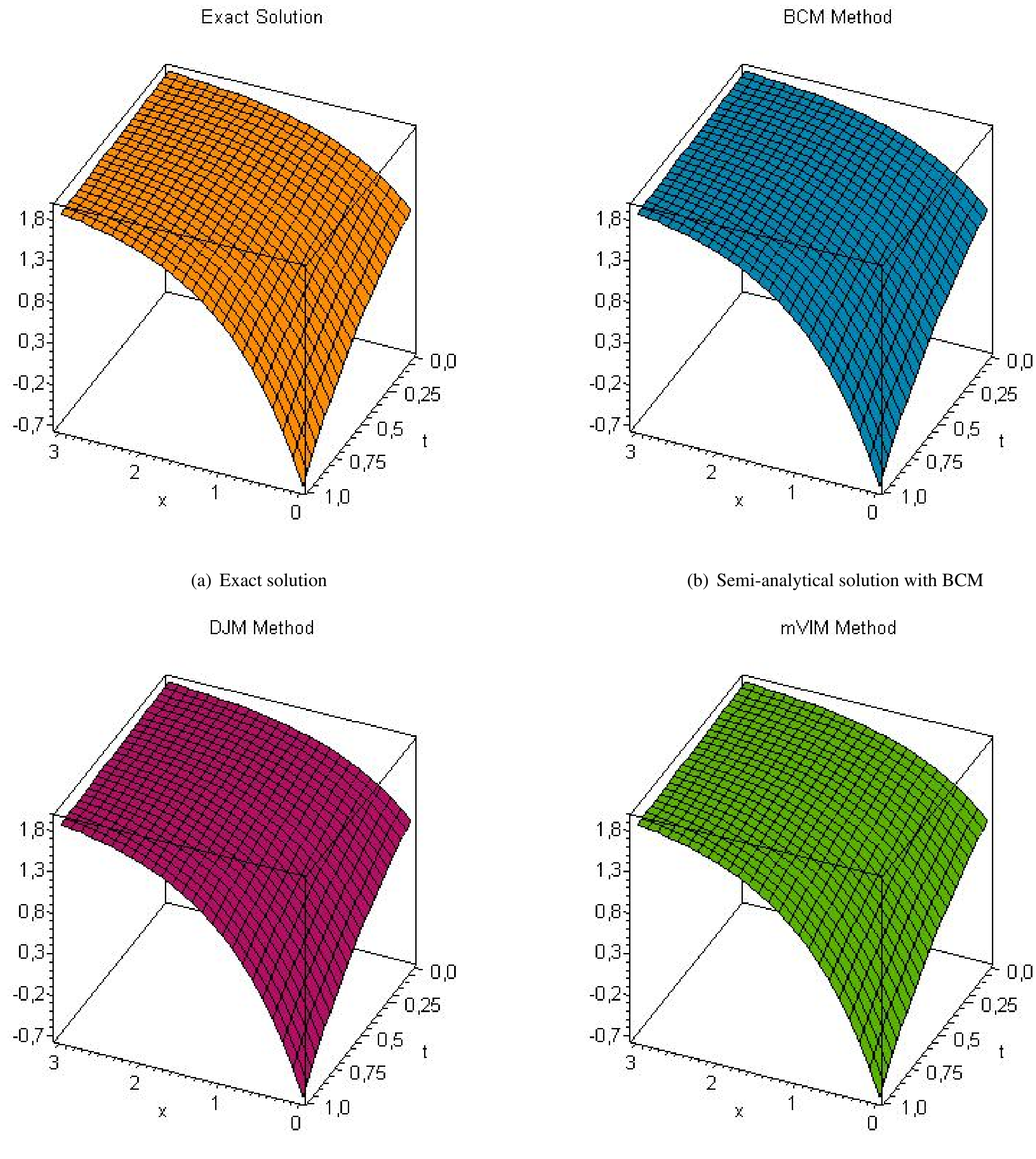
$$\begin{aligned} \theta_0 &= 2 - e^{-x}, \\ \theta_1 &= -e^{-x}, \\ \theta_2 &= -\frac{1}{2}e^{-x}t^2, \\ \theta_3 &= -\frac{1}{6}e^{-x}t^3, \\ \theta_4 &= -\frac{1}{24}e^{-x}t^4, \\ \theta_5 &= -\frac{1}{120}e^{-x}t^5. \end{aligned}$$

The 6-term semi-analytical solution for Equation (3.1) is obtained as follows:

$$\theta_6(x,t) = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3 - \frac{1}{24}e^{-x}t^4 - \frac{1}{120}e^{-x}t^5.$$

Continuing this process, it is found that r-term approximate converges to the exact solution of the Kudryashov-Sinelshchikov dynamical equation. That is,

$$\lim_{r\to\infty} \theta_r = \Theta(x,t) = 2 - e^{-x+t}.$$



(a) Exact solution

(b) Semi-analytical solution with BCM

(c) Semi-analytical solution with DJM

(d) Semi-analytical solution with MVIM

FIGURE 1. Comparison of BCM, DJM and MVIM solutions with the exact solution of Example (3.1)

The semi-analytical solutions with 5 iterations obtained by MVIM for the Kudryashov-Sinelshchikov dynamical equation with initial value problems are:

$$\theta_0 = 2 - e^{-x},$$
$$\theta_1 = 2 - e^{-x} - e^{-x}t,$$
$$\theta_2 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2,$$
$$\theta_3 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3,$$
$$\theta_4 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3 - \frac{1}{24}e^{-x}t^4,$$
$$\theta_5 = 2 - e^{-x} - e^{-x}t - \frac{1}{2}e^{-x}t^2 - \frac{1}{6}e^{-x}t^3 - \frac{1}{24}e^{-x}t^4 - \frac{1}{120}e^{-x}t^5,$$
$$\vdots$$

Similarly, it is said that rth sequence of the iteration converges to the exact solution of the Kudryashov-Sinelshchikov dynamical equation. That is,

$$\lim_{r\to\infty} \theta_r = \Theta(x,t) = 2 - e^{-x+t}.$$

The surface graphs of the exact solution or semi-analytical solutions for Equation (3.1) are shown in Figure (1).

Example 3.2. Let $\alpha = 1, \beta = 1, \gamma = -3$ and $\Theta(x,0) = \frac{5}{6} + \frac{e^{-x}}{(1+e^{-x})^2}$ in Equation (1.1) for all semi-analytical methods.

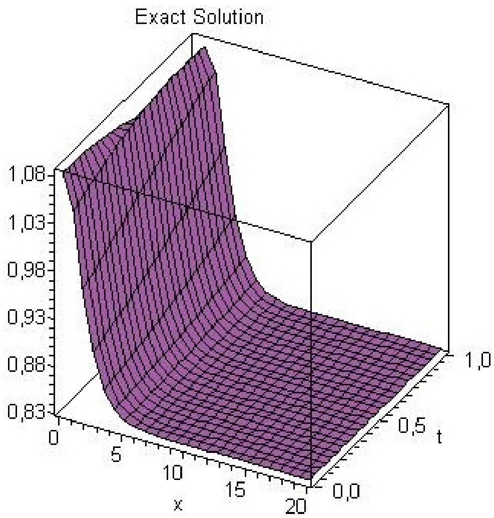
$$\begin{cases} \Theta_t + \Theta_x + \Theta_{xxx} - (\Theta_x\Theta_{xx}) + 3\Theta_x\Theta_{xx} = 0, \\ \Theta(x,0) = \frac{5}{6} + \frac{e^{-x}}{(1+e^{-x})^2}. \end{cases} \tag{3.2}$$

Exact solution given in [29] of the Kudryashov-Sinelshchikov dynamical equation with initial value problem is

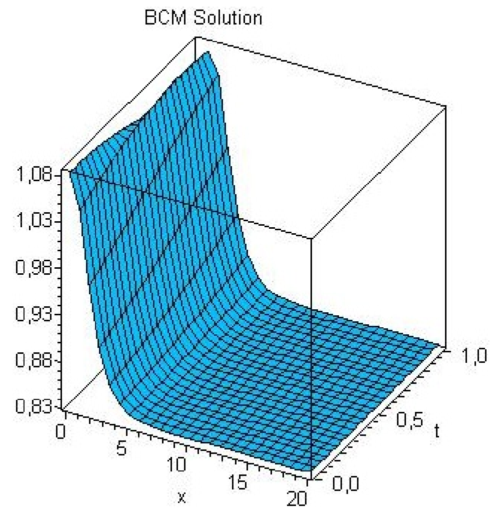
$$\Theta(x,t) = \frac{5}{6} + \frac{e^{t-x}}{(1+e^{t-x})^2}.$$

Some semi-analytical solutions obtained by BCM for the Kudryashov-Sinelshchikov dynamical equation with initial value problems are

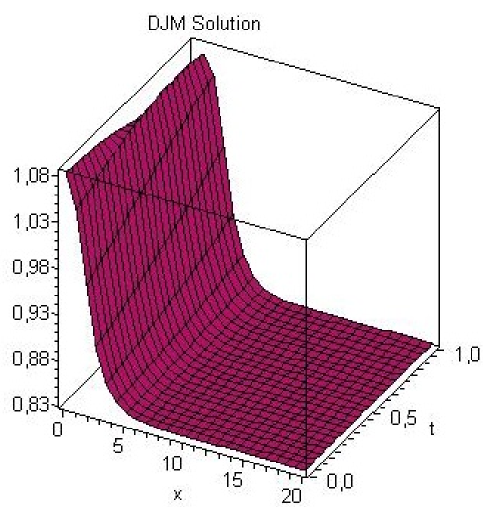
$$\theta_0 = \frac{5}{6} + \frac{e^{-x}}{(1+e^{-x})^2},$$
$$\theta_1 = \frac{1}{6}\frac{5 + 16e^{-x} + 5e^{-2x}}{(1+e^{-x})^2},$$
$$\theta_2 = \frac{1}{6}\frac{5 + 5e^{-3x} + 21e^{-x} + 21e^{-2x} - 6te^{-2x} + 6te^{-x}}{(1+e^{-x})^3},$$
$$\theta_3 = \frac{1}{6.(1+e^{-x})^9}(5 + 222e^{-7x} + 30te^{-4x} + 3t^2e^{-x} - 54te^{-6x} - 30te^{-5x} + 54te^{-3x} + 30te^{-2x} + 3t^2e^{-2x} - 75t^2e^{-5x}$$
$$-27t^2e^{-6x} - 27t^2e^{-3x} + 5e^{-9x} + 51e^{-8x} - 12t^3e^{-3x} - 30te^{-7x} - 6te^{-8x} + 3t^2e^{-8x} + 51e^{-x} + 4t^3e^{-2x} - 32t^3e^{-5x} + 222e^{-2x}$$
$$+3t^2e^{-7x} + 546e^{-3x} + 840e^{-4x} - 4t^3e^{-7x} + 840e^{-5x} - 75t^2e^{-4x} + 546e^{-6x} + 32t^3e^{-4x} + 6te^{-x} + 12t^3e^{-6x}),$$
$$\vdots$$



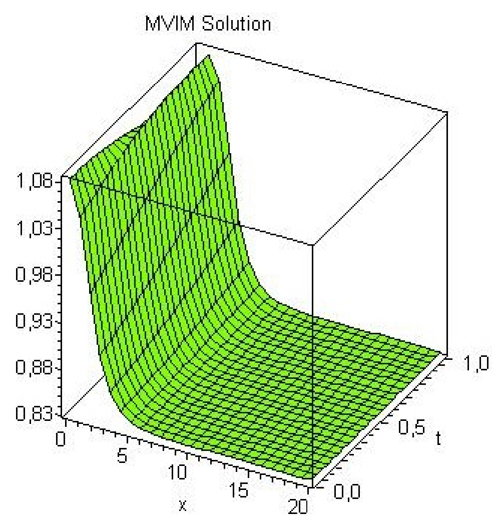
(a) Exact solution



(b) Semi-analytical solution with BCM



(c) Semi-analytical solution with DJM



(d) Semi-analytical solution with MVIM

FIGURE 2. Comparison of BCM, DJM and MVIM solutions with the exact solution of Example (3.2)

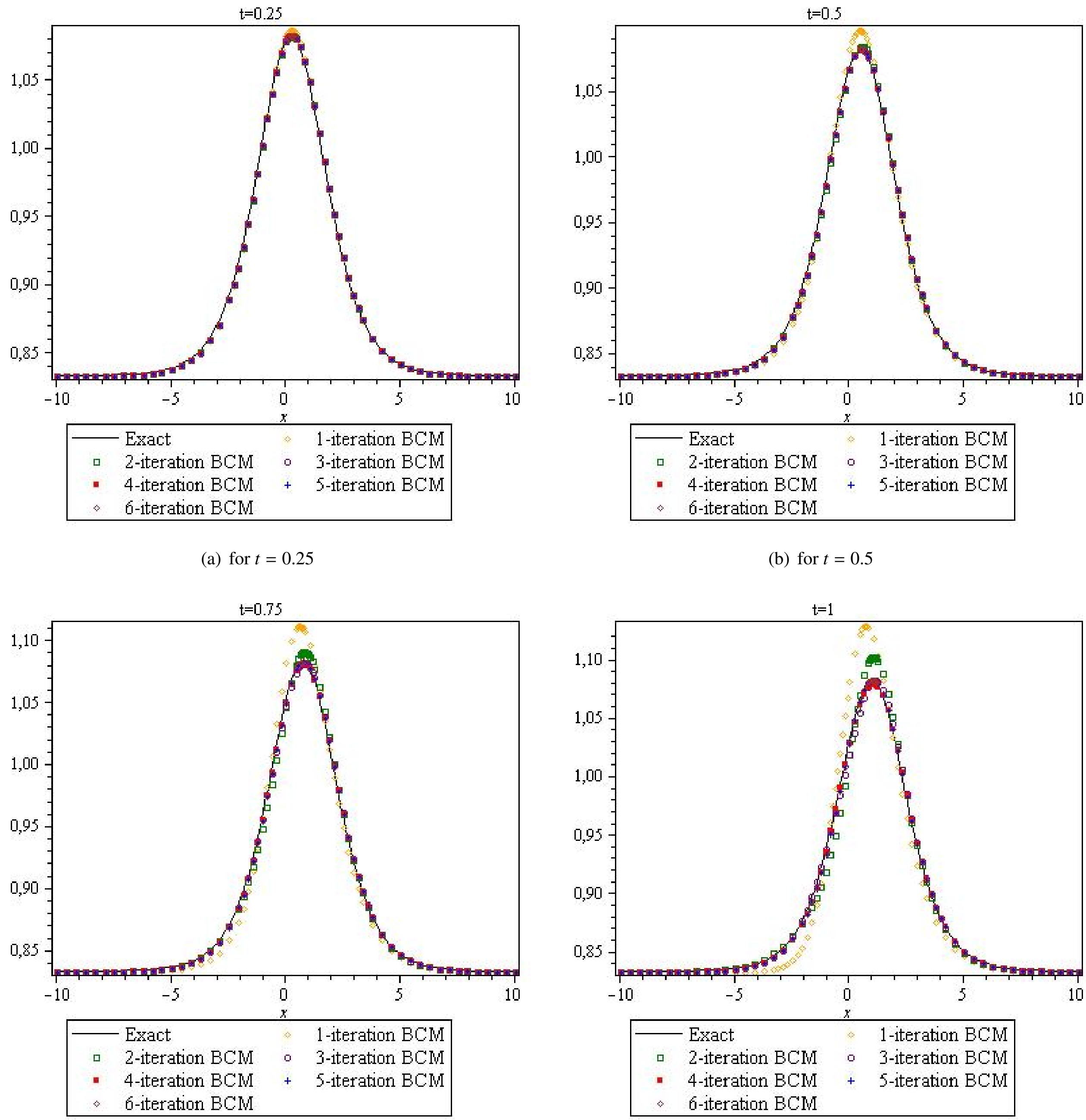
Continuing like this, other semi-analytical solutions with BCM can be obtained. Some semi-analytical solution obtained with DJM for Equation (3.2) is obtained as follows:

$$\theta_0 = \frac{5}{6} + \frac{e^{-x}}{(1+e^{-x})^2},$$

$$\theta_1 = \frac{1}{6}\frac{5+16e^{-x}+5e^{-2x}}{(1+e^{-x})^2},$$

$$\theta_2 = \frac{1}{6}\frac{5+5e^{-3x}+21e^{-x}+21e^{-2x}-6te^{-2x}+6te^{-x}}{(1+e^{-x})^3},$$

$$\theta_3 = \frac{1}{6.(1+e^{-x})^9}(5+222e^{-7x}+30te^{-4x}+3t^2e^{-x}-54te^{-6x}-30te^{-5x}+54te^{-3x}+30te^{-2x}+3t^2e^{-2x}-75t^2e^{-5x}$$
$$-27t^2e^{-6x}-27t^2e^{-3x}+5e^{-9x}+51e^{-8x}-12t^3e^{-3x}-30te^{-7x}-6te^{-8x}+3t^2e^{-8x}+51e^{-x}+4t^3e^{-2x}-32t^3e^{-5x}+222e^{-2x}+...).$$



(a) for $t = 0.25$

(b) for $t = 0.5$

(c) for $t = 0.75$

(d) for $t = 1$

FIGURE 3. Comparison of the semi-analytical solutions obtained with BCM as the number of iteration increases for Equation (3.2) (a) for $t = 0.25$ (b) for $t = 0.5$ (c) for $t = 0.75$ (d) for $t = 1$

Continuing like this, other semi-analytical solutions with DJM can be obtained. Some semi-analytical solutions obtained by MVIM for the Kudryashov-Sinelshchikov dynamical equation with initial value problems are:

$$\theta_0 = \frac{5}{6} + \frac{e^{-x}}{(1+e^{-x})^2}$$

$$\theta_1 = \frac{5 + 5e^{-3x} + 21e^{-2x} - 6te^{-2x} + 21e^{-x} + 6te^{-x}}{6.(1+e^{-x})^3}$$

$$\theta_2 = \frac{5 + 5e^{-4x} + 3t^2e^{-3x} - 6te^{-3x} + 26e^{-3x} + 42e^{-2x} - 12t^2e^{-2x} + 26e^{-x} + 3t^2e^{-x} + 6te^{-x}}{6.(1+e^{-x})^4}$$

$$\vdots$$

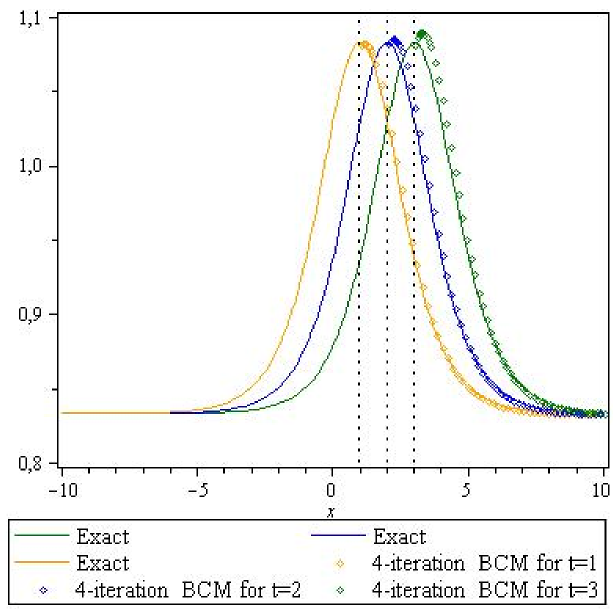


Figure 4. Comparison of the analytical solutions with BCM as the t values increases for Equation (3.2)

x	$Exact$	BCM	DJM	$MVIM$
0	1.029945267	1.029042833	1.029042833	1.029774306
2	1.029945267	1.029781448	1.029781448	1.029813155
4	0.8785099931	0.8785165972	0.8785165972	0.8785183078
6	0.8399813900	0.8399823176	0.8399823180	0.8399812127
8	0.8342435545	0.8342434563	0.8342434563	0.8342434838
10	0.8334567126	0.8334566924	0.8334566924	0.8334567019
12	0.8333500344	0.8333500666	0.8333500666	0.8333500345
14	0.8333355936	0.8333355195	0.8333355195	0.8333355903
16	0.8333336392	0.8333335664	0.8333335664	0.8333336362
18	0.8333333747	0.8333334112	0.8333334112	0.8333333763
20	0.8333333389	0.8333333486	0.8333333486	0.8333333394

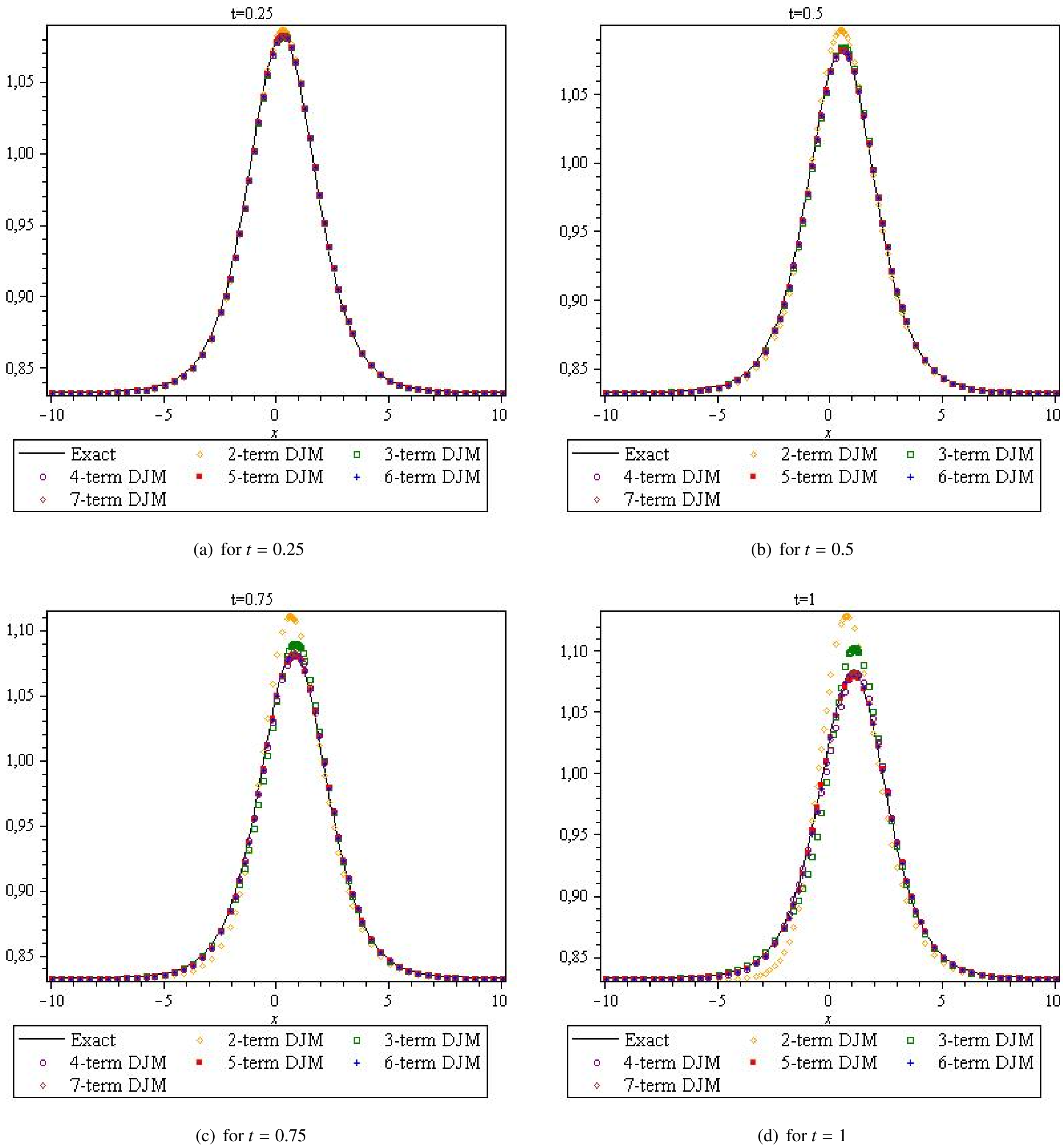
Table 1: Comparison of exact solution and semi-analytical solutions for Equation (3.2)

Continuing like this, other semi-analytical solutions with MVIM can be obtained. When $t = 1, 0 < x < 20$, comparison of analytical solutions of BCM with 6 iterations, DJM with 7 terms and MVIM with 6 iterations is given in Table 1. Comparison of errors of semi-analytical solutions is given in Table 2. The surface graphs of the exact solution or semi-analytical solutions for Equation (3.2) are shown in Figure 2.

As the number of iteration or term increases, the comparison of the semi-analytical solutions obtained with BCM, DJM and MVIM with the exact solution is given in Figure 3,4,5,6,7,8 respectively.

4. Conclusion

In this study, it has been shown that both analytical and semi-analytical solutions of the K-S dynamical equation can be found with BCM, DJM, and MVIM. It has been determined that the semi-analytical solutions obtained with BCM, DJM and MVIM converge rapidly to the exact solution of the K-S dynamical equation from Figure 3,5,7 as the number of iteration or terms increases. When at least 4 iterations or terms are taken, it is seen that the results converging to



(a) for $t = 0.25$

(b) for $t = 0.5$

(c) for $t = 0.75$

(d) for $t = 1$

Figure 5. Comparison of the semi-analytical solutions obtained with DJM as the number of iteration increases for Equation (3.2) (a) for $t = 0.25$ (b) for $t = 0.5$ (c) for $t = 0.75$ (d) for $t = 1$

the exact solution is obtained. This situation is very beneficial in terms of computer memory and processing capacity. Moreover, when $0 \leq t \leq 1$ it has been observed that the fluctuation that occurs around $x = 0$ damped as time passes, although the number of iterations and terms increases. Also, it has been observed that the fluctuations in the semi-analytical solutions obtained with BCM, DJM and MVIM shift harmoniously as the fluctuation in the exact solution

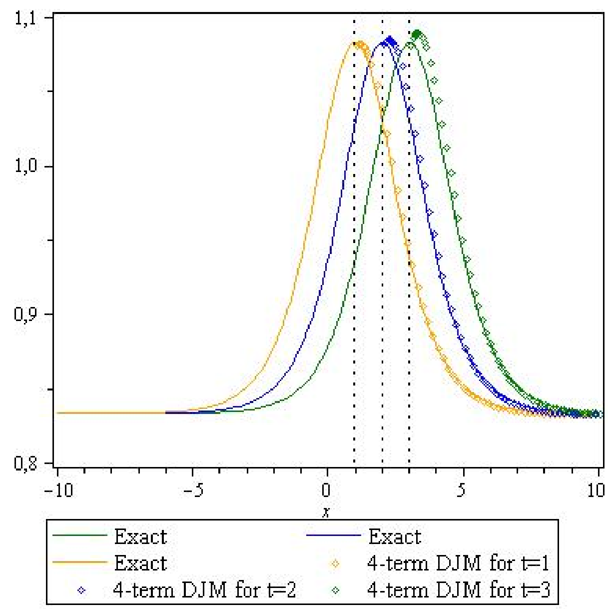
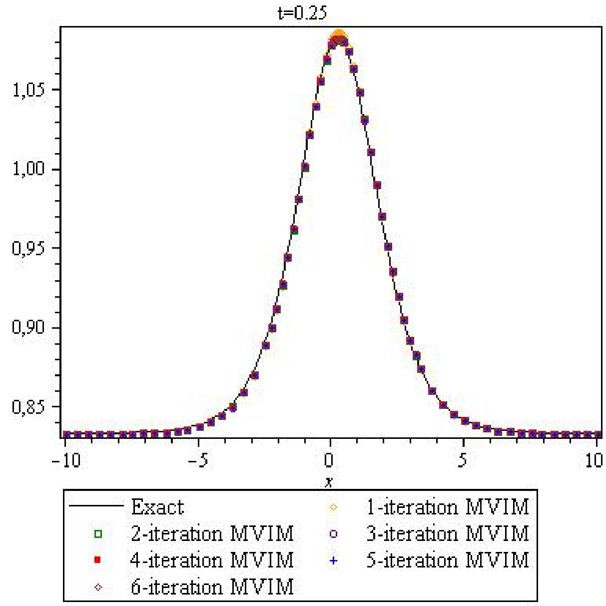


Figure 6. Comparison of the analytical solutions with DJM as the t values increases for Equation (3.2)

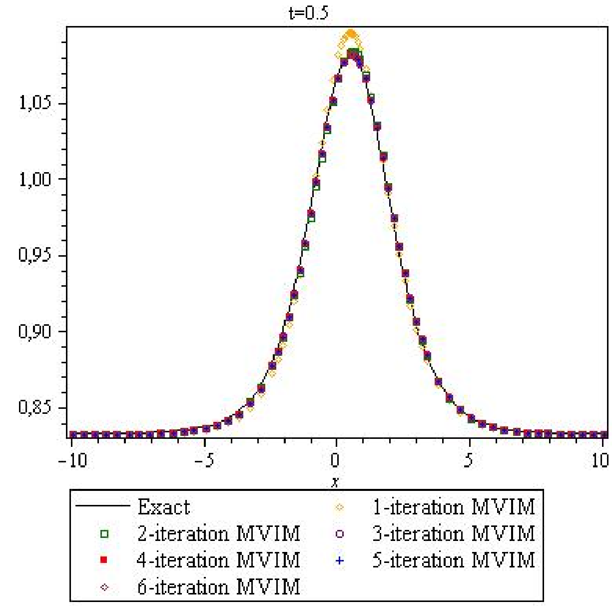
x	$\Theta_{Exact} - \Theta_{BCM}$	$\Theta_{Exact} - \Theta_{DJM}$	$\Theta_{Exact} - \Theta_{MVIM}$
0	0.0009024340	0.0009024340	0.0001709610
2	0.0001638190	0.0001638190	0.0001321120
4	0.0000066041	0.0000066041	0.0000083147
6	0.0000009276	0.0000009280	0.0000001173
8	0.0000000982	0.0000000982	0.0000000707
10	0.0000000202	0.0000000202	0.0000000107
12	0.0000000322	0.0000000322	0.0000000001
14	0.0000000741	0.0000000741	0.0000000033
16	0.0000000728	0.0000000728	0.0000000030
18	0.0000000365	0.0000000365	0.0000000016
20	0.0000000097	0.0000000097	0.0000000005

Table 2: Comparison of errors of semi-analytical methods for Equation (3.2)

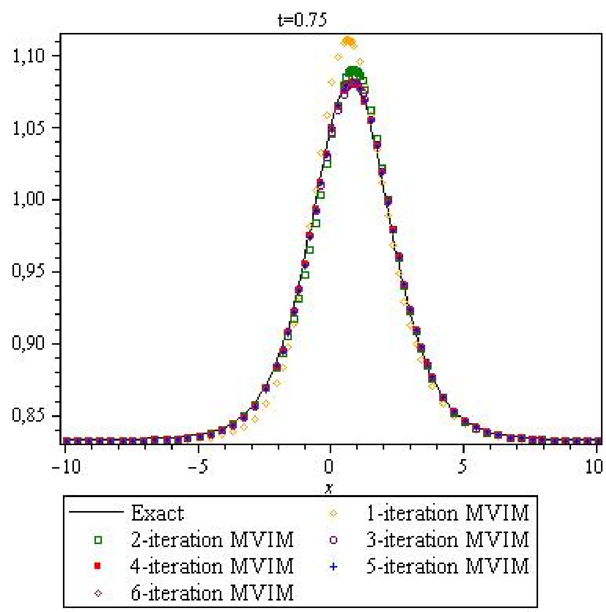
shifts when $t > 1$ in Figure 4,6,8. In addition, the convergence of the semi-analytical solutions to the exact solution is shown using the tables and 3D figures. Maple package program is used for all calculations and visualizations. As a result of this study, it is concluded that for the Kudryashov-Sinelshchikov dynamical equation with initial value problem, exact solutions can be found with semi-analytical methods instead of the methods in which it is difficult and tiring to find exact solution in Example (3.1)-(3.2). In addition, MVIM is found to give better results than BCM and DJM for solving the initial value problem in Example (3.2), although the exact solution of K-S dynamical equation is found with all the methods in Example (3.1). As can be seen in tables and surface figures, it has been determined from the error table that there is a serious decrease in the error values due to the increase in the time and spatial values. As can be seen, it can be said that the proposed semi-analytical methods are equivalent to each other. In addition, it is concluded that BCM, DJM and MVIM are extremely efficient and useful in terms of memory and computational cost to solve the K-S dynamical equation. Semi-analytical solutions for the K-S dynamical equation signify density for compression of waves in liquid with gas-bubbles taking into account to heat transfer and viscosity of liquids.Exact solution methods related to the K-S dynamical equation have been used in the literature, but numerical methods have



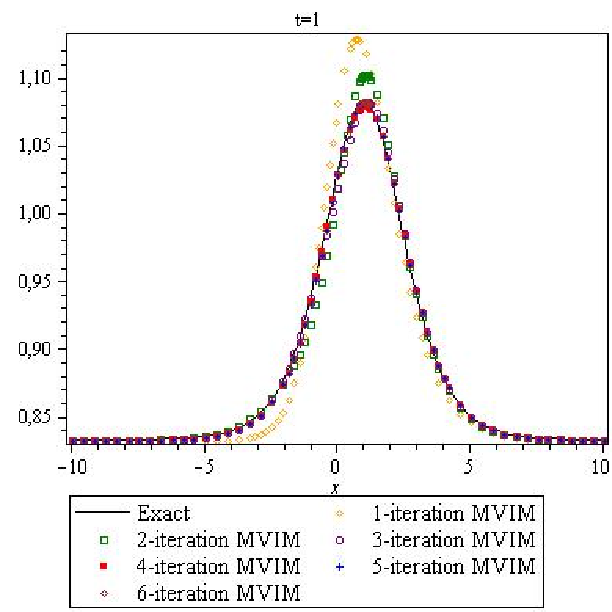
(a) for $t = 0.25$



(b) for $t = 0.5$



(c) for $t = 0.75$



(d) for $t = 1$

FIGURE 7. Comparison of the semi-analytical solutions obtained with MVIM as the number of iteration increases for Equation (3.2) (a) for $t = 0.25$ (b) for $t = 0.5$ (c) for $t = 0.75$ (d) for $t = 1$

not been emphasized much. Therefore, semi analytical methods used to solve the K-S dynamical equation and to show effectiveness of this type of methods. Also, the obtained semi-analytical solutions for the K-S dynamical equation may be used instead of analytical solution for understanding of the mechanism of complicated non-linear physical phenomena in wave interaction. The utilization of these semi-analytical solutions could get more advantageous in the

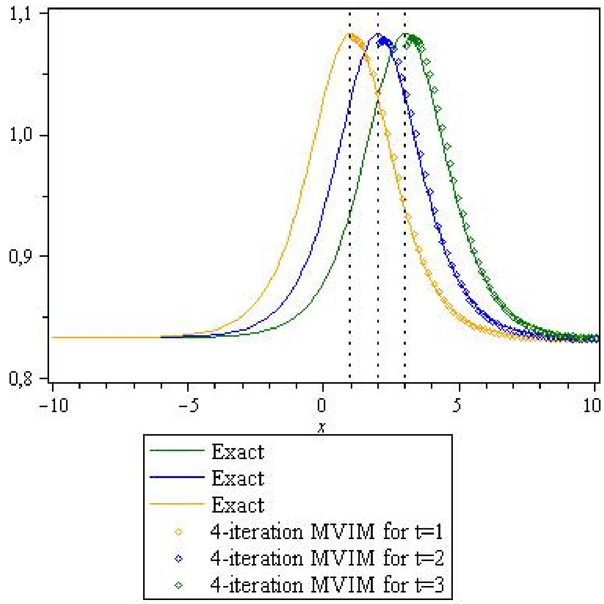


Figure 8. Comparison of the analytical solutions with MVIM as the t values increases for Equation (3.2)

examination of optical fibers, quantum plasma, fluid dynamics, soliton dynamics, mathematical physics, biomedical issues, engineering, and various other fields. The inherent composition of semi-analytical solutions demonstrates the efficacy and potency of these approaches. This work can be studied on semi-analytical solutions for the Kudryashov-Sinelshchikov dynamical equation and further generalized.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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