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New Notions From (α,β)-Generalised Fuzzy Preopen Sets

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Abstract

The present article discuss (α,β)-generalised fuzzy pre-border, (α,β)-generalised fuzzy pre-exterior and (α,β)-generalised fuzzy pre-frontier in double fuzzy topologies. Furthermore, some characterisations of generalised double fuzzy pre-continuous, generalised double fuzzy preopen, generalised double fuzzy preclosed and generalised double fuzzy preclosure-irresolute functions are studied and investigated. Moreover, the interrelations among the new concepts are discussed with some necessary examples.

1. INTRODUCTION

Fuzzy theory [1] was initiated in 1965. Later on, Chang [6] utilized it to define fuzzy topological spaces. Intuitionistic fuzzy topologies introduced by Çoker et al. [8, 10] based on intuitionistic fuzzy sets. In [12], the authors succeeded to intuitionism the structure of topology. The name "intuitionistic" is no longer used in mathematics because of some suspicions about the convenience of this term. This suspicions were refuted in [13], when the authors substituted the term "intuitionistic" by "double" and renamed its related topologies. The concept of intuitionistic gradation of openness has been renamed to"double fuzzy topological spaces".

M. Sudha et al. [16] studied notions via generalised fuzzy preopen sets in Chang's fuzzy topologies. In [14, 15], weak continuity has been introduced based on the notions (α,β)-fuzzy preopen and (α,β)-fuzzy preclosed sets in double fuzzy topologies.

There are a variety of applications for Topology. For instance, ophthalmologists have found that when the sight is restored to a blind person, the person will have a topological vision for some time. During this short time, the person who has recovered from the blindness cannot differentiate between a circle and a square and any closed curves. He has to exercise for some time to precisely characterize various closed curves. Based on this idea, Zeeman has constructed a topological model of the brain and the visual perception [7].

Topological psychology [9, 11] is a dialectical topic in which mathematicians have different opinions on it. The German scientist Lewin studied topology and applied the topological notions in his theories in psychology. He tried hard to formalize his theories into an evident form in order to avoid the rigidity and obstinacy of the results. Lewin presented concepts incidentally and progressively developed them through experimental and observation methods.

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The topological relationships between spatial objects are essential information used in Geographic information systems (GIS), along with positional and attribute information. Information on topological relationships can be used for spatial queries, spatial analyses, data quality control, and others. Topological relationships may be fuzzy or crisp based on the certainty or uncertainty of spatial objects and the nature of their relationships (see [17]).

This paper takes some investigations on (α,β)-generalised fuzzy pre-border, (α,β)-generalised fuzzy pre-exterior and (α,β)-generalised fuzzy pre-frontier in double fuzzy topological spaces. Moreover, some characteristic properties of generalised double fuzzy pre-continuous, generalised double fuzzy preopen, generalised double fuzzy preclosed, and generalised double fuzzy preclosure-irresolute functions are studied and investigated. Also, we present the relationships between the new notions and already known functions. Our new results are a generalization to the corresponding notions in topology and fuzzy topology and therefore we believe that it will be more useful and applicable for GIS modeling and for other scientific fields.

2. PRELIMINARIES

In this paper $X \neq \varnothing$, $I_0 = (0,1]$, $I_1 = [0,1)$ and $I = [0,1]$. I^X is the family of fuzzy sets defined on the universal set X. $P_t(X)$ refers to the collection of all fuzzy points defined on X. $\underline{0}$ and $\underline{1}$ refer to the smallest and the largest elements in I^X, respectively. The complement of $A \in I^X$ is denoted by $\underline{1} - A$. For any crisp map $f : X \to Y$, the direct image $f(A)$ and inverse image $f^{-1}(A)$ of f are given by $f(A)(y) = \bigvee_{f(x)=y} A(x)$ and $f^{-1}(B)(x) = B(f(x))$ for all $A \in I^X$, $B \in I^Y$ and $x \in X$, respectively.

Definition 2.1 *[12, 13] The pair of functions τ, $\tau^* : I^X \to I$ is a double fuzzy topology on X if and only if it achieves the next statements:*

(1) $\tau(A) \leq \underline{1} - \tau^*(A)$ for every $A \in I^X$.

(2) $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$ and $\tau^*(A_1 \wedge A_2) \leq \tau^*(A_1) \vee \tau^*(A_2)$ for every A_1, $A_2 \in I^X$.

(3) $\tau(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \tau(A_i)$ and $\tau^*(\bigvee_{i\in\Gamma} A_i) \leq \bigvee_{i\in\Gamma} \tau^*(A_i)$ for every $A_i \in I^X$, $i \in \Gamma$.

By (X,τ,τ^*), we denotes the double fuzzy topological space (briefly, dfts). A is said to be an (α,β)-fuzzy open (briefly, (α,β)-fo) if and only if $\tau(A) \geq \alpha$ and $\tau^*(A) \leq \beta$, and A is said to be an (α,β)-fuzzy closed (briefly, (α,β)-fc) if and only if $\underline{1} - A$ is an (α,β)-fo set. Moreover, a fuzzy set B is called (α,β)-fuzzy clopen (briefly, (α,β)-fco) if and only if B is (α,β)-fo and (α,β)-fc, simultaneously. For any two dfts's (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*), a double fuzzy continuous $f : X \to Y$ is a function that satisfies $\tau_1(f^{-1}(D)) \geq \tau_2(D)$ and $\tau_1^*(f^{-1}(D)) \leq \tau_2^*(D)$ for every $D \in I^Y$.

Theorem 2.1 *[2, 3] For a dfts (X,τ,τ^*), $A \in I^X$, $\alpha \in I_0$, and $\beta \in I_1$, we define the map $C_{\tau,\tau^*} : I^X \times I_0 \times I_1 \to I^X$ by:*

$$C_{\tau,\tau^*}(A,\alpha,\beta) = \bigwedge\{B \in I^X \mid A \leq B, \tau(\underline{1} - B) \geq \alpha, \tau^*(\underline{1} - B) \leq \beta\}.$$

The map C_{τ,τ^*} achieves the following conditions, for all $A, B \in I^X$, $\alpha, \alpha_1, \alpha_2 \in I_0$ and $\beta, \beta_1, \beta_2 \in I_1$:

(1) $C_{\tau,\tau^*}(\underline{0},\alpha,\beta) = \underline{0}$.

(2) $A \leq C_{\tau,\tau^*}(A,\alpha,\beta)$.

(3) $C_{\tau,\tau^*}(A,\alpha,\beta) \vee C_{\tau,\tau^*}(B,\alpha,\beta) = C_{\tau,\tau^*}(A \vee B,\alpha,\beta)$.

(4) If $\alpha_1 \leq \alpha_2$, $\beta_1 \geq \beta_2$, then $C_{\tau,\tau^*}(A,\alpha_1,\beta_1) \leq C_{\tau,\tau^*}(A,\alpha_2,\beta_2)$.

(5) $C_{\tau,\tau^*}(C_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = C_{\tau,\tau^*}(A,\alpha,\beta)$.

Theorem 2.2 *[2, 3] For a dfts* (X,τ,τ^*), $A \in I^X$, $\alpha \in I_0$, *and* $\beta \in I_1$, *we define the map* $I_{\tau,\tau^*}: I^X \times I_0 \times I_1 \to I^X$ *by:*

$$I_{\tau,\tau^*}(A,\alpha,\beta) = \bigvee\{B \in I^X \mid B \leq A, \tau(B) \geq \alpha, \tau^*(B) \leq \beta\}.$$

The map I_{τ,τ^*} achieves the following conditions, for every $A, B \in I^X$, $\alpha,\alpha_1,\alpha_2 \in I_0$ and $\beta,\beta_1,\beta_2 \in I_1$:

(1) $I_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) = \underline{1} - C_{\tau,\tau^*}(A,\alpha,\beta)$.

(2) $I_{\tau,\tau^*}(\underline{1},\alpha,\beta) = \underline{1}$.

(3) $I_{\tau,\tau^*}(A,\alpha,\beta) \leq A$.

(4) $I_{\tau,\tau^*}(A,\alpha,\beta) \wedge I_{\tau,\tau^*}(B,\alpha,\beta) = I_{\tau,\tau^*}(A \wedge B,\alpha,\beta)$.

(5) $I_{\tau,\tau^*}(A,\alpha_1,\beta_1) \geq I_{\tau,\tau^*}(A,\alpha_2,\beta_2)$ if $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$.

(6) $I_{\tau,\tau^*}(I_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = I_{\tau,\tau^*}(A,\alpha,\beta)$.

(7) If $I_{\tau,\tau^*}(C_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = A$, then $C_{\tau,\tau^*}(I_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta),\alpha,\beta) = \underline{1} - A$.

Definition 2.2 *[14, 4, 5] For a dfts* (X,τ,τ^*), A, $B \in I^X$, $\alpha \in I_0$ *and* $\beta \in I_1$, *we have:*

(1) A is said to be an (α,β)-fuzzy preopen (briefly, (α,β)-fpo) if $A \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta)$. Ù□A fuzzy set is said to be (α,β)-fuzzy preclosed (briefly, (α,β)-fpc) if and only if its complement is (α,β)-fpo set.

(2) (α,β)-fuzzy preclosure of A is defined by $PC_{\tau,\tau^*}(A,\alpha,\beta) = \bigwedge\{B \in I^X \mid A \leq B,\ B$ is (α,β)-fpc$\}$.

(3) (α,β)-fuzzy preinterior of A is given by $PI_{\tau,\tau^*}(A,\alpha,\beta) = \bigvee\{B \in I^X \mid B \leq A,\ B$ is (α,β)-fpo$\}$.

(4) A is an (α,β)-generalised fuzzy closed (briefly, (α,β)-gfc) if $C_{\tau,\tau^*}(A,\alpha,\beta) \leq B$ such that $A \leq B$, $\tau(B) \geq r$ and $\tau^*(B) \leq s$. Ù□A fuzzy set is said to be (α,β)-generalised fuzzy open (briefly, (α,β)-gfo) if and only if its complement is (α,β)-gfc set.

(5) $GC_{\tau,\tau^*}(A,\alpha,\beta) = \bigwedge\{B \in I^X \mid A \leq B,\ B\ is\ (\alpha,\beta) - gfc\}$, where $GC_{\tau,\tau^*}(A,\alpha,\beta)$ is an (α,β)-fuzzy generalised closure of A.

(6) $GI_{\tau,\tau^*}(A,\alpha,\beta) = \bigvee\{B \in I^X \mid B \leq A\ and\ B\ is\ (\alpha,\beta) - gfo\}$, where $GI_{\tau,\tau^*}(A,\alpha,\beta)$ is an (α,β)-fuzzy generalised interior of A.

3. PROPERTIES OF (α,β)-GENERALISED FUZZY PREOPEN SETS

In this section, we study (α,β)-generalised fuzzy pre-border, (α,β)-generalised fuzzy pre-exterior and (α,β)-generalised fuzzy pre-frontier. Some of its interesting properties and characterizations are examined.

Definition 3.1 For any dfts (X,τ,τ^*), $A\in I^X$, $\alpha\in I_0$ and $\beta\in I_1$, we have:

(1) A is said to be (α,β)-generalised fuzzy preclosed (briefly, (α,β)-gfpc) if $C_{\tau,\tau^*}(A,\alpha,\beta)\le B$, whenever $A\le B$ and B is an (α,β)-fpo. A fuzzy set is said to be an (α,β)-generalised fuzzy preopen (briefly, (α,β)-gfpo) if and only if its complement is (α,β)-gfpc set.

(2) $GPC_{\tau,\tau^*}(A,\alpha,\beta) = \bigwedge\{B\in I^X \mid A\le B \ and \ B \ is\,(\alpha,\beta)-gfpc\}$, where $GPC_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-fuzzy generalised preclosure of A.

(3) $GPI_{\tau,\tau^*}(A,\alpha,\beta) = \bigvee\{B\in I^X \mid B\le A \ and \ B \ is\,(\alpha,\beta)-gfpo\}$, where $GPI_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-fuzzy generalised preinterior of A.

Proposition 3.1 For any dfts (X,τ,τ^*) and for all A, $B\in I^X$, $\alpha\in I_0$ and $\beta\in I_1$, we have:

(1) $GPI_{\tau,\tau^*}(A,\alpha,\beta)$ is the largest (α,β)-gfpo set such that $GPI_{\tau,\tau^*}(A,\alpha,\beta)\le A$.

(2) $A = GPI_{\tau,\tau^*}(A,\alpha,\beta)$, if A is an (α,β)-gfpo set.

(3) $GPI_{\tau,\tau^*}(GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = GPI_{\tau,\tau^*}(A,\alpha,\beta)$, if A is an (α,β)-gfpo set.

(4) $\underline{1}-GPI_{\tau,\tau^*}(A,\alpha,\beta) = GPC_{\tau,\tau^*}(\underline{1}-A,\alpha,\beta)$.

(5) $\underline{1}-GPC_{\tau,\tau^*}(A,\alpha,\beta) = GPI_{\tau,\tau^*}(\underline{1}-A,\alpha,\beta)$.

(6) If $A\le B$, then $GPI_{\tau,\tau^*}(A,\alpha,\beta)\le GPI_{\tau,\tau^*}(B,\alpha,\beta)$.

(7) If $A\le B$, then $GPC_{\tau,\tau^*}(A,\alpha,\beta)\le GPC_{\tau,\tau^*}(B,\alpha,\beta)$.

(8) $GPI_{\tau,\tau^*}(A,\alpha,\beta)\wedge GPI_{\tau,\tau^*}(B,\alpha,\beta) = GPI_{\tau,\tau^*}(A\wedge B,\alpha,\beta)$.

(9) $GPI_{\tau,\tau^*}(A,\alpha,\beta)\vee GPI_{\tau,\tau^*}(B,\alpha,\beta) = GPI_{\tau,\tau^*}(A\vee B,\alpha,\beta)$.

Proof. (1) and (2) follows from the definitions and (3) follows from (2).

(4)

$$\begin{aligned}
GPC_{\tau,\tau^*}(\underline{1}-A,\alpha,\beta) &= \bigwedge\{B \mid \ B \ is\ (\alpha,\beta)-gfpc\ set,\ B\ge \underline{1}-A\} \\
&= \underline{1}-\bigvee\{\underline{1}-B \mid \underline{1}-B \ \ is\ (\alpha,\beta)-gfpo\ set,\ \underline{1}-B\le A\} \\
&= \underline{1}-GPI_{\tau,\tau^*}(A,\alpha,\beta).
\end{aligned}$$

(5) It is similar to (4).

(6) It is clear that

$$GPI_{\tau,\tau^*}(A,\alpha,\beta) = \bigvee\{D \,|D \; is \; an \; (\alpha,\beta)-gfpo \; and \; D \le A\}$$
$$\le \bigvee\{D \,|\, D \le B \; and \; D \; is \; an \; (\alpha,\beta)-gfpo\}$$
$$= GPI_{\tau,\tau^*}(B,\alpha,\beta),$$

if $A \le B$.

It is similar to (6).

(8)

$$GPI_{\tau,\tau^*}(A \wedge B,\alpha,\beta) = \bigvee\{D \,|\; D \; is \; an \; (\alpha,\beta)-gfpo \; and \; D \le (A \wedge B)\}$$
$$= \left(\bigvee\{D \,|D \; is \; an \; (\alpha,\beta)-gfpo \; and \; D \le A\}\right)$$
$$\wedge \left(\bigvee\{D \,|\; D \; is \; an \; (\alpha,\beta)-gfpo \; and \; D \le B\}\right)$$
$$= (GPI_{\tau,\tau^*}(A,\alpha,\beta)) \wedge (GPI_{\tau,\tau^*}(B,\alpha,\beta)).$$

(9) It is similar to (8).

Definition 3.2 For any dfts (X,τ,τ^*), $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

(1) $PB_{\tau,\tau^*}(A,\alpha,\beta) = A - PI_{\tau,\tau^*}(A,\alpha,\beta),$ where $PB_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-fuzzy pre-border of A.

(2) $GB_{\tau,\tau^*}(A,\alpha,\beta) = A - GI_{\tau,\tau^*}(A,\alpha,\beta),$ where $GB_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy border of A.

(3) $GPB_{\tau,\tau^*}(A,\alpha,\beta) = A - GPI_{\tau,\tau^*}(A,\alpha,\beta),$ where $GPB_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy pre-border of A.

Proposition 3.2 For any dfts (X,τ,τ^*), for all $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

(1) $GPB_{\tau,\tau^*}(A,\alpha,\beta) \le PB_{\tau,\tau^*}(A,\alpha,\beta)$.

(2) If A is an (α,β)-gfpo, then $GPB_{\tau,\tau^*}(A,\alpha,\beta) = \underline{0}$.

(3) $GPB_{\tau,\tau^*}(A,\alpha,\beta) \le GPC_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta)$.

(4) $GPI_{\tau,\tau^*}(GPB_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \le A$.

(5) $GPB_{\tau,\tau^*}(A \vee B) \le (GPB_{\tau,\tau^*}(A,\alpha,\beta)) \vee (GPB_{\tau,\tau^*}(B,\alpha,\beta))$.

(6) $GPB_{\tau,\tau^*}(A \wedge B) \ge (GPB_{\tau,\tau^*}(A,\alpha,\beta)) \wedge (GPB_{\tau,\tau^*}(B,\alpha,\beta))$.

Proof. (1) For any $A \in I^X$, since

$$PI_{\tau,\tau^*}(A,\alpha,\beta) \le GPI_{\tau,\tau^*}(A,\alpha,\beta),$$
$$A - GPI_{\tau,\tau^*}(A,\alpha,\beta) \le A - PI_{\tau,\tau^*}(A,\alpha,\beta).$$

Therefore,

$$GPB_{\tau,\tau^*}(A,\alpha,\beta) \le PB_{\tau,\tau^*}(A,\alpha,\beta).$$

(2) For any an (α,β)-gfpo set $A \in I^X$, we have

$$A = GPI_{\tau,\tau^*}(A,\alpha,\beta).$$

Thus,

$$GPB_{\tau,\tau^*}(A,\alpha,\beta) = \underline{0}.$$

(3)

$$\begin{aligned} GPB_{\tau,\tau^*}(A,\alpha,\beta) &= A - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= A - (\underline{1} - GPC_{\tau,\tau^*}(\underline{1} - A)) \\ &\le \underline{1} - \underline{1} + GPC_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) \\ &= GPC_{\tau,\tau^*}(\underline{1} - A). \end{aligned}$$

(4)

$$\begin{aligned} GPI_{\tau,\tau^*}(GPB_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) &= GPI_{\tau,\tau^*}(A - GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &\le A - GPI_{\tau,\tau^*}(A,\alpha,\beta) \le A, \end{aligned}$$

by (1) of Proposition 3.1. Therefore,

$$GPI_{\tau,\tau^*}(GPB_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \le A.$$

(5)

$$\begin{aligned} GPB_{\tau,\tau^*}(A \vee B,\alpha,\beta) &= (A \vee B) - GPI_{\tau,\tau^*}(A \vee B) \\ &= (A \vee B) - (GPI_{\tau,\tau^*}(A,\alpha,\beta)) \vee (GPI_{\tau,\tau^*}(B,\alpha,\beta)) \\ &\le (A - GPI_{\tau,\tau^*}(A,\alpha,\beta)) \vee (B - GPI_{\tau,\tau^*}(B,\alpha,\beta) \\ &= (GPB_{\tau,\tau^*}(A,\alpha,\beta)) \vee (GPB_{\tau,\tau^*}(B,\alpha,\beta)). \end{aligned}$$

Therefore,

$$GPB_{\tau,\tau^*}(A \vee B,\alpha,\beta) \le (GPB_{\tau,\tau^*}(A,\alpha,\beta)) \vee (GPB_{\tau,\tau^*}(B,\alpha,\beta)).$$

(6) It is similar to (5).

Definition 3.3 For any dfts (X,τ,τ^*), $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

(1) $PF_{\tau,\tau^*}(A,\alpha,\beta) = PC_{\tau,\tau^*}(A,\alpha,\beta) - PI_{\tau,\tau^*}(A,\alpha,\beta)$, where $PF_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-fuzzy pre-frontier of A.

(2) $GF_{\tau,\tau^*}(A,\alpha,\beta) = GC_{\tau,\tau^*}(A,\alpha,\beta) - GI_{\tau,\tau^*}(A,\alpha,\beta)$, where $GF_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy frontier of A.

(3) $GPF_{\tau,\tau^*}(A,\alpha,\beta) = GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta)$, where $GPF_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy pre-frontier of A.

Proposition 3.3 For any dfts (X,τ,τ^*), for every $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

(1) $GPF_{\tau,\tau^*}(A,\alpha,\beta) \le PF_{\tau,\tau^*}(A,\alpha,\beta)$.

(2) $GPB_{\tau,\tau^*}(A,\alpha,\beta) \le GPF_{\tau,\tau^*}(A,\alpha,\beta)$.

(3) $GPF_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) = GPF_{\tau,\tau^*}(A,\alpha,\beta)$.

(4) $GPF_{\tau,\tau^*}(GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \le GPF_{\tau,\tau^*}(A,\alpha,\beta)$.

(5) $GPF_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \le GPF_{\tau,\tau^*}(A,\alpha,\beta)$.

(6) $A - GPF_{\tau,\tau^*}(A,\alpha,\beta) \le GPI_{\tau,\tau^*}(A,\alpha,\beta)$.

(7) $GPF_{\tau,\tau^*}(A \vee B,\alpha,\beta) \le GPF_{\tau,\tau^*}(A,\alpha,\beta) \vee GPF_{\tau,\tau^*}(B,\alpha,\beta)$.

(8) $GPF_{\tau,\tau^*}(A \wedge B,\alpha,\beta) \ge GPF_{\tau,\tau^*}(A,\alpha,\beta) \wedge GPF_{\tau,\tau^*}(B,\alpha,\beta)$.

Proof. (1)

$$\begin{aligned} GPF_{\tau,\tau^*}(A,\alpha,\beta) &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &\le PC_{\tau,\tau^*}(A,\alpha,\beta) - PI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= PF_{\tau,\tau^*}(A,\alpha,\beta). \end{aligned}$$

(2)

$$\begin{aligned} GPB_{\tau,\tau^*}(A,\alpha,\beta) &= A - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &\le GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta), \end{aligned}$$

since

$$A \le GPC_{\tau,\tau^*}(A,\alpha,\beta) = GPF_{\tau,\tau^*}(A,\alpha,\beta).$$

Therefore

$$GPB_{\tau,\tau^*}(A,\alpha,\beta) \le GPF_{\tau,\tau^*}(A,\alpha,\beta).$$

(3)

$$\begin{aligned} GPF_{\tau,\tau^*}(A,\alpha,\beta) &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - (\underline{1} - GPC_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta)) \\ &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - \underline{1} + GPC_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) \\ &= -GPI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) + GPC_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) \\ &= GPF_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta). \end{aligned}$$

(4)

$$\begin{aligned} GPF_{\tau,\tau^*}(GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) &= GPC_{\tau,\tau^*}(GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &\quad - GPI_{\tau,\tau^*}(GPI_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &\leq GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPF_{\tau,\tau^*}(A,\alpha,\beta). \end{aligned}$$

(5)

$$\begin{aligned} GPF_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) &= GPC_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &\quad - GPI_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &\geq GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPF_{\tau,\tau^*}(A,\alpha,\beta). \end{aligned}$$

(6)

$$\begin{aligned} A - GPF_{\tau,\tau^*}(A,\alpha,\beta) &= A - (GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta)) \\ &\leq GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPC_{\tau,\tau^*}(A,\alpha,\beta) + GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPI_{\tau,\tau^*}(A,\alpha,\beta). \end{aligned}$$

(7)

$$\begin{aligned} GPF_{\tau,\tau^*}(A \vee B,\alpha,\beta) &= GPC_{\tau,\tau^*}(A \vee B,\alpha,\beta) - GPI_{\tau,\tau^*}(A \vee B,\alpha,\beta) \\ &= GPC_{\tau,\tau^*}(A \vee B,\alpha,\beta) - (GPI_{\tau,\tau^*}(A,\alpha,\beta) \vee GPI_{\tau,\tau^*}(B,\alpha,\beta)) \\ &= (GPC_{\tau,\tau^*}(A,\alpha,\beta) \vee GPC_{\tau,\tau^*}(B,\alpha,\beta)) \\ &\quad - (GPI_{\tau,\tau^*}(A,\alpha,\beta) \vee GPI_{\tau,\tau^*}(B,\alpha,\beta)) \\ &\leq (GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta)) \\ &\quad \vee (GPC_{\tau,\tau^*}(B,\alpha,\beta) - GPI_{\tau,\tau^*}(B,\alpha,\beta)) \\ &= GPF_{\tau,\tau^*}(A,\alpha,\beta) \vee GPF_{\tau,\tau^*}(B,\alpha,\beta). \end{aligned}$$

(8) It is similar to (7).

Definition 3.4 For any dfts (X,τ,τ^*), for each $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

1. $PE_{\tau,\tau^*}(A,\alpha,\beta) = PI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta),$

where $PE_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-fuzzy pre-exterior of A .

2. $GE_{\tau,\tau^*}(A,\alpha,\beta) = GI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta),$

where $GE_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy exterior of A .

3. $GPE_{\tau,\tau^*}(A,\alpha,\beta) = GPI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta),$

where $GPE_{\tau,\tau^*}(A,\alpha,\beta)$ is the (α,β)-generalised fuzzy pre-exterior of A.

Proposition 3.4 For any dfts (X,τ,τ^*), for every $A \in I^X$, $\alpha \in I_0$ and $\beta \in I_1$, we have:

(1) $PE_{\tau,\tau^*}(A,\alpha,\beta) \le GPE_{\tau,\tau^*}(A,\alpha,\beta)$.

(2) $GPE_{\tau,\tau^*}(A,\alpha,\beta) = \underline{1} - GPC_{\tau,\tau^*}(A,\alpha,\beta)$.

(3) $GPE_{\tau,\tau^*}(GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = GPI_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta)$.

(4) If $A \le B$, then $GPE_{\tau,\tau^*}(A,\alpha,\beta) \ge GPE_{\tau,\tau^*}(B,\alpha,\beta)$.

(5) $GPE_{\tau,\tau^*}(\underline{1},\alpha,\beta) = \underline{0}$.

(6) $GPE_{\tau,\tau^*}(\underline{0},\alpha,\beta) = \underline{1}$.

(7) $GPI_{\tau,\tau^*}(A,\alpha,\beta) \le GPE_{\tau,\tau^*}(GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta)$.

(8) $GPE_{\tau,\tau^*}(A \vee B,\alpha,\beta) = GPE_{\tau,\tau^*}(A,\alpha,\beta) \wedge GPE_{\tau,\tau^*}(B,\alpha,\beta)$.

(9) $GPE_{\tau,\tau^*}(A \wedge B,\alpha,\beta) = GPE_{\tau,\tau^*}(A,\alpha,\beta) \vee GPE_{\tau,\tau^*}(B,\alpha,\beta)$.

Proof. (1) Since

$$GPC_{\tau,\tau^*}(A,\alpha,\beta) \le PC_{\tau,\tau^*}(A,\alpha,\beta),$$
$$\underline{1} - GPC_{\tau,\tau^*}(A,\alpha,\beta) \ge \underline{1} - PC_{\tau,\tau^*}(A,\alpha,\beta)$$
$$\Rightarrow GPI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) \ge PI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta).$$

Therefore by definition,

$$GPE_{\tau,\tau^*}(A,\alpha,\beta) \ge PE_{\tau,\tau^*}(A,\alpha,\beta).$$

(2) It follows from the definitions.

(3)

$$\begin{aligned} GPE_{\tau,\tau^*}(GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) &= GPI_{\tau,\tau^*}(\underline{1} - GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) \\ &= GPI_{\tau,\tau^*}(\underline{1} - GPI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta),\alpha,\beta). \end{aligned}$$

Therefore,

$$GPE_{\tau,\tau^*}(GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta) = GPI_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta).$$

(4) Let $A \le B$. By using Proposition 3.1(1),

$$GPC_{\tau,\tau^*}(A,\alpha,\beta) \le GPC_{\tau,\tau^*}(B,\alpha,\beta).$$

Therefore,

$$\underline{1} - GPC_{\tau,\tau^*}(A,\alpha,\beta) \geq \underline{1} - GPC_{\tau,\tau^*}(B,\alpha,\beta).$$

But,

$$GPI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) \geq GPI_{\tau,\tau^*}(\underline{1} - B,\alpha,\beta).$$

Hence,

$$GPE_{\tau,\tau^*}(A,\alpha,\beta) \geq GPE_{\tau,\tau^*}(B,\alpha,\beta).$$

(5) By (2),

$$\begin{aligned} GPE_{\tau,\tau^*}(\underline{1},\alpha,\beta) &= \underline{1} - GPC_{\tau,\tau^*}(\underline{1},\alpha,\beta) \\ &= \underline{1} - \underline{1} \\ &= \underline{0}. \end{aligned}$$

(6) It is similar to (5).

(7) Since $A \leq GPC_{\tau,\tau^*}(A,\alpha,\beta)$, and

$$GPI_{\tau,\tau^*}(A,\alpha,\beta) \leq GPI_{\tau,\tau^*}(GPC_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta).$$

Then by (3),

$$GPI_{\tau,\tau^*}(A,\alpha,\beta) \leq GPE_{\tau,\tau^*}(GPE_{\tau,\tau^*}(A,\alpha,\beta),\alpha,\beta).$$

(8)

$$\begin{aligned} GPE_{\tau,\tau^*}(A \vee B,\alpha,\beta) &= GPI_{\tau,\tau^*}(\underline{1} - (A \vee B),\alpha,\beta) \\ &= GPI_{\tau,\tau^*}((\underline{1} - A) \wedge (\underline{1} - B),\alpha,\beta) \\ &= (GPI_{\tau,\tau^*}(\underline{1} - A),\alpha,\beta) \wedge (GPI_{\tau,\tau^*}(\underline{1} - B),\alpha,\beta) \\ &= GPE_{\tau,\tau^*}(A,\alpha,\beta) \wedge GPE_{\tau,\tau^*}(B,\alpha,\beta). \end{aligned}$$

(9) Obvious.

4. CHARACTERISATIONS OF GENERALISED DOUBLE FUZZY PRECLOSURE-IRRESOLUTE AND GENERALISED DOUBLE FUZZY PRE-CONTINUOUS FUNCTIONS

In this section, some characterisations of generalised double fuzzy pre-continuous, generalised double fuzzy preopen, generalised double fuzzy preclosed, and generalised double fuzzy preclosure-irresolute functions are studied.

Definition 4.1 *A map $f:(X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ between any two dfts's (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*) is called:*

(1) generalised double fuzzy preopen (briefly, gdfp-open) if for every (α,β)-gfpo set $A \in I^X$, $\beta \in I_1$ and $\alpha \in I_0$, $f(A)$ is an (α,β)-gfpo in I^Y.

(2) generalised double fuzzy preclosed (briefly, gdfp-closed) if for every (α,β)-gfpc set $A \in I^X$, $\beta \in I_1$ and $\alpha \in I_0$, $f(A)$ is an (α,β)-gfpc in I^Y.

(3) generalised double fuzzy pre-continuous (gdfp-continuous, for short) if for every (α,β)-fo set $A \in I^Y$, $\alpha \in I_0$ and $\beta \in I_1$, $f^{-1}(A)$ is an (α,β)-gfpo in I^X.

(4) generalised double fuzzy preclosure-Irresolute (gdfpc-Irr, for short) if $f^{-1}(GPC_{\tau,\tau^*}(f(A),\alpha,\beta)$ is an (α,β)-gfpc set for every (α,β)-gfpc set $A \in I^Y$, $\alpha \in I_0$ and $\beta \in I_1$.

Theorem 4.1 *For any two dfts's* (X,τ_1,τ_1^*) *and* (Y,τ_2,τ_2^*) *and any bijective map* $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$, *we have:*

(1) f is gdfpc-Irr function.

(2) For every fuzzy set A in I^X, $f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) \le GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)$.

(3) For every fuzzy set B in I^Y, $GPC_{\tau_1,\tau_1^*}(f^{-1}(B),\alpha,\beta) \le f^{-1}(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta))$.

Proof. $(1)\Rightarrow(2)$ Suppose $A \in I^X$ and $GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta) \in I^Y$ is an (α,β)-gfpc, then by (1), we have $f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)) \in I^X$ is an (α,β)-gfpc set, $\alpha \in I_0$ and $\beta \in I_1$. Therefore,

$$GPC_{\tau_2,\tau_2^*}(f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)),\alpha,\beta) = f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)).$$

Since $A \le f^{-1}(f(A))$ and

$$GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta) \le GPC_{\tau_2,\tau_2^*}(f^{-1}(f(A),\alpha,\beta)).$$

Also,

$$f(A) \le GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

Then

$$\begin{aligned} GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta) &\le GPC_{\tau_2,\tau_2^*}(f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta),\alpha,\beta)) \\ &= f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)). \end{aligned}$$

$(2)\Rightarrow(3)$ Suppose $B \in I^Y$, by (2),

$$\begin{aligned} f(GPC_{\tau_2,\tau_2^*}(f^{-1}(B),\alpha,\beta)) &\le GPC_{\tau_2,\tau_2^*}(f(f^{-1}(B)),\alpha,\beta) \\ &\le GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta). \end{aligned}$$

That is,

$$f(GPC_{\tau_1,\tau_1^*}(f^{-1}(B),\alpha,\beta)) \le GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta).$$

Therefore,

$$f^{-1}(f(GPC_{\tau_2,\tau_2^*}(f^{-1}(B),\alpha,\beta))) \le f^{-1}(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta)).$$

Hence,

$$GPC_{\tau_1,\tau_1^*}(f^{-1}(B),\alpha,\beta) \le f^{-1}(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta)).$$

$(3)\Rightarrow(1)$ Suppose $B \in I^Y$ is an (α,β)-gfpc set. Then, $GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta) = B$. By (3),

$$GPC_{\tau_1,\tau_1^*}(f^{-1}(B),\alpha,\beta) \leq f^{-1}(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta))$$
$$= f^{-1}(B).$$

But

$$f^{-1}(B) \leq GPC_{\tau_1,\tau_1^*}(f^{-1}(B),\alpha,\beta).$$

Therefore,

$$f^{-1}(B) = GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta)$$

i.e., $f^{-1}(B) \in I^X$ is (α,β)-gfpc. Thus f is gdfpc-Irr map.

Proposition 4.1 *The map* $f:(X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ *between any two dfts's* (X,τ_1,τ_1^*) *and* (Y,τ_2,τ_2^*) *is a gdfp-closed iff for each* $A \in I^X$ *,*

$$GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta) \leq f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)).$$

Proof. Suppose that f is a gdfp-closed function and A is any fuzzy set in X. Then $f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta))$ is an (α,β)-gfpc in I^Y. Therefore,

$$GPC_{\tau_1,\tau_1^*}(f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)),\alpha,\beta) = f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)).$$

Since

$$A \leq GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta) \Rightarrow f(A) \leq f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)).$$

Therefore,

$$GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta) \leq GPC_{\tau_1,\tau_1^*}(f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)),\alpha,\beta)$$
$$= f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)).$$

Hence, for every fuzzy set $A \in I^X$,

$$GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta) \leq f(GPC_{\tau_1,\tau_1^*}(A),\alpha,\beta).$$

Conversely, suppose that for every fuzzy set $A \in I^X$, $GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta) \leq f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta))$. Since A is an (α,β)-gfpc set, we have

$$GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta) = A.$$

Therefore,

$$f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) = f(A) \leq GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

Hence,

$$f(A) = f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) = GPC_{\tau_1,\tau_1^*}(f(A),\alpha,\beta),$$

which implies that $f(A) \in I^Y$ is an (α,β)-gfpc set, i.e, f is gdfp-closed function.

Proposition 4.2 *For any two dfts's* (X,τ_1,τ_1^*) *and* (Y,τ_2,τ_2^*) *and any gdfpc-Irr function* $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$, $GPB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$ *is zero for every* (α,β)*-gfpo set* $A\in I^Y$.

Proof. Let A be an (α,β)-gfpo set in I^Y, $f^{-1}(A)\in I^X$ is (α,β)-gfpo. Therefore,

$$GPI_{\tau_2,\tau_2^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A).$$

By definition,

$$GPB_{\tau_2,\tau_2^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A) - GPI_{\tau_2,\tau_2^*}(f^{-1}(A),\alpha,\beta).$$

Hence,

$$GPB_{\tau_2,\tau_2^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A) - f^{-1}(A) = \underline{0}.$$

Definition 4.2 *A dfts* (X,τ,τ^*) *is said to be a double fuzzy pre-*$(\tau,\tau^*)_{1/2}$ *space (briefly, dfp-*$(\tau,\tau^*)_{1/2}$ *), if each* (α,β)*-fpc set is* (α,β)*-fc set in* X.

Proposition 4.3 *For any dfts* (X,τ_1,τ_1^*) *and dfp-*$(\tau,\tau^*)_{1/2}$ *space* (Y,τ_2,τ_2^*), *if the map* $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ *is a bijective, the next statements will be equivalent:*

(1) f and f^{-1} are gdfpc-Irr.

(2) f is gdfp-continuous and gdfp-open.

(3) f is gdfp-continuous and gdfp-closed.

(4) $f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) = GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)$, for every $A\in I^X$.

Proof. $(1)\Rightarrow(2)$ Suppose B is an (α,β)-gfpo set in X. Since f^{-1} is gdfpc-Irr, $(f^{-1})^{-1}(B)\in I^Y$ is (α,β)-gfpo set, so f is gdfp-open. Now, let $\nu\in I^Y$ be an (α,β)-fpo set, then it is an (α,β)-gfpo. But by hypothesis, f^{-1} are gdfpc-Irr, then $f^{-1}(\nu)\in I^X$ is an (α,β)-gfpo, i.e, f is gdfp-continuous.

$(2)\Rightarrow(3)$ Let $A\in I^X$ is an (α,β)-gfpc set, then $\underline{1}-A\in I^X$ is an (α,β)-gfpo set. By (2), $\underline{1}-f(A) = f(\underline{1}-A)$ is an (α,β)-gfpo set in Y, which implies that $f(A)$ is an (α,β)-gfpc set. Hence f is a gdfp-closed function.

$(3)\Rightarrow(4)$ Let $A\in I^X$, we have

$$A\le f^{-1}(f(A))$$

and

$$f(A)\le GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)\Rightarrow A\le f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

Now, $GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)\in I^Y$ is an (α,β)-gfpc set. But (Y,τ_2,τ_2^*) is a dfp-$(\tau,\tau^*)_{1/2}$ space, and $GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)$ is an (α,β)-fc set, then $GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)\in I^Y$ is an (α,β)-fpc set. Since f is gdfp-continuous, $f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta))$ is an (α,β)-gfpc set, which implies,

$$GPC_{\tau_2,\tau_2^*}(f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)),\alpha,\beta) = f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)).$$

But

$$GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta) \leq GPC_{\tau_2,\tau_2^*}(f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)),\alpha,\beta).$$

and

$$GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta) \leq f^{-1}(GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)).$$

Therefore,

$$f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) \leq GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

Also,

$$GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta) \leq f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta))$$
$$\Rightarrow f(GPC_{\tau_1,\tau_1^*}(A,\alpha,\beta)) = GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

$(4) \Rightarrow (1)$ Let $A \in I^X$, by hypothesis of (4),

$f(GPC_{\tau_1,\tau_1^*}(A),\alpha,\beta) = GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$

Therefore,

$$f(GPC_{\tau_1,\tau_1^*}(A),\alpha,\beta) \leq GPC_{\tau_2,\tau_2^*}(f(A),\alpha,\beta).$$

Then by Proposition 4.1, f is a gdfpc-Irr function. Now, suppose $B \in I^Y$ is (α,β)-gfpc. Then

$$GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta) = B \Rightarrow f(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta)) = f(B).$$

But, by (4),

$$GPC_{\tau_1,\tau_1^*}(f(B),\alpha,\beta) = f(GPC_{\tau_2,\tau_2^*}(B,\alpha,\beta)).$$

Therefore,

$$GPC_{\tau_1,\tau_1^*}(f(B),\alpha,\beta) = f(B),$$

then $f(B) \in I^Y$ is an (α,β)-gfpc set. Therefore, f^{-1} is a gdfpc-Irr.

5. INTERRELATIONS

In this section, we present the relationships among the concepts introduced in sections 3 and 4.

Proposition 5.1 *If* A *is an* (α,β)*-gfpc set in a dfts* (X,τ,τ^*) *then:*

(1) $GPB_{\tau,\tau^*}(A,\alpha,\beta) = GPF_{\tau,\tau^*}(A,\alpha,\beta)$.

(2) $GPE_{\tau,\tau^*}(A,\alpha,\beta) = \underline{1} - A$.

Proof. (1) Let $A \in I^X$ be an (α,β)-gfpc, we have

$$GPC_{\tau,\tau^*}(A,\alpha,\beta) = A.$$

But by definition,

$$\begin{aligned} GPB_{\tau,\tau^*}(A,\alpha,\beta) &= A - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPC_{\tau,\tau^*}(A,\alpha,\beta) - GPI_{\tau,\tau^*}(A,\alpha,\beta) \\ &= GPF_{\tau,\tau^*}(A,\alpha,\beta). \end{aligned}$$

Therefore,

$$GPB_{\tau,\tau^*}(A,\alpha,\beta) = GPF_{\tau,\tau^*}(A,\alpha,\beta)$$

(2) Let A be an (α,β)-gfpc set, we get

$$GPC_{\tau,\tau^*}(A,\alpha,\beta) = A \Rightarrow GPI_{\tau,\tau^*}(\underline{1}-A,\alpha,\beta) = \underline{1}-A$$

Therefore by definition,

$$GPE_{\tau,\tau^*}(A,\alpha,\beta) = \underline{1}-A.$$

Proposition 5.2 *For any two dfts's* (X,τ_1,τ_1^*) *and* (Y,τ_2,τ_2^*) *and any map* $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$.

(1) If f is any function, then $GPE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) \le GPC_{\tau_1,\tau_1^*}(\underline{1}-f^{-1}(A),\alpha,\beta)$, for each fuzzy set $A\in I^Y$.

(2) If f is a gdfp-continuous function, then for every (α,β)-fpc set $A\in I^Y$, we have $GPB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GPF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$.

Proof. (1) Let $A\in I^Y$. Then, by definition

$$\begin{aligned} GPE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) &= GPI_{\tau_1,\tau_1^*}(\underline{1}-f^{-1}(A),\alpha,\beta) \\ &\le \underline{1}-f^{-1}(A), \end{aligned}$$

Also

$$\begin{aligned} GPE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) &\le \underline{1}-GPI_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) \\ &= GPC_{\tau_1,\tau_1^*}(\underline{1}-f^{-1}(A),\alpha,\beta). \end{aligned}$$

Therefore

$$GPE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) \le GPC_{\tau_1,\tau_1^*}(\underline{1}-f^{-1}(A),\alpha,\beta).$$

(2) Let A be an (α,β)-fpc set in Y. Then, $f^{-1}(A)$ is an (α,β)-gfpc set in X. Therefore,

$$GPC_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A).$$

Hence,

$$GPF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GPC_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) - GPI_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$$
$$= f^{-1}(A) - GPI_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$$
$$= GPB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta).$$

Therefore,

$$GPF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GPB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta).$$

Definition 5.1 *A dfts (X,τ,τ^*) is said to be a g^* double fuzzy pre-$(\tau,\tau^*)_{1/2}$ space (briefly, g^*dfp-$(\tau,\tau^*)_{1/2}$), if each (α,β)-gfpc set in X is an (α,β)-gfc set.*

Proposition 5.3 *Let (X,τ,τ^*) be a g^*dfp-$(\tau,\tau^*)_{1/2}$ space and A be an (α,β)-gfpc set in X. Then the following statements hold:*

(1) $GB_{\tau,\tau^*}(A,\alpha,\beta) = GF_{\tau,\tau^*}(A,\alpha,\beta)$,

(2) $GE_{\tau,\tau^*}(A,\alpha,\beta) = \underline{1} - A$.

Proof. Let $A \in I^X$ be an (α,β)-gfpc set. Then A is an (α,β)-gfc set in X, which implies $GC_{\tau,\tau^*}(A,\alpha,\beta) = A$. But by definition,

$$GB_{\tau,\tau^*}(A,\alpha,\beta) = A - GI_{\tau,\tau^*}(A,\alpha,\beta)$$
$$= GC_{\tau,\tau^*}(A,\alpha,\beta) - GI_{\tau,\tau^*}(A,\alpha,\beta)$$
$$= GF_{\tau,\tau^*}(A,\alpha,\beta).$$

(2) By definition,

$$GE_{\tau,\tau^*}(A,\alpha,\beta) = GI_{\tau,\tau^*}(\underline{1} - A,\alpha,\beta) = \underline{1} - A.$$

Proposition 5.4 *Let (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*) be dfts's and let $f:(X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ be gdfpc-Irr function such that (X,τ_1,τ_1^*) is g^*dfp-$(\tau,\tau^*)_{1/2}$ space. Then, for every an (α,β)-gfpc set $A \in I^Y$, the following statements hold:*

(1) $GB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$.

(2) $GE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = \underline{1} - f^{-1}(A)$.

Proof. Suppose $A \in I^Y$ is an (α,β)-gfpc set. Then, $f^{-1}(A)$ is an (α,β)-gfpc set in X. Since by hypothesis, (X,τ_1,τ_1^*) is g^*dfp-$(\tau,\tau^*)_{1/2}$ space, $f^{-1}(A)$ is an (α,β)-gfc set in X. Therefore,

$$GC_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A).$$

Also,

$$GB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = f^{-1}(A) - GI_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$$
$$= GC_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) - GI_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$$
$$= GF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta)$$

Therefore

$$GB_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GF_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta).$$

(2) By definition,

$$GE_{\tau_1,\tau_1^*}(f^{-1}(A),\alpha,\beta) = GI_{\tau_1,\tau_1^*}(\underline{1} - f^{-1}(A),\alpha,\beta) = \underline{1} - f^{-1}(A).$$

The above statement is not true if (X,τ_1,τ_1^*) is not $g^*dfp\text{-}(\tau,\tau^*)_{1/2}$ space, as shown in the following Example.

Example 5.1 *Let* $X = \{a,b\}$ *and* $f:(X,\tau_1,\tau_1^*) \to (X,\tau_2,\tau_2^*)$ *be the identity map. Define* A_1, A_2, A_3, B_1 *and* B_2 *as follows:*

$$A_1(a) = 0.67, A_1(b) = 0.64,$$
$$A_2(a) = 0.67, A_2(b) = 0.35,$$
$$A_3(a) = 0.33, A_3(b) = 0.34,$$
$$B_1(a) = 0.75, B_1(b) = 0.67,$$
$$B_2(a) = 0.67, B_2(b) = 0.49.$$

and define (τ_1,τ_1^*) and (τ_2,τ_2^*) as follows:

$$\tau_1(A) = \begin{cases} 1, & if \quad A \in \{\underline{0},\underline{1}\}, \\ \frac{1}{4}, & if \quad A = A_1, \\ \frac{1}{2}, & if \quad A = A_2, \\ \frac{3}{4}, & if \quad A = A_3, \\ 0, & otherwise. \end{cases} \qquad \tau_1^*(A) = \begin{cases} 0, & if \quad A \in \{\underline{0},\underline{1}\}, \\ \frac{3}{4}, & if \quad A = A_1, \\ \frac{1}{2}, & if \quad A = A_2, \\ \frac{1}{4}, & if \quad A = A_3, \\ 1, & otherwise. \end{cases}$$

and

$$\tau_2(A) = \begin{cases} 1, & if \quad A \in \{\underline{0},\underline{1}\}, \\ \frac{1}{4}, & if \quad A = B_1, \\ \frac{1}{8}, & if \quad A = B_2, \\ 0, & otherwise. \end{cases} \qquad \tau_2^*(A) = \begin{cases} 0, & if \quad A \in \{\underline{0},\underline{1}\} \\ \frac{1}{8}, & if \quad A = B_1, \\ \frac{1}{4}, & if \quad A = B_2, \\ 1, & otherwise. \end{cases}$$

Then, A is an (α,β)-gfpc set in (X,τ_2,τ_2^*) and $f^{-1}(A) = A$ is an (α,β)-gfpc set which is not (α,β)-gfc set in (X,τ_1,τ_1^*). Therefore, f is gdfpc-Irr function, but (X,τ_1,τ_1^*) is not $g^*dfp\text{-}(\tau,\tau^*)_{1/2}$ space.

Proposition 5.5 *Let $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ be gdfp-closed map such that (Y,τ_2,τ_2^*) is g^*dfp-$(\tau,\tau^*)_{1/2}$ space. Then, for each an (α,β)-gfpc set A in I^X, the following statements hold:*

(1) $GB_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)=GF_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)$.

(2) $GE_{\tau_2,\tau_2^*}(f(A),\alpha,\beta)=\underline{1}-f(A)$.

Proof. Similar to the proof of Proposition 5.4.

The above statement is not true if (Y,τ_2,τ_2^*) is not g^*dfp-$(\tau,\tau^*)_{1/2}$ space, as shown in the following Example.

Example 5.2 *Let $X=\{a,b\}$ and $f:(X,\tau_1,\tau_1^*)\to(X,\tau_2,\tau_2^*)$ be the identity function. Define A_1, A_2, A_3, B_1 and B_2 are as in Example 5.1 and define (τ_1,τ_1^*) and (τ_2,τ_2^*) as follows:*

$$\tau_1(A)=\begin{cases}1, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & if \quad A=B_1,\\ \frac{1}{8}, & if \quad A=B_2,\\ 0, & otherwise.\end{cases} \qquad \tau_1^*(A)=\begin{cases}0, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{8}, & if \quad A=B_1,\\ \frac{1}{4}, & if \quad A=B_2,\\ 1, & otherwise.\end{cases}$$

and

$$\tau_2(A)=\begin{cases}1, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & if \quad A=A_1,\\ \frac{1}{2}, & if \quad A=A_2,\\ \frac{3}{4}, & if \quad A=A_2,\\ 0, & otherwise.\end{cases} \qquad \tau_2^*(A)=\begin{cases}0, & if \quad A\in\{\underline{0},\underline{1}\}\\ \frac{3}{4}, & if \quad A=A_1,\\ \frac{1}{2}, & if \quad A=A_2,\\ \frac{1}{4}, & if \quad A=A_3,\\ 1, & otherwise.\end{cases}$$

Then, A which is as in Example 5.1 is an (α,β)-gdfp-closed in (X,τ_1,τ_1^*). Therefore, f is a gdfp-closed function, but $f(A)=A$ is not (α,β)-gfc set in (X,τ_2,τ_2^*). Hence (X,τ_2,τ_2^*) is not a g^*dfp-$(\tau,\tau^*)_{1/2}$ space.

Proposition 5.6 *Let (X,τ_1,τ_1^*), (Y,τ_2,τ_2^*) and (Z,τ_3,τ_3^*) be dfts's, $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ and $g:(Y,\tau_2,\tau_2^*)\to(Z,\tau_3,\tau_3^*)$ be gdfpc-Irr functions such that (X,τ_1,τ_1^*) is g^*dfp-$(\tau,\tau^*)_{1/2}$ space. Then, for each an (α,β)-gfpc set in Z, the following statements hold:*

(1) $GB_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)=GF_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)$.

(2) $GE_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)=\underline{1}-(g\circ f)^{-1}(A)$.

Proof. (1) Let A be an (α,β)-gfpc set in Z. Then by hypothesis of g is gdfpc-Irr, $g^{-1}(A)\in I^Y$ is an (α,β)-gfpc set. Also, f is gdfpc-Irr, so $f^{-1}(g^{-1}(A))\in I^X$ is (α,β)-gfpc set. Thus $(g\circ f)^{-1}(A)\in I^X$ is an (α,β)-gfpc. Since (X,τ_1,τ_1^*) is g^*dfp-$(\tau,\tau^*)_{1/2}$, $(g\circ f)^{-1}(A)$ is an (α,β)-gfc in X. So by definition, we have

$$GB_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta) = (g\circ f)^{-1}(A) - GI_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)$$
$$= GC_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta) - GI_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)$$
$$= GF_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta).$$

(2) By definition,

$$GE_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta) = GI_{\tau_1,\tau_1^*}(\underline{1} - (g\circ f)^{-1}(A),\alpha,\beta)$$
$$= \underline{1} - GC_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta)$$
$$= \underline{1} - (g\circ f)^{-1}(A),$$

since $(g\circ f)^{-1})(A)$ is an (α,β)-gfc set. Therefore,

$$GE_{\tau_1,\tau_1^*}((g\circ f)^{-1}(A),\alpha,\beta) = \underline{1} - (g\circ f)^{-1}(A).$$

The above statement is not true if (X,τ_1,τ_1^*) is not a g^*dfp-$(\tau,\tau^*)_{1/2}$ space, as shown in the following Example.

Example 5.3 *Let* $X=\{a,b\}$, $f:(X,\tau_1,\tau_1^*)\to(X,\tau_2,\tau_2^*)$ *and* $g:(X,\tau_2,\tau_2^*)\to(X,\tau_3,\tau_3^*)$ *be the identity functions. Define* A_1, A_2, A_3, B_1 *and* B_2 *as in Example 5.1 and* δ_1, δ_1 *as follows:*

$$\delta_1(a)=0.75, \delta_1(b)=0.75,$$
$$\delta_2(a)=0.67, \delta_2(b)=0.40.$$

also, define (τ_1,τ_1^*) and (τ_2,τ_2^*) as follows:

$$\tau_1(A)=\begin{cases}1, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & if \quad A=A_1,\\ \frac{1}{2}, & if \quad A=A_2,\\ \frac{3}{4}, & if \quad A=A_3,\\ 0, & otherwise.\end{cases} \qquad \tau_1^*(A)=\begin{cases}0, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{3}{4}, & if \quad A=A_1,\\ \frac{1}{2}, & if \quad A=A_2,\\ \frac{1}{4}, & if \quad A=A_3,\\ 1, & otherwise.\end{cases}$$

,

$$\tau_2(A)=\begin{cases}1, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & if \quad A=B_1,\\ \frac{1}{8}, & if \quad A=B_2,\\ 0, & otherwise.\end{cases} \qquad \tau_2^*(A)=\begin{cases}0, & if \quad A\in\{\underline{0},\underline{1}\}\\ \frac{1}{8}, & if \quad A=B_1,\\ \frac{1}{4}, & if \quad A=B_2,\\ 1, & otherwise.\end{cases}$$

and

$$\tau_3(A)=\begin{cases}1, & if \quad A\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & if \quad A=\delta_1,\\ \frac{1}{8}, & if \quad A=\delta_2,\\ 0, & otherwise.\end{cases} \qquad \tau_3^*(A)=\begin{cases}0, & if \quad A\in\{\underline{0},\underline{1}\}\\ \frac{1}{8}, & if \quad A=\delta_1,\\ \frac{1}{4}, & if \quad A=\delta_2,\\ 1, & otherwise.\end{cases}$$

Then, f and g are gdfpc-Irr function, but (X,τ_1,τ_1^*) is not g^*dfp-$(\tau,\tau^*)_{1/2}$ space as $(g\circ f)^{-1}(A)$ is an (α,β)-gfpc set but not an (α,β)-gfc set.

6. CONCLUSION

In this article, we studied (α,β)-generalised fuzzy pre-border, (α,β) -generalised fuzzy pre-exterior and (α,β)-generalised fuzzy pre-frontier in double fuzzy topological spaces. Also some characteristic properties of generalised double fuzzy pre-continuous, generalised double fuzzy preopen, generalised double fuzzy preclosed, and generalised double fuzzy preclosure-irresolute functions are studied and investigate. We also introduced the relationships between these new notions with the other already defined notions. We wish the outcomes of this paper will encourage the mathematicians to conclude more important relationships between topology and fuzzy theory.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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