Mixed Graph Representation and Mixed Graph Isomorphism

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Article Info	Abstract
	Mixed graph is a graph containing oriented as well as un-oriented set of edges. The structural information conveyed by mixed graph is basically information where the edges of the graph are representations of roads, cables, telephone lines and other structures while the vertices are representations of high way intersections/squares, computers, telephone outlet and others. In this paper results related to degreesof vertices, numbers of edges, adjacency matrix representation of mixed graphshave been introduced and illustrated. These results and notations have alsobeen used to formulate a relationship between degrees of vertices and number of edges in a mixed graph as well as to prove that a set of isomorphic mixedgraphs is an equivalence relation. It is, therefore, hoped that these results are helpful for further studies of graph theory, and specifically in the studyof mixed graph.

1. INTRODUCTION

Mixed graph $G_m = (V, E, \vec{E})$) is a graph containing oriented, $\vec{E}$, as well as un-oriented, E, set of edges [2]. Mixed graph is used to represent physical situations with some of whose tasks can be carried out in one direction while others are in two directions. Mixed graph is often used to model problems in engineering, physical science, biological science, communication technology, computer technology and others [1]. For instance, we use mixed graph to model a computer network containing links that could be operated in both directions (represented by undirected edge) and other links that could be operated only in one direction (represented by directed edge).

Many researchers have done much work in graphs and their properties. Representation of graphs, degrees of vertices, isomorphism of graphs, paths of graphs, connectedness of graphs, planarity of graphs, homeomorphism of graphs are among others. However, the study of mixed graphs has been given less attention in recent publications and many contemporary researches [1, 2, 3, 4 and publications not mentioned in the reference section].

This manuscript, therefore, introduces adjacency matrix of mixed graph and isomorphism of mixed graphs. In addition this manuscript proves two corollaries and gives different examples for illustrations. For all graph related theoretical terms, concepts, definitions, properties and theorems not stated here, the reader is referred to [2].

2. DEGREE OF VERTEX AND NUMBER OF EDGES IN MIXED GRAPH

Degree of a vertex v in undirected graph $G_u = (V, E)$, denoted by *deg (v),* is the number of edges incident with the vertex v except a loop at that vertex contributes twice to the degree of that vertex. Furthermore, in a directed graph $G_d = (V, \vec{E})$ with directed edges $\vec{E}$, the in-degree of a vertex v denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex and the out degree of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex [2]. For example if a telephone call is represented using

directed edges, then $deg^-(v)$ represents the number of calls v received while $deg^+(v)$ represents the number of calls v made; but if the call is represented using un-directed graph , $deg(v)$ represents the number of calls either made or received by v. The known theorem, Handshaking theorem in [2], relates the number of edges and degrees of vertices in un-directed $G_u = (V_1, E)$ and directed $G_u = (V_2, \overrightarrow{E})$ graphs respectively as:

$$2|E| = \sum_{v \in V} \deg(v) \text{ and } \left|\overrightarrow{E}\right| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Now the problem is related with degrees of vertices in a mixed graph. In fact there are researchers who use the total number of edges (un-oriented plus oriented edges) incident to the same vertex v as the degree of the vertex, for example [3]. Sometimes, however, we find it essential to treat the number of orientated and un-oriented edges separately; this is because for some problems the meaning of "degree of a vertex" is beyond numbers for that reason a discussion of graphs that ignores the orientation of their edges may miss many essential points in the physical interpretation of the graph models. For illustration purposes consider the following road network containing both one carriageway (represented by directed edge) and dual carriageway (represented by un-directed edges).
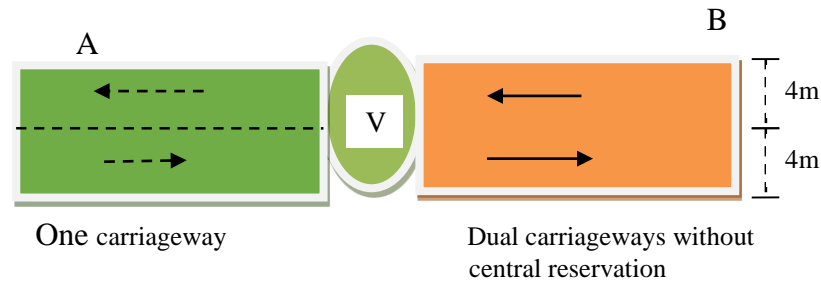

Figure 1

The degree of the vertex v in figure 1, irrespective of the orientation of the edges, is simply the number of roads incident with the vertex v. This, however, doesn't depict the required information that drivers need to get about the square while driving from A to v and from v to B; this is because for some reasons vehicles in dual carriageways of the form segment Bv could unintentionally left the carriageway and limits a space for the vehicles travelling in the opposing directions. Contrarily, if the number of one way roads that takes to and away from the square v and the number of two way roads whose intersection is v are treated separately, the drivers could simply imagine the traffic in v so that delay could be minimized through different alternatives. This section, therefore, introduces a definition for degree of vertex in mixed graph and proves a corollary on the relationship between degrees of vertices with number of edges in a mixed graph.

Definition 2.1 In a mixed graph the degree of a vertex v, denoted by deg (v), is the number of undirected edges incident with v while the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex and the out degree of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

Corollary 2.1 Let $V_m = \{v_1, v_2, v_3, \ldots v_n\}$ be the vertex set of mixed graph $G_m = (V, E, \overrightarrow{E})$). The total number of edges $|E_m|$ in the mixed graph G_m is given by

$$|E_m| = \frac{1}{2}\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg^-(v) \quad or \quad |E_m| = \frac{1}{2}\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg^+(v)$$

Proof .Let $G_d = (V, \overrightarrow{E})$ be directed and $G_u = (V, E)$ undirected edge disjoint sub graphs of the mixed graph $G_m = (V, E, \overrightarrow{E})$

such that $E_m = \mathrm{E} \cup \vec{E}$. From Handshaking theorem we have

$$|E| = \frac{1}{2} \sum \deg(v) \text{ and } \left|\vec{E}\right| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) \qquad (1)$$

Again from elementary counting principle follows

$$|E_m| = \left|E \cup \vec{E}\right| = \left|\vec{E}\right| + |E| - \left|E \cap \vec{E}\right| \qquad (2)$$

Since $G_d = (V, \vec{E})$ and $G_u = (V, E)$ are edge disjoint, $\left|E \cap \vec{E}\right| = 0$ Thus (2) is reduced to

$$\left|E_m\right| = |E| + \left|\vec{E}\right| \qquad (3)$$

Combination of (1) and (3) yields

$$|E_m| = \sum_{v \in V} \deg^-(v) + \frac{1}{2} \sum_{v \in V} \deg(v)$$

$$= \sum_{v \in V} \deg^+(v) + \frac{1}{2} \sum_{v \in V} \deg(v) \quad \because \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v)$$

3. AUGMENTED ADJACENCY MATRIX OF MIXED GRAPH

Graph is completely determined by either its adjacency or its incidences [4]. One way of representing graphs without multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph. However, if there are many edges in a graph, for simplicity, we use adjacency matrix and incidence matrix representations. One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges. Unlike the representation of graphs by adjacency list, matrices representation is applicable to represent multi-graphs with both directed and un-directed edges.

Now suppose $G_u = (V, E)$ is un-directed graph with $|V| = n$. Again let the set of vertices V are arbitrarily listed as $v_1, v_2 \ldots v_n$. Then the adjacency matrix $\mathbf{A_{Gu}}$ of G_u with respect to the listing of the vertices is given by n by n zero-one matrix with the entries

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of the graph} \\ 0 & \text{otherwise} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (*)$$

Furthermore, a directed graph $G_d = (V, \vec{E})$ with arbitrarily listed vertices $v_1, v_2 \ldots v_n$ also has n by n zero-one adjacency matrix $\mathbf{A_{Gd}}$ with the entries

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (**)$$

Example 3.1 Use adjacency matrix to represent the graphs below
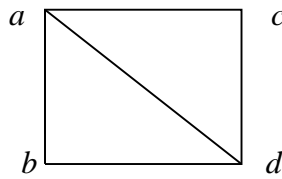

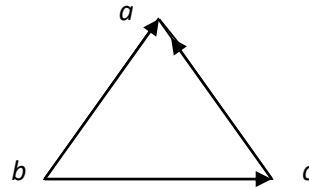***Figure2.*** *undirected graph*   ***Figure3.****directed graph*

Solution. The adjacency matrix of the graph in *Figure 2* with the arbitrary ordering of vertices *a,b,c,d* is given by:

$$\mathbf{A_{Gu}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Similarly the adjacency matrix of the graph in *Figure 3* with the arbitrary ordering of vertices *a,b,c* is given by

$$\mathbf{A_{Gd}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let $V = \{ v_1, v_2, v_3, \ldots v_n \}$ be a set of vertex of a mixed graph $G_m = (V, E, \overrightarrow{E})$. Again let $G_u = (V, E)$ and $G_d = (V, \overrightarrow{E})$, respectively, be an un-directed and directed edge disjoint sub graphs of the mixed graph G_m. This section therefore presents an *n by 2n* augmented adjacency matrix, denoted by $\mathbf{A_{Gm}} = \left(\mathbf{A_{Gu}} / \mathbf{A_{Gd}} \right)$, to represent the mixed graph $G_m = (V, E, \overrightarrow{E})$ where $\mathbf{A_{Gu}}$ and $\mathbf{A_{Gd}}$ are *n* by *n* are adjacency matrices of the edge disjoint sub graphs.

Example 3.2 Use augmented adjacency matrix to represent the mixed graph below.
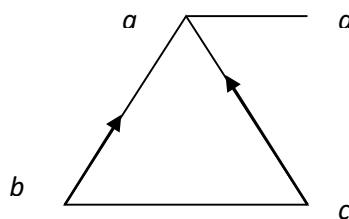

Figure 4. *Mixed graph*

Solution. Let *a,b,c,d* be an arbitrary ordering of the vertices. Again let $\mathbf{A_{Gm}} = \left(\mathbf{A_{Gu}} / \mathbf{A_{Gd}} \right)$ be the augmented adjacency matrix, where $\mathbf{A_{Gu}}$ and $\mathbf{A_{Gd}}$ respectively are 4 by 4 adjacency matrices of $G_u = (V, E)$ and $G_d = (V, \overrightarrow{E})$. Then, the 4 by 8 augmented adjacency matrix $\mathbf{A_{Gm}}$ is given by

$$\mathbf{A_{Gm}} = (\mathbf{A_{Gu}}/\mathbf{A_{Gd}}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3.3 Draw the graph represented by the augmented adjacency matrix given below.

$$\mathbf{A_{Gm}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution. Let *a,b,c ,d* be the ordering of the vertices with the given adjacency matrix. Then the graph is given by
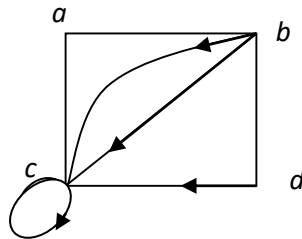

Figure 5. *Mixed graph*

Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Therefore, for a graph with *n* vertices, there are *n* ! different adjacency matrices of the given graph, because there are *n* ! different ordering of the *n* vertices [2].

4. MIXED GRAPH ISOMORPHISM

We often need to know if two or more graph models have the same structure. For instance two cities may have the same road network but different structure or they can have exactly the same road network in the sense that there is one to one correspondence between their squares/intersections that preserves roads. Graphs with the same structure are said to be isomorphic graphs.

Two undirected graphs $G_u = (V_1, E_1)$ and $H_u = (V_2, E_2)$ are said to be isomorphic if there is one to one and on-to mapping $f : V_1 \to V_2$ with the property that for all vertices *u, v* in V_1, the number of edges between *u* and *v* in G_u is the same as the number of edges between $f(u)$ and $f(v)$ in H_u. Furthermore, the directed graphs $G_d = (V_1, E_1)$ and $H_d = (V_2, E_2)$ are said to be isomorphic if the mapping $f : V_1 \to V_2$ is one to one and on-to and for all vertices *u,v* in V1, the number of edges from u to v in G_d is the same as the number of edges from $f(u)$ *to* $f(v)$ in H_d [2]. Based on the isomorphism of directed and undirected graphs and the introduction of augmented adjacency matrix in section 3, mixed graph isomorphism is presented as follows.

 Now Consider mixed graphs $G_m = (V_1, E, \vec{E})$ and $H_m = (V_2, E, \vec{E})$. Then G_m and H_m are said to be isomorphic if there is one to one and on-to function $f : V_1 \to V_2$ with the property that for all vertices *u, v* in V_1, the number of un-directed edges between *u* and *v* in G_m is the same as the number of un-directed edges between $f(u)$ and $f(v)$ in H_m and for all vertices *u, v* in V_1, the number of directed edges from *u* to *v* in G_m is the same as the number of directed edges from $f(u)$ to $f(v)$ in H_m. The number of vertices, number of edges and number of vertices of each degree are all invariants under isomorphism. Besides the degrees, in-degrees and out-degrees of the vertices in isomorphic mixed graphs must be the same. That is,

a vertex of degree d_1, in-degree d_2 and out-degree d_3 in G_m must correspond to a vertex $f(v)$ of degree d_1, in-degree d_2 and out-degree d_3 in H_m; that is because a vertex u in G_m is adjacent to v if and only if $f(u)$ and $f(v)$ are adjacent in H_m. Note that if any of the invariants differ in the two mixed graphs, these graphs cannot be isomorphic. However, when these invariants are the same, it does not necessarily mean that the two mixed graphs are isomorphic. To show a function f from the vertex set of graph G_m to the vertex set of a graph H_m is an isomorphism, we need to show that f preserves edges. One helpful way to do this is to use augmented adjacency matrices. Particularly to show that f preserves edges, we can show that the augmented adjacency matrix of G_m is the same with the augmented adjacency matrix of H_m when rows and columns are labeled to correspond to the images under f of the vertices in G_m that are labeled of these rows and columns in the augmented adjacency matrix of G_m.

Example 4.1 Determine whether the graphs in figure 6 are isomorphic or not.

Graph G_m   Graph H_m

***Figure 6.** Mixed graphs*
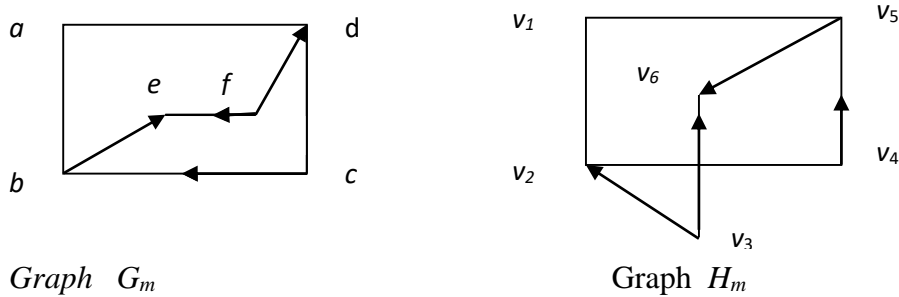
Solution. The number of edges, degree of vertices contributed by directed edges (in degree and out degree) and undirected edges (degree) are the invariants of the graphs G_m and H_m. As can be seen from the table below the mixed graphs G_m and H_m agree with respect to the invariants. Now it is reasonable to try to find an isomorphism between them.

Let $f: V_1 \rightarrow V_2$ be a function defined to determine whether it is an isomorphism where V_1 and V_2 are the vertices sets of the mixed graphs G_m and H_m respectively.

***Table 1.** Invariants of Mixed graphs G_m and H_m*

Mixed Graph G_m			
Vertex	$\deg(v)$	$\deg^-(v)$	$\deg^+(v)$
a	2	0	0
b	1	1	1
c	1	0	1
d	2	1	0
e	0	2	0
f	0	0	2

Mixed Graph H_m			
Vertex	$\deg(v)$	$\deg^-(v)$	$\deg^+(v)$
v_1	2	0	0
v_2	2	1	0
v_3	0	0	2
v_4	1	0	1
v_5	1	1	1
v_6	0	2	0

Using the above table as guide and adjacency of the vertices of graph G_m and the degree, in-degree and out-degrees, we have

$$f(a) = v_1 \ , f(b) = v_5 \ , f(c) = v_4 \ , f(d) = v_2 \ , f(e) = v_6 \text{ and } f(f) = v_3.$$

Again to see whether f preserves edges, we examine the augmented adjacency matrix of G_m in the ordering $a,b,c,d,e\ ,f$.

$$A_{Gm} = \left(A_{Gu}/A_{Gd}\right) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The augmented adjacency matrix of H_m with the rows and columns labeled by the images of the corresponding vertices in G_m is given in the ordering of $v_1, \ v_5, \ v_4, v_2 \ v_6 \ , v_3$

$$A_{Hm} = \left(A_{Hu}/A_{Hd}\right) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$A_{Gm} = A_{Hm}$, it follows that f preserves edges. Thus, f is an isomorphism and the mixed graphs G_m and H_m are isomorphic.

Corollary 4.1 Isomorphism of mixed graphs is an equivalence relation**.**

Proof. To show isomorphism of mixed graphs is an equivalence relation, we need to show mixed graph isomorphism is reflexive, symmetric and transitive.

Let $f\colon G_m \to G_m$ be a mapping, clearly f is identity function, so isomorphism is reflexive.

For isomorphic mixed graphs G_m and H_m, there exists a one to one correspondence $f\colon G_m \to H_m$ that preserves adjacency. Since f is one to one correspondence from G_m to H_m, there is one to one correspondence f^1 from H_m to G_m that preserves adjacency. Thus, isomorphism of mixed graphs is symmetric.

Suppose G_m is isomorphic to H_m and H_m is isomorphic to K_m. Then there are one to one correspondences f and h from G_m to H_m and from H_m to K_m , respectively, that preserves adjacency. Hence it follows *hof* is a one to one correspondence from G_m to K_m that preserves adjacency. Hence isomorphism is transitive.

Thus, isomorphism of mixed graphs is an equivalence relation**.**

5. FINAL REMARKS

To solve physical problems of different origin, sometimes we find it necessary to use mixed graphs. However, the studies of graphs that have been made before give more attention to directed and undirected graphs. This manuscript, therefore, has tried to introduce the relationship between degree of vertices and number of edges in mixed graph. Further, it presents adjacency matrix representation of mixed graphs using augmented matrix and isomorphism of mixed graphs.

Finally this manuscript suggests that the study of the concepts and properties of mixed graph need to give due emphasis; because examining the properties of mixed graph from its underlying fully oriented or fully un-oriented graph, may, lead to miss many essential points in the physical interpretation of the mixed graph.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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