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# Principal Functions of Boundary-Value Problem with Quadratic Spectral Parameter in Boundary Conditions

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**Abstract**

In this paper, we determine the principal functions corresponding to the eigenvalues and the spectral singularities of the boundary value problem (BVP)

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ = [0,\infty]$$

$$(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y'(0) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y(0) = 0,$$

where $q$ is a complex-valued function, $\alpha_i$, $\beta_i \in \mathbb{C}$, $i = 0,1,2$ and $\lambda$ is a eigenparameter, and introduce the convergence properties of principal functions.

## 1. INTRODUCTION

The spectral analysis of a non-self-adjoint Sturm-Liouville equation (SLE) with continuous and discrete spectrum was investigated by Naimark $[1]$. He proved the existence of the spectral singularities in the continuous spectrum of SLE. Lyance showed that the spectral singularities play an important role in the spectral theory of SLE $[2]$. He also studied the effect of the spectral singularities in the spectral expansion of SLE in terms of the principal functions. The spectral analysis of the non-selfadjoint operators with purely dicrete spectrum has been considered by Keldysh$[3,4]$. He studied the spectrum and principal functions of operators involving a polynomial dependent on the spectral parameter, and also showed the completeness of the principal functions of these operators in Hilbert space. Some problems of the spectral analysis of a non-selfadjoint Schrödinger, Dirac and Klein-Gordon differential and difference equations with spectral singularities were studied in $[5-13]$. The spectral analysis of quadratic eigenparameter dependent non-selfadjoint Sturm Liouville equation has been studied in $[14-15]$.

Let us consider the following BVP

$$-y'' + q(x)y = \lambda^2 y, \; x \in \mathbb{R}_+, \tag{1.1}$$

$$(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y'(0) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y(0) = 0, \tag{1.2}$$

where $q$ is complex-valued function also absolutely continuous in each finite subinterval of $\mathbb{R}_+$ and $\alpha_i$, $\beta_i \in \mathbb{C}$, $i = 0,1,2$, with $|\alpha_2| + |\beta_2| \neq 0$. It is clear that, the BVP $(1.1)$ and $(1.2)$ is non-selfadjoint.

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Differently other studies in the literature, the specific feature of this paper which is one of the articles have applicability in study areas such as physics, engineering, mathematics is the presence of the spectral parameter not only in the Sturm-Liouville equation but also in the boundary condition at quadratic form.

In this paper, which is extention of $\left[15\right]$, we aim to determine of the principal functions corresponding to eigenvalues and spectral singularities of the BVP $(1.1)-(1.2)$ and investigate of their convergence properties.

## 2. DISCRETE SPECTRUM OF (1.1) AND (1.2)

We consider the equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ \tag{2.1}$$

related to the operator $L$ .

The complex-valued function $q$ is assumed to satisfy the condition

$$\int_0^\infty x\left|q(x)\right|dx < \infty. \tag{2.2}$$

Let $\varphi(x,\lambda)$ and $e(x,\lambda)$ denote the solutions of (2.1) subject to the conditions

$$\varphi(0,\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2, \quad \varphi'(0,\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2,$$

$$\lim_{x\to\infty} e(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+,$$

respectively. The solution $e(x,\lambda)$ is called the Jost solution of $(2.1)$. Therefore, under the condition $(2.2)$, the solution $\varphi(x,\lambda)$ is an entire function of $\lambda$ and the Jost solution is an analytic function of $\lambda$ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and continuous in $\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$ .

In addition, Jost solution has the following representation:

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.3}$$

where the kernel $K(x,t)$ is expressed in terms of $q$ , and is continuously differentiable with respect to its arguments.

On the other hand, $K(x,t)$ satisfies

$$\left|K(x,t)\right| \leq c\sigma(\frac{x+t}{2}), \tag{2.4}$$

$$\left|K_x(x,t)\right|, \left|K_t(x,t)\right| \leq \frac{1}{4}\left|q(\frac{x+t}{2})\right| + c\sigma(\frac{x+t}{2}), \tag{2.5}$$

where $c > 0$ is a constant and $\sigma(x) = \int_x^\infty \left|q(s)\right|ds$ .

Let $e^{\pm}(x,\lambda)$ denote the solutions of (2.1) subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\lambda x} e^{\pm}(x,\lambda) = 1, \quad \lim_{x\to\infty} e^{\pm i\lambda x} e_x^{\pm}(x,\lambda) = \pm i\lambda, \quad \lambda \in \overline{\mathbb{C}}_{\pm}.$$

Then

$$\begin{gathered} W\left[e(x,\lambda), e^{\pm}(x,\lambda)\right] = \mp 2i\lambda, \quad \lambda \in \mathbb{C}_{\pm} \\ W[e(x,\lambda), e(x,-\lambda)] = -2i\lambda, \quad \lambda \in \mathbb{R} = \ -\infty,\infty \end{gathered} \tag{2.6}$$

where $W\left[f_1, f_2\right]$ is the Wronskian of $f_1$ and $f_2$ .

Let us define the following functions:

$$\begin{gathered} N^{+}(\lambda) := (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e^{'}(0,\lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)e(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_{+}, \\ N^{-}(\lambda) := (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e^{'}(0,-\lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)e(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_{-}, \end{gathered} \tag{2.7}$$

where $\overline{\mathbb{C}}_{-} = \ \lambda \ : \ \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \le 0$ . It is obvious that the functions $N^{+}$ and $N^{-}$ are analytic in $\mathbb{C}_{+}$ and $\mathbb{C}_{-} = \ \lambda \ : \ \lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0$ , respectively, and continuous on the real axis. The functions $N^{+}$ and $N^{-}$ are called Jost functions of $L$ .

The resolvent of $L$ defined by

$$R_{\lambda}(L)f = \int_0^{\infty} R(x,t;\lambda) f(t)\,dt, \quad f \in L_2(\mathbb{R}_{+})$$

where $R(x,t;\lambda)$ is Green's function given by

$$R(x,t;\lambda) = \begin{cases} R^{+}(x,t;\lambda), & \lambda \in \mathbb{C}_{+} \\ R^{-}(x,t;\lambda), & \lambda \in \mathbb{C}_{-} \end{cases} \tag{2.8}$$

and

$$\begin{gathered} R^{+}(x,t;\lambda) = \begin{cases} -\frac{\varphi(t,\lambda)e(x,\lambda)}{N^{+}(\lambda)}, & 0 \le t \le x \\ -\frac{\varphi(x,\lambda)e(t,\lambda)}{N^{+}(\lambda)}, & x \le t < \infty \end{cases} \\ R^{-}(x,t;\lambda) = \begin{cases} -\frac{\varphi(t,\lambda)e(x,-\lambda)}{N^{-}(\lambda)}, & 0 \le t \le x \\ -\frac{\varphi(x,\lambda)e(t,-\lambda)}{N^{-}(\lambda)}, & x \le t < \infty. \end{cases} \end{gathered} \tag{2.9}$$

in which $(2.7)$ is taking into account.

We denote the set of eigenvalues and spectral singularities of $L$ by $\sigma_d(L)$ and $\sigma_{ss}(L)$ , respectively. From the definition of the eigenvalues and spectral singularities, we have $\left[17\right]$

$$\begin{gathered} \sigma_d(L) = \ \lambda \ : \ \lambda \in \mathbb{C}_{+}, N^{+}(\lambda) = 0 \ \cup \ \lambda \ : \ \lambda \in \mathbb{C}_{-}, N^{-}(\lambda) = 0 \ , \\ \sigma_{ss}(L) = \ \lambda \ : \ \lambda \in \mathbb{R}^{*}, N^{+}(\lambda) = 0 \ \cup \ \lambda \ : \ \lambda \in \mathbb{R}^{*}, N^{-}(\lambda) = 0 \ , \end{gathered} \tag{2.10}$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

**Definition 1.** The multiplicity of a zero of $N^+$( or $N^-$) in $\overline{\mathbb{C}}_+$ ( or $\overline{\mathbb{C}}_-$) is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP $(1.2)$ and (1.3).

*We see from (2.6) that the functions* $\psi^{\pm}(x,\lambda)$ *and* $\psi(x,\lambda)$, *defined by*

$$\psi^+(x,\lambda) = \frac{\tilde{N}^+(\lambda)}{2i\lambda} e(x,\lambda) - \frac{N^+(\lambda)}{2i\lambda} e^+(x,\lambda), \quad \lambda \in \mathbb{C}_+$$

$$\psi^-(x,\lambda) = \frac{\tilde{N}^-(\lambda)}{2i\lambda} e(x,-\lambda) - \frac{N^-(\lambda)}{2i\lambda} e^-(x,\lambda), \quad \lambda \in \mathbb{C}_-$$

$$\psi(x,\lambda) = \frac{N^+(\lambda)}{2i\lambda} e(x,-\lambda) - \frac{N^-(\lambda)}{2i\lambda} e(x,\lambda), \quad \lambda \in \mathbb{R}^*$$

*are the solutions of the boundary value problem* $(1.2)-(1.3)$ *where*

$$\tilde{N}^{\pm}(\lambda) = (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) e^{\pm'}(0,\lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2) e^{\pm}(0,\lambda).$$

*Now let us assume that*

$$q, q' \in AC(\mathbb{R}_+), \quad \lim_{x\to\infty} \left|q(x)\right| + \left|q'(x)\right| = 0, \quad \sup_{x\in\mathbb{R}_+} \left[e^{\varepsilon\sqrt{x}} \left|q''(x)\right|\right] < \infty, \quad \varepsilon > 0. \quad (2.11)$$

**Theorem 1.** ([15]). Under the condition (2.11) the operator $L$ has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

## 3. THE PRINCIPAL FUNCTIONS OF (1.1) AND (1.2)

In this section, we determine the principal vectors of the operator $L$ corresponding to its eigenvalues and spectral singularities. We start with the following definition.

**Definition 2.** Let $\lambda = \lambda_0$ be an eigenvalue of L . If the functions

$$y_0(x,\lambda_0), y_1(x,\lambda_0), \ldots, y_s(x,\lambda_0)$$

*satisfy the equations*

$$\left[-\frac{d^2}{dx^2} + q(x) - \lambda_0\right] y_0(x,\lambda_0) = 0$$

$$\left[-\frac{d^2}{dx^2} + q(x) - \lambda_0\right] y_n(x,\lambda_0) - y_{n-1}(x,\lambda_0) = 0$$

for $n = 0,,1,\ldots,s,$ *then the function* $y_0(x,\lambda_0)$ *is said to be the eigenfunction corresponding to the eigenvalue* $\lambda = \lambda_0$ *of* $L.$ *The functions* $y_1(x,\lambda_0),\ldots,y_s(x,\lambda_0)$ *are called associated functions corresponding to the eigenvalue* $\lambda = \lambda_0$. *The eigenfunctions and associated functions corresponding to* $\lambda = \lambda_0$ *are called the principal functions of the eigenvalue* $\lambda = \lambda_0$. *The principal functions corresponding to the spectral singularities are defined similarly.*

Henceforth, we assume that the condition $(2.11)$ holds. Let $\lambda_1,\dots,\lambda_j$ and $\lambda_{j+1},\dots,\lambda_k$ denote the zeros of the functions $N^+$ in $\mathbb{C}_+$ and $N^-$ in $\mathbb{C}_-$ (which are the eigenvalues of $L$) with multiplicities $m_1,\dots,m_j$ and $m_{j+1},\dots,m_k$, respectively. It is obvious that

$$\left\{\frac{d^n}{d\lambda^n}W\left[\psi^+(x,\lambda),e(x,\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^n}{d\lambda^n}N^+(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{3.1}$$

for $n=0,1,\dots,m_p-1,\;\; p=1,2,\dots,j,$ and

$$\left\{\frac{d^n}{d\lambda^n}W\left[\psi^-(x,\lambda),e(x,-\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^\nu}{d\lambda^\nu}N^-(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{3.2}$$

for $n=0,1,\dots,m_p-1,\;\; p=j+1,\dots,k.$

**Theorem 2.** The formula

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}\binom{n}{k}a_{n-k}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{3.3}$$

for $n=0,1,\dots,m_p-1,\;\; p=1,2,\dots,j$, and

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}\binom{n}{k}b_{n-k}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{3.4}$$

for $n=0,1,\dots,m_p-1,\;\; p=j+1,\dots,k,$ hold, where the constants $a_o,a_1,\dots,a_n$ and $b_o,b_1,\dots,b_n$ depend on $\lambda_p$, respectively.

Proof. We will prove only (3.3) using the mathematical induction, because the case of (3.4) is similar. Let $n=0.$ It is evident from (3.1) that

$$\psi^+(x,\lambda_p)=a_0(\lambda_p).e(x,\lambda_p)$$

for $p=1,2,\dots,j$. Suppose that (3.3) holds for an arbitrary integer $n_0$ such that $1\le n_0\le m_p-2.$ Differentiating the equation

$$\left[-\frac{d^2}{dx^2}+q\;\;x\;\;-\lambda^2\right]y\;\;x,\lambda\;\;=0 \tag{3.5}$$

with respect to $\lambda$ and substituting $\lambda=\lambda_p$, we have

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda_p\right]\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}y(x,\lambda_p)=2\lambda_p\;\;n_0+1\;\;\frac{\partial^{n_0}}{\partial\lambda^{n_0}}y(x,\lambda_p)+n_0(n_0+1)\frac{\partial^{n_0-1}}{\partial\lambda^{n_0-1}}y(x,\lambda_p). \tag{3.6}$$

As $\psi^+(x,\lambda)$ and $e(x,\lambda)$ solve the equation (3.5), we obtain by (3.3), (3.5)-(3.6) that

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda^2\right]g_{n_0}(x,\lambda_p)=0,\ for\ p=1,2,\ldots,j,$$

where

$$g_{n_0}(x,\lambda_p)=\left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_p}-\sum_{k=0}^{n_0+1}\binom{n_0+1}{k}a_{n_0+1-k}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{3.7}$$

On the other hand, $(3.1)$ and $(3.7)$ imply that

$$W\left[g_{n_0}(x,\lambda),e(x,\lambda)\right]_{\lambda=\lambda_p}=\frac{d^{n_0+1}}{d\lambda^{n_0+1}}W\left[\psi^+(x,\lambda),e(x,\lambda)\right]_{\lambda=\lambda_p}=0,$$

and therefore,

$$g_{n_0}(x,\lambda_p)=a_{n_0+1}(\lambda_p)e(x,\lambda_p)$$

for $p=1,2,\ldots,j.$ The proof is complete.

Using the notations $$A_{n-k}\ \lambda_p\ =\frac{a_{n-k}\ \lambda_p}{n-k\ !}$$

and

$$B_{v-j}\ \lambda_p\ =\frac{b_{v-j}\ \lambda_p}{v-j\ !}$$

we can write $3.3$ and $3.4$ as

$$\frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}A_{n-k}\ \lambda_p\ \frac{1}{k!}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,\lambda)\right\}_{\lambda=\lambda_p},$$

$n=0,1,\ldots,m_p-1,\ \ p=1,2,\ldots,j,$

$$\frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}B_{n-k}\ \lambda_p\ \frac{1}{k!}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,-\lambda)\right\}_{\lambda=\lambda_p},$$

$n=0,1,\ldots,m_p-1,\ \ p=j+1,\ldots,k.$

Now we introduce the functions

$$U_{n,p}(x,\lambda_p)=\frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}A_{n-k}\ \lambda_p\ \frac{1}{k!}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{3.8}$$

$n=0,1,\ldots,m_p-1,\ p=1,2,\ldots,j$ and

$$U_{n,p}(x,\lambda_p)=\frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{k=0}^{n}B_{n-k}\ \lambda_p\ \frac{1}{k!}\left\{\frac{\partial^k}{\partial\lambda^k}e(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{3.9}$$

$n = 0,1,...,m_p - 1, \quad p = j+1,...,k.$

It follows from the definition that the functions $U^{+}_{n,p}(x,\lambda_p), n = 0,1,...,m_p - 1, \quad p = 1,2,...,j,$ $j+1,...,k$ are the principal functions corresponding to the eigenvalues of $L$ .

**Theorem 3.**

$$U_{n,p}(x,\lambda_p) \in L^2(\mathbb{R}_+), \quad \text{for } n = 0,1,...,m_p - 1, \; p = 1,2,...,j, j+1,...,k. \tag{3.10}$$

Proof. *Let* $0 \le n \le m_p - 1$ *and* $1 \le p \le j$. *From (2.3), (2.4) and (2.11) we obtain*

$$\left|K(x,t)\right| \le c e^{-\varepsilon\sqrt{\frac{x+t}{2}}},$$

*and therefore,*

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n} e(x,\lambda)\right\}_{\lambda=\lambda_p}\right| \le x^n e^{-x \operatorname{Im}\lambda_p} + c\int_x^{\infty} t^n e^{-\varepsilon\sqrt{\frac{x+t}{2}}} e^{-t \operatorname{Im}\lambda_p} dt, \tag{3.11}$$

*where* $c > 0$ *is a constant. Since* $\operatorname{Im}\lambda_p > 0$ *for the eigenvalues* $\lambda_p$, $p = 1,2,...,j,$ *(3.10) follows from (3.8) and (3.11). Similarly we prove the results for* $0 \le n \le m_p - 1$ , $j+1 \le p \le k.$

Let $\mu_1,...,\mu_v$ and $\mu_{v+1},...,\mu_l$ be the zeros of $N^{+}$ and $N^{-}$ in $\mathbb{R}^*$ (which are the spectral singularities of $L$) with multiplicities $n_1,...,n_v$ and $n_{v+1},...,n_l,$ respectively. We can show

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{k=0}^{n}\binom{n}{k} c_{n-k}\left\{\frac{\partial^k}{\partial\lambda^k} e(x,\lambda)\right\}_{\lambda=\mu_p}, \tag{3.12}$$

$n = 0,1,...,n_p - 1, \quad p = 1,2,...,v,$ and

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{k=0}^{n}\binom{n}{k} d_{n-k}\left\{\frac{\partial^k}{\partial\lambda^k} e(x,-\lambda)\right\}_{\lambda=\mu_p}, \tag{3.13}$$

$n = 0,1,...,n_p - 1, \quad p = v+1,...,l.$

Now define the generalized eigenfunctions and generalized associated functions corresponding to the spectral singularities of $L$ by the following:

$$v_{n,p}(x,\mu_p) = \frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{k=0}^{n} C_{n-k}(\mu_p)\frac{1}{k!}\left\{\frac{\partial^k}{\partial\lambda^k} e(x,\lambda)\right\}_{\lambda=\mu_p} \tag{3.14}$$

$n = 0,1,...,n_p - 1, \quad p = 1,2,...,v,$

$$v_{n,p}(x,\mu_p) = \frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\mu_p} = \sum_{j=0}^{\nu} D_{v-j}(\mu_p)\frac{1}{j!}\left\{\frac{\partial^j}{\partial\lambda^j} e(x,-\lambda)\right\}_{\lambda=\mu_p} \tag{3.15}$$

$n = 0,1,\ldots,n_p - 1, \quad p = v+1,\ldots,l.$

Consequently, the functions $v_{n,p}(x,\mu_p),\ n = 0,1,\ldots,n_p - 1, \quad p = 1,2,\ldots,v,v+1,\ldots,l$ are the principal functions corresponding to the spectral singularities of $L$ .

**Theorem 4.**

$$v_{n,p}(x,\mu_p) \notin L^2(\mathbb{R}_+), \quad \text{for } n = 0,1,\ldots,n_p - 1, \quad p = 1,2,\ldots,v,v+1,\ldots,l. \tag{3.16}$$

Proof. If we consider $(3.11)$ for the principal functions corresponding to the spectral singularities $\lambda = \mu_p,\ p = 1,2,\ldots,v,v+1,\ldots,l,$ of $L$ and consider that $\operatorname{Im}\mu_p = 0$ for the spectral singlarities, then we have $(3.16)$, by $(3.14)$ and $(3.15)$.

Define the following Hilbert spaces:

$$H_n = \left\{ f \ : \ \int_0^\infty (1+x)^{2n} \left|f(x)\right|^2 dx < \infty \right\} \text{ and } H_{-n} = \left\{ g \ : \ \int_0^\infty (1+x)^{-2n} \left|g(x)\right|^2 dx < \infty \right\}$$

for $n = 0,1,2,\ldots,$ with norms

$$\left\|f\right\|_n^2 = \int_0^\infty (1+x)^{2n} \left|f(x)\right|^2 dx \text{ and } \left\|g\right\|_{-n}^2 = \int_0^\infty (1+x)^{-2n} \left|g(x)\right|^2 dx,$$

respectively. It is evident that

$$H_0 = L_2(\mathbb{R}_+) \text{ and } H_{n+1} \subsetneqq H_n \subsetneqq L^2(\mathbb{R}_+) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}. \tag{3.17}$$

Also $H_{-n}$ is isormorphic to the dual of $H_n$.

**Theorem 5.**

$v_{n,p}(x,\mu_p) \in H_{-(n+1)}, \quad n = 0,1,\ldots,n_p - 1, \quad p = 1,2,\ldots,v,v+1,\ldots,l.$

Proof. For $0 \le n \le n_p - 1$ and $1 \le p \le v$ using (2.3) we get

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e(x,\lambda) \right\}_{\lambda=\lambda_i} \right| \le x^n + \int_x^\infty t^n \left|K(x,t)\right| dt.$$

By the definition of the space $H_{-(n+1)}$ and using (2.4) and (3.14), we arrive at $v_{n,p}(x,\mu_p) \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $1 \le p \le v$. In the same manner, we obtain $v_{n,p}(x,\mu_p) \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $v+1 \le p \le l$ .

As a consequence of the preceding theorem we have the folowing:

**Corollary 1.**

$v_{n,p}(x,\mu_p) \in H_{-n_0}$ for $n = 0,1,\ldots,n_p - 1, \ p = 1,2,\ldots,v,v+1,\ldots,l,$

*where*

$$n_0 = \max \quad n_1, n_{2,\ldots,} n_v, n_{v+1,\ldots,} n_l \quad .$$

## CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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