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On Absolute Matrix Summability Factors of Infinite Series and Fourier Series

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Abstract

In this paper, a known theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series have been generalized for $|A, \theta_n|_k$ summability factors. Using this theorem, some new results dealing with Fourier series have been obtained.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \quad n \to \infty, \; (P_{-i} = p_{-i} = 0, \; i \geq 1). \tag{1}$$

The sequence –to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (T_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ (resp. $|\bar{N}, p_n|$) summability.

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Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{4}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t). \tag{5}$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}, \quad \phi_1(t) = \frac{1}{t}\int_0^t \phi(u)du. \tag{6}$$

Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\overline{A}=(\overline{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v=0,1,\ldots \quad \overline{\Delta} a_{nv} = a_{nv} - a_{n-1,v}, \quad a_{-1,0} = 0 \tag{7}$$

and

$$\hat{a}_{00} = \overline{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \overline{\Delta}\overline{a}_{nv}, \quad n = 1,2,\ldots \tag{8}$$

It may be noted that $\overline{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \overline{a}_{nv} a_v \tag{9}$$

$$\overline{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{10}$$

Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A,\theta_n|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} \theta_n^{\,k-1} |\overline{\Delta} A_n(s)|^k < \infty \tag{11}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Remark. If we take $\theta_n = \frac{P_n}{p_n}$, then $|A,\theta_n|_k$ summability reduces to $|A, p_n|_k$ summability (see [11]). Also, if we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $\left|\bar{N}, p_n\right|_k$ summability. Furthermore, if we take $\theta_n = n,\ a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A,\theta_n|_k$ reduces to $\left|C,1\right|_k$ summability (see [6]). Finally, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [3]).

2.THE KNOWN RESULTS

The following theorems are known dealing with Fourier series (see [2]).

Theorem 2.1. Let (p_n) be a sequence of positive numbers such that

$$P_n = O(n\,p_n) \text{ as } n \to \infty, \tag{12}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{13}$$

If $\phi_1(t)$ is of bounded variation in $(0,\pi)$ and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k < \infty \tag{14}$$

and

$$\sum_{n=1}^{\infty} \left|\Delta\lambda_n\right| < \infty, \tag{15}$$

then the series $\sum C_n(t)\frac{\lambda_n P_n}{np_n}$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem 2.2. If the sequences (p_n) and (λ_n) satisfy the conditions (12)-(15) of Theorem 2.1 and

$$B_n \equiv \sum_{v=1}^{n} v a_v = O(n), \quad n \to \infty, \tag{16}$$

then the series $\sum a_n \frac{\lambda_n P_n}{np_n}$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

3. THE MAIN RESULTS

Many studies have been done for Riesz summability and matrix generalization of infinite series and Fourier series (see [4], [5], [9], [12]). The aim of this paper is to generalize Theorem 2.1 and Theorem 2.2 under suitable and different conditions using general summability factors for $|A,\theta_n|_k$ summability methods.

Now, we shall prove the following theorems.

Theorem 3.1. Let $A=(a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0}=1, \quad n=0,1,..., \tag{17}$$

$$a_{n-1,v}\geq a_{nv}, \text{ for } n\geq v+1, \tag{18}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{19}$$

$$\hat{a}_{n,v+1}=O(v|\bar{\Delta}a_{nv}|). \tag{20}$$

Let $\phi_1(t)$ be of bounded variation in $(0,\pi)$ and $(\theta_n a_{nn})$ be a non-increasing sequence. If the conditions (12), (13), (15) of Theorem 2.1 are satisfied and (θ_n) is any sequence of positive constants such that

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\frac{|\lambda_n|^k}{n^k}<\infty, \tag{21}$$

then the series $\sum C_n(t)\frac{\lambda_n P_n}{np_n}$ is summable $|A,\theta_n|_k, k\geq 1.$

Theorem 3.2. If the conditions (12), (13) and (15-21) are satisfied and $(\theta_n a_{nn})$ is a non increasing sequence, then the series $\sum a_n\frac{\lambda_n P_n}{np_n}$ is summable $|A,\theta_n|_k, k\geq 1.$

Remark. It should be noted that in the above theorems, if we take $\theta_n=\frac{P_n}{p_n}$ and $a_{nv}=\frac{p_v}{P_n}$, then we get Theorem 2.1 and Theorem 2.2. In this case, condition (21) reduces to condition (14).

We need the following lemmas for the proof of our theorems.

Lemma 3.3 [8] If $\phi_1(t)$ is of bounded variation in $(0,\pi)$, then

$$\sum_{n=1}^{\infty}vC_v(x)=O(n) \quad as \quad n\to\infty. \tag{22}$$

Lemma 3.4 [2] If the sequence (p_n) such that conditions (12) and (13) of Theorem 2.1 are satisfied, then

$$\Delta\left\{\frac{P_n}{p_n n^2}\right\}=O\left(\frac{1}{n^2}\right). \tag{23}$$

4. PROOF OF THEOREM 3.2.

Let (I_n) denotes the A-transform of the series $\sum a_n P_n\lambda_n(np_n)^{-1}$. Then, by (9) and (10), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1}. \tag{24}$$

Applying Abel's transformation to this sum, we get that

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1}$$

$$= \sum_{v=1}^{n-1} \Delta\left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r.a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r.a_r$$

$$= \left\{ \sum_{v=1}^{n-1} \frac{\bar{\Delta} a_{nv} P_v \lambda_v}{v^2 p_v} + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{v^2 p_v} \Delta\lambda_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\left(\frac{P_v}{v^2 p_v}\right) \right\} \sum_{r=1}^{v} r.a_r + \frac{a_{nn} \lambda_n P_n}{n^2 p_n} \sum_{r=1}^{n} r.a_r$$

$$= \frac{a_{nn} \lambda_n P_n}{n^2 p_n} B_n + \sum_{v=1}^{n-1} \frac{\bar{\Delta} a_{nv} P_v \lambda_v}{v^2 p_v} B_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\left(\frac{P_v}{v^2 p_v}\right) B_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{v^2 p_v} \Delta\lambda_v B_v$$

$$= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.$$

To complete the proof of Theorem 3.2, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} | I_{n,r} |^k < \infty, \text{ for } r = 1, 2, 3, 4. \tag{25}$$

Firstly, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} \left| I_{n,1} \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| \frac{a_{nn} \lambda_n P_n}{n^2 p_n} B_n \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} | \lambda_n |^k | B_n |^k \frac{1}{n^{2k}}$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} \frac{|\lambda_n|^k}{n^k} = O(1) \ \ as \ m \to \infty,$$

by virtue of the hypotheses of Theorem 3.2. Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} | I_{n,2} |^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \bar{\Delta} a_{nv} \frac{P_v \lambda_v}{v^2 p_v} B_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} \bar{\Delta} a_{nv} \left(\frac{P_v}{v^2 p_v}\right)^k | \lambda_v |^k | B_v |^k \right\} \times \left\{ \sum_{v=1}^{n-1} | \bar{\Delta} a_{nv} | \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} | \bar{\Delta} a_{nv} | \left(\frac{P_v}{v^2 p_v}\right)^k | \lambda_v |^k | B_v |^k$$

$$=O(1)\sum_{v=1}^{m}|\lambda_v|^k|B_v|^k\left(\frac{P_v}{v^2p_v}\right)^k\sum_{n=v+1}^{m+1}\left(\theta_n a_{nn}\right)^{k-1}|\bar{\Delta}a_{nv}|$$

$$=O(1)\sum_{v=1}^{m}\left(\theta_v a_{vv}\right)^{k-1}|\lambda_v|^k|B_v|^k\left(\frac{P_v}{v^2p_v}\right)^k\sum_{n=v+1}^{m+1}|\bar{\Delta}a_{nv}|$$

$$=O(1)\sum_{v=1}^{m}\left(\theta_v a_{vv}\right)^{k-1}\frac{|\lambda_v|^k}{v^k}\left(\frac{P_v}{p_v}\right)^{k-1}=O(1)\sum_{v=1}^{m}\theta_v^{\,k-1}\frac{|\lambda_v|^k}{v^k}=O(1)\text{ as } m\to\infty,$$

by virtue of the hypotheses of Theorem 3.2. On the other hand, since $\Delta\left\{\frac{P_v}{v^2p_v}\right\}=O\left(\frac{1}{v^2}\right)$ by Lemma 3.4, we obtain

$$\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left|I_{n,3}\right|^k=\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left|\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\Delta\left(\frac{P_v}{v^2p_v}\right)B_v\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{1}{v}\right\}\times\left\{\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}|}{v}\right\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{1}{v}\right\}\times\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\right\}^{k-1}$$

$$=O(1)\sum_{v=1}^{m}\left|\lambda_{v+1}\right|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\left(\theta_n a_{nn}\right)^{k-1}\left|\hat{a}_{n,v+1}\right|=O(1)\sum_{v=1}^{m}\left(\theta_v a_{vv}\right)^{k-1}\left|\lambda_{v+1}\right|^k\frac{1}{v+1}\left(1+\frac{1}{v}\right)$$

$$=O(1)\left(\theta_1 a_{11}\right)^{k-1}\sum_{v=1}^{m}\left|\lambda_{v+1}\right|^k\frac{1}{v}=O(1)\ \ as\ \ m\to\infty,$$

by virtue of the hypotheses of Theorem 3.2 and Lemma 3.3. Finally, we get

$$\sum_{n=2}^{m+1}\theta_n^{\,k-1}|I_{n,4}|^k=\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}P_v}{v^2p_v}\Delta\lambda_v B_v\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}}{v}\Delta\lambda_v B_v\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{\,k-1}\left\{\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\frac{\left|B_v\right|^k}{v^k}\right\}\times\left\{\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\right\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{\,k-1}a_{nn}^{\,k-1}\left\{\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\frac{\left|B_v\right|^k}{v^k}\right\}\times\left\{\sum_{v=1}^{n-1}\left|\Delta\lambda_v\right|\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|\frac{|B_v|^k}{v^k}\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|\Delta\lambda_v|\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|$$

$$= O(1)\left(\theta_1 a_{11}\right)^{k-1}\sum_{v=1}^{m}|\Delta\lambda_v| = O(1) \text{ as } m\to\infty,$$

by virtue of the hypotheses of Theorem 3.2.

This completes the proof of Theorem 3.2.

Proof of Theorem 3.1. Theorem 3.1 is a direct consequence of Theorem 3.2 and Lemma 3.3.

5. CONCLUSIONS

If we take $\theta_n = \frac{P_n}{p_n}$ in Theorem 3.1 and Theorem 3.2, then we get two theorems dealing with $|A, p_n|_k$ summability (see [13]). Also, if we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.1 and Theorem 2.2. Additionally, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a theorem dealing with $|R, p_n|_k$ summability. Finally, if we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C,1|_k$ summability.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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