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Some Notes On The Sequence Spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$

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Abstract

In this work, we introduce the sequence spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$ derived by the domain of the composition of m-th order generalized difference matrix and lambda matrix. Moreover, we determine some topological properties and examine inclusion relations related to these spaces. Furthermore, we give Schauder basis for the space $l_p^\lambda(G^m)$. Finally, we determine α-, β- and γ-duals of the spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$.

1. INTRODUCTION

The set of all sequences $x=(x_k)$ with $x_k\in\mathbb{C}$ for all $k\in\mathbb{N}=\{0,1,2,\dots\}$ is represented with w, where $\mathbb{C}$ is a family of all complex numbers. The set w becomes a vector space over $\mathbb{C}$ under point-wise addition and scalar multiplication. Every vector subspace X of w is called a sequence space.

We use the notations l_∞, c, c_0 and l_p for the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively, where $0<p<\infty$. Also, the symbols bv and bv_0 stand for the spaces consisting of all sequences $x=(x_k)$ such that $(x_k-x_{k+1})\in l_1$ and intersection of the spaces bv and c_0, respectively.

A sequence space X with a linear topology is called a K-space provided each of the maps $p_i: X\longrightarrow\mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$. It is assumed that w is always endowed with its locally convex topology generated by the sequence $\{p_n\}_{n=0}^\infty$ of seminorms on w where $p_n(x)=|x_n|$, $n=0,1,2,\dots$. A K-space X is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK-space [1].

The classical sequence spaces l_∞, c and c_0 equipped with the usual sup-norm defined by $\|x\|_\infty=\sup_{k\in\mathbb{N}}|x_k|$ are BK-spaces. Also, l_p is a BK-space with its l_p-norm defined by

$$\|x\|_{l_p}=\left(\sum_{k=0}^{\infty}|x_k|^p\right)^{\frac{1}{p}}$$

where $1\le p<\infty$. In case of $0<p<1$, l_p is a complete p-normed space according to the usual p-norm defined by

$$\|x\|_p=\sum_{k=0}^{\infty}|x_k|^p$$

(see [2]).

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To use the theory of matrix transformation was motivated by special and classical results in summability theory which were obtained by Cesàro, Borel, Norlund, Riesz and others. Because of the most general linear operator on one sequence space into another is actually given by an infinite matrix, matrix transformations are of great interest in the study of sequence spaces.

For an infinite matrix $A=(a_{nk})$ and a sequence $x=(x_k)$, $n,k\in\mathbb{N}$ of complex numbers, the A-transform of $x=(x_k)$ is written by $y=Ax$ and is defined by

$$y_n=(Ax)_n=\sum_{k=0}^{\infty}a_{nk}x_k \tag{1.1}$$

for all $n\in\mathbb{N}$ and each of these series being assumed convergent. A sequence $x=(x_k)$ is said to be A-summable to l if Ax converges to l, which is called A-limit of x [3].

Given two sequence spaces X and Y, the set of all infinite matrices $A=(a_{nk})$ such that $Ax\in Y$ for all $x\in X$ is denoted by $(X:Y)$.

For an arbitrary sequence space X, the set X_A is called matrix domain of an infinite matrix $A=(a_{nk})$ and is defined by

$$X_A=\{x=(x_k)\in w : Ax\in X\} \tag{1.2}$$

which is a sequence space also.

We write bs and cs for the sequence spaces of all bounded and convergent series, respectively. By using the notation (1.2) and summation matrix $S=(s_{nk})$, the sequence spaces bs and cs are defined by

$$bs=\left\{x=(x_k)\in w : \left(\sum_{k=0}^{n}x_k\right)\in l_\infty\right\}=(l_\infty)_S$$

and

$$cs=\left\{x=(x_k)\in w : \left(\sum_{k=0}^{n}x_k\right)\in c\right\}=c_S$$

respectively, where $S=(s_{nk})$ is defined by

$$s_{nk}=\begin{cases}1 & , \quad 0\le k\le n\\ 0 & , \quad k>n\end{cases}$$

for all $n,k\in\mathbb{N}$.

A matrix $A=(a_{nk})$ is called a triangle if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$ for all $n\in\mathbb{N}$. Also a triangle matrix $A=(a_{nk})$ uniquely has an inverse A^{-1} which is a triangle matrix.

In the next sections, unless stated otherwise, the summation without limits runs from 0 to ∞ and any term with negative subscript is assumed equal to zero, such that $x_{-1}=0$.

To define new sequence spaces, most of time, many authors use the notion of the matrix domain of an infinite matrix. For example: $(l_\infty)_{N_q}$ and c_{N_q} in [4], X_p and X_∞ in [5], r_∞^t, r_0^t and r_c^t in [6], $c_0(\Delta), c(\Delta)$ and $l_\infty(\Delta)$ in [7], $c_0(\Delta^2), c(\Delta^2)$ and $l_\infty(\Delta^2)$ in [8], $c_0(\Delta^m), c(\Delta^m)$ and $l_\infty(\Delta^m)$ in [9], $r^q(p,B^m)$ in [10], $c_0(B), c(B), l_\infty(B)$ and $l_p(B)$ in [11].

In this work, we introduce the sequence spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$ derived by the domain of the composition of m-th order generalized difference matrix and lambda matrix. Moreover, we determine some topological properties and examine inclusion relations related to these spaces. Furthermore, we give Schauder basis for the space $l_p^\lambda(G^m)$. Finally, we determine α-, β- and γ- duals of the spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$.

2. THE SEQUENCE SPACES $l_p^\lambda(G^m)$ AND $l_\infty^\lambda(G^m)$

In this section, we define the sequence spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$. Also, we determine some topological properties related to these spaces.

By using the matrix domain of lambda matrix $\Lambda = (\lambda_{nk})$, the sequence spaces l_p^λ and l_∞^λ are first introduced by M. Mursaleen and A. K. Noman in [12] and [13]. They defined the sequence spaces l_p^λ and l_∞^λ as follows:

$$l_p^\lambda = \left\{ x = (x_k) \in \mathrm{w} : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}$$

where $0 < p < \infty$ and

$$l_\infty^\lambda = \left\{ x = (x_k) \in \mathrm{w} : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}$$

respectively, where $\lambda = (\lambda_k)$ consist of positive reals such that

$$0 < \lambda_0 < \lambda_1 < \cdots \quad and \quad \lim_{k\to\infty} \lambda_k = \infty$$

and the lambda matrix $\Lambda = (\lambda_{nk})$ is defined by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Afterwards, F. Başar and A. Karaisa followed them and improved their work by defining the sequence spaces $l_p^\lambda(B)$ and $l_\infty^\lambda(B)$ in [14]. The sequence spaces $l_p^\lambda(B)$ and $l_\infty^\lambda(B)$ are defined by

$$l_p^\lambda(B) = \left\{ x = (x_k) \in \mathrm{w} : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(b_1 x_k + b_2 x_{k-1}) \right|^p < \infty \right\}$$

where $0 < p < \infty$ and

$$l_\infty^\lambda(B) = \left\{ x = (x_k) \in \mathrm{w} : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(b_1 x_k + b_2 x_{k-1}) \right| < \infty \right\}$$

respectively, where $B = B(b_1, b_2)$ is called double band(generalized difference) matrix and is defined by

$$b_{nk} = \begin{cases} b_1 & , \quad k = n \\ b_2 & , \quad k = n-1 \\ 0 & , \quad otherwise \end{cases}$$

for all $n, k \in \mathbb{N}$.

For given two non-zero real numbers r and s, m-th order generalized difference matrix $G^m(r,s) = \left(g_{nk}^m(r,s)\right)$ is defined by

$$g_{nk}^m(r,s) = \begin{cases} \binom{m-1}{n-k} r^{m-n+k-1} s^{n-k} & , \quad max\{0, n-m+1\} \le k \le n \\ 0 & , \quad otherwise \end{cases}$$

for all $n, k \in \mathbb{N}$ and $m \in \mathbb{N}_2 = \{2,3,4,\ldots\}$ [10]. Here we want to point out that $G^2(r,s) = B(b_1,b_2)$, $G^3(r,s) = B(b_1,b_2,b_3)$, $G^4(r,s) = B(b_1,b_2,b_3,b_4)$, … where $B(b_1,b_2)$, $B(b_1,b_2,b_3)$, $B(b_1,b_2,b_3,b_4)$, … are double band(generalized difference), triple band, quadruple band, …matrix, respectively. Moreover, $G^m(1,-1) = \Delta^m$, $G^3(1,-1) = \Delta^2$ and $G^2(1,-1) = \Delta$. So, our results obtained from the matrix domain of the m-th order difference matrix $G^m(r,s) = \left(g_{nk}^m(r,s)\right)$ are more general and more extensive than the results on the matrix domain of $B(b_1,b_2)$, $B(b_1,b_2,b_3)$, $B(b_1,b_2,b_3,b_4)$, …, Δ^m, Δ^2 and Δ.

For a given arbitrary sequence $x=(x_k)$, the $G^m(r,s)$-transform of x is the sequence $\xi=(\xi_k)$ and is defined by

$$\xi_k=\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta}r^{m-\vartheta-1}s^{\vartheta}x_{k-\vartheta}$$

for all $k\in\mathbb{N}$.

Now, by considering the sequence $\xi=(\xi_k)$ defined above, we define the sequence spaces $l_p^{\lambda}(G^m)$ and $l_\infty^{\lambda}(G^m)$ by means of m-th order generalized difference matrix and lambda matrix as follows:

$$l_p^{\lambda}(G^m)=\left\{x=(x_k)\in \mathrm{w}:\ \sum_{n=0}^{\infty}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})\xi_k\right|^p<\infty\right\}$$

where $0<p<\infty$ and

$$l_\infty^{\lambda}(G^m)=\left\{x=(x_k)\in \mathrm{w}:\ \sup_{n\in\mathbb{N}}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})\xi_k\right|<\infty\right\}$$

respectively.

If we consider the notation (1.2), the sequence spaces $l_p^{\lambda}(G^m)$ and $l_\infty^{\lambda}(G^m)$ are redefined by

$$l_p^{\lambda}(G^m)=\left(l_p^{\lambda}\right)_{G^m}\ \text{and}\ l_\infty^{\lambda}(G^m)=\left(l_\infty^{\lambda}\right)_{G^m}\tag{2.1}$$

respectively. Moreover, by using a same way, we can redefine the sequence spaces $l_p^{\lambda}(G^m)$ and $l_\infty^{\lambda}(G^m)$ by means of the infinite matrix $T^{m\lambda}(r,s)=\left(t_{nk}^{m\lambda}(r,s)\right)$ as follows:

$$l_p^{\lambda}(G^m)=\left(l_p\right)_{T^{m\lambda}}\ \text{and}\ l_\infty^{\lambda}(G^m)=\left(l_\infty\right)_{T^{m\lambda}}\tag{2.2}$$

respectively, where the infinite matrix $T^{m\lambda}(r,s)=\left(t_{nk}^{m\lambda}(r,s)\right)$ that is composition of m-th order generalized difference matrix and lambda matrix is defined by

$$t_{nk}^{m\lambda}=\begin{cases}\dfrac{1}{\lambda_n}\displaystyle\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta}r^{m-\vartheta-1}s^{\vartheta}(\lambda_{k+\vartheta}-\lambda_{k+\vartheta-1}) & ,\quad k<n-m+2\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{\vartheta=1}^{m-1}\binom{m-1}{\vartheta-1}r^{m-\vartheta}s^{\vartheta-1}(\lambda_{n-m+\vartheta+1}-\lambda_{n-m+\vartheta}) & ,\quad k=n-m+2\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{\vartheta=2}^{m-1}\binom{m-1}{\vartheta-2}r^{m-\vartheta+1}s^{\vartheta-2}(\lambda_{n-m+\vartheta+1}-\lambda_{n-m+\vartheta}) & ,\quad k=n-m+3\\ \vdots & \quad\vdots\\ \dfrac{r^{m-1}(\lambda_{n-1}-\lambda_{n-2})+(m-1)r^{m-2}s(\lambda_n-\lambda_{n-1})}{\lambda_n} & ,\quad k=n-1\\ \dfrac{r^{m-1}(\lambda_n-\lambda_{n-1})}{\lambda_n} & ,\quad k=n\\ 0 & ,\quad k>n\end{cases}$$

for all $n,k\in\mathbb{N}$ and $m\in\mathbb{N}_2$.

For a given arbitrary sequence $x=(x_k)$, the $T^{m\lambda}$-transform of x is defined by

$$y_k = \left(T^{m\lambda}x\right)_k = \frac{1}{\lambda_k}\sum_{j=0}^{k}\left(\lambda_j - \lambda_{j-1}\right)\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta} r^{m-\vartheta-1}s^{\vartheta}x_{j-\vartheta} \tag{2.3}$$

or

$$y_k = \frac{1}{\lambda_k}\sum_{j=0}^{k-m+1}\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta} r^{m-\vartheta-1}s^{\vartheta}\left(\lambda_{j+\vartheta} - \lambda_{j+\vartheta-1}\right)x_j + \cdots + \frac{r^{m-1}(\lambda_k - \lambda_{k-1})}{\lambda_k}x_k \tag{2.4}$$

for all $k \in \mathbb{N}$.

Theorem 2.1 *The following statements hold.*

(a) *In case of* $0 < p < 1$, $l_p^{\lambda}(G^m)$ *is a complete* p*-normed space according to its* p*-norm defined by*

$$\|x\|_{l_p^{\lambda}(G^m)} = \left\|T^{m\lambda}x\right\|_p = \sum_{n=0}^{\infty}\left|\left(T^{m\lambda}x\right)_n\right|^p$$

(b) *In case of* $1 \le p < \infty$, $l_p^{\lambda}(G^m)$ *is a* BK*-space with its* l_p*-norm defined by*

$$\|x\|_{l_p^{\lambda}(G^m)} = \left\|T^{m\lambda}x\right\|_{l_p} = \left(\sum_{n=0}^{\infty}\left|\left(T^{m\lambda}x\right)_n\right|^p\right)^{\frac{1}{p}}$$

(c) *The sequence space* $l_{\infty}^{\lambda}(G^m)$ *is a* BK*-space according to its sup-norm defined by*

$$\|x\|_{l_{\infty}^{\lambda}(G^m)} = \left\|T^{m\lambda}x\right\|_{\infty} = \sup_{n\in\mathbb{N}}\left|\left(T^{m\lambda}x\right)_n\right|$$

Proof It is known that l_p is a complete p-normed space with its p-norm and a BK-space with its l_p-norm in case of $0 < p < 1$ and in case of $1 \le p < \infty$, respectively. Also, the sequence space l_{∞} equipped with its usual sup-norm is a BK-space. Moreover, (2.2) holds and $T^{m\lambda}(r,s) = \left(t_{nk}^{m\lambda}(r,s)\right)$ is a triangle matrix. By combining these five facts and Theorem 4.3.12 of Wilansky [3], we deduce that (a), (b) and (c) hold. This step completes the proof.

Theorem 2.2 *In the event of* $0 < p \le \infty$, *the sequence space* $l_p^{\lambda}(G^m)$ *is linearly isomorphic to the sequence space* l_p, *namely* $l_p^{\lambda}(G^m) \cong l_p$.

Proof For the proof, the existence of a linear bijection between $l_p^{\lambda}(G^m)$ and l_p is necessary. We define a transformation L such that $L: l_p^{\lambda}(G^m) \longrightarrow l_p$, $L(x) = T^{m\lambda}x$. Then, it is clear that $L(x) = T^{m\lambda}x \in l_p$ for all $x \in l_p^{\lambda}(G^m)$. Also, it is trivial that L is a linear transformation and $x = \theta$ whenever $L(x) = \theta$. Because of this L is injective.

Moreover, given a sequence $y = (y_k) \in l_p$, we define a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{r^{m-1}}\sum_{j=0}^{k}\binom{m+k-j-2}{m-2}\left(-\frac{s}{r}\right)^{k-j}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{\lambda_j - \lambda_{j-1}}y_i$$

for all $k \in \mathbb{N}$ and $m \in \mathbb{N}_2$. Then, for every $k \in \mathbb{N}$, we obtain

$$\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta} r^{m-\vartheta-1}s^{\vartheta}x_{k-\vartheta} = \sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k - \lambda_{k-1}}y_i$$

If we consider the equality above, we obtain

$$\left(T^{m\lambda}x\right)_n = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\sum_{\vartheta=0}^{m-1}\binom{m-1}{\vartheta} r^{m-\vartheta-1}s^{\vartheta}x_{k-\vartheta}$$

$$= \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})\sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k-\lambda_{k-1}}y_i$$
$$= \frac{1}{\lambda_n}\sum_{k=0}^{n}\sum_{i=k-1}^{k}(-1)^{k-i}\lambda_i y_i$$
$$= y_n$$

for all $n \in \mathbb{N}$. So, $T^{m\lambda}x = y$ and since $y \in l_p$, we conclude that $T^{m\lambda}x \in l_p$. This shows that $x \in l_p^{\lambda}(G^m)$ and $L(x) = y$. Thus L is surjective. From the Theorem 2.1 , we have

$$\|L(x)\|_{l_p} = \|T^{m\lambda}x\|_{l_p} = \|x\|_{l_p^{\lambda}(G^m)}$$

for all $x \in l_p^{\lambda}(G^m)$ and $0 < p \leq \infty$. So, L is norm preserving. As a results of these L is a linear bijection. This last step shows that $l_p^{\lambda}(G^m)$ and l_p are linearly isomorphic in case of $0 < p \leq \infty$. This step completes the proof.

Theorem 2.3 *The sequence space $l_p^{\lambda}(G^m)$ is not a Hilbert space whenever $p \in [1,\infty)\backslash\{2\}$.*

Proof From the Theorem 2.1 (b), we know that $l_2^{\lambda}(G^m)$ is a BK-space with its l_2-norm defined by $\|x\|_{l_2^{\lambda}(G^m)} = \|T^{m\lambda}x\|_{l_2}$, where l_2-norm can be obtained from an inner product on l_2 such that

$$\|x\|_{l_2^{\lambda}(G^m)} = \langle x, x\rangle^{\frac{1}{2}} = \langle T^{m\lambda}x, T^{m\lambda}x\rangle_{l_2}^{\frac{1}{2}}$$

for all $x \in l_2^{\lambda}(G^m)$. If we consider this fact, we deduce that $l_2^{\lambda}(G^m)$ is a Hilbert space.

Now, by taking into account $p \in [1,\infty)\backslash\{2\}$, we define two sequences $b = (b_k)$ and $d = (d_k)$ as follows:

$$b_k = \begin{cases} \dfrac{1}{r^{m-1}} & , \quad k = 0 \\ \dfrac{r+(1-m)s}{r^m} & , \quad k = 1 \\ \dfrac{1}{r^{m-1}}\left(-\dfrac{s}{r}\right)^{k-2}\left[\dfrac{s^2}{r^2}\binom{m+k-2}{m-2} - \dfrac{s}{r}\binom{m+k-3}{m-2} - \dfrac{\lambda_1}{\lambda_2-\lambda_1}\binom{m+k-4}{m-2}\right] & , \quad k > 1 \end{cases}$$

and

$$d_k = \begin{cases} \dfrac{1}{r^{m-1}} & , \quad k = 0 \\ -\dfrac{1}{r^{m-1}}\left[\dfrac{(m-1)s}{r} + \dfrac{\lambda_1+\lambda_0}{\lambda_1-\lambda_0}\right] & , \quad k = 1 \\ \dfrac{1}{r^{m-1}}\left(-\dfrac{s}{r}\right)^{k-2}\left[\dfrac{s^2}{r^2}\binom{m+k-2}{m-2} + \dfrac{s}{r}\binom{m+k-3}{m-2}\dfrac{\lambda_1+\lambda_0}{\lambda_1-\lambda_0} + \dfrac{\lambda_1}{\lambda_2-\lambda_1}\right] & , \quad k > 1 \end{cases}$$

for all $k \in \mathbb{N}$ and $m \in \mathbb{N}_2$. Then we write

$$T^{m\lambda}b = (1, 1, 0, 0, \ldots) \text{ and } T^{m\lambda}d = (1, -1, 0, 0, \ldots)$$

If we consider the norm of the space $l_p^{\lambda}(G^m)$, we obtain

$$\|b+d\|_{l_p^{\lambda}(G^m)}^2 + \|b-d\|_{l_p^{\lambda}(G^m)}^2 = 8 \neq 2^{\frac{2}{p}+2} = 2\left(\|b\|_{l_p^{\lambda}(G^m)}^2 + \|d\|_{l_p^{\lambda}(G^m)}^2\right)$$

whenever $p \in [1,\infty)\backslash\{2\}$. So, the parallelogram equality does not hold. As a result of this, the norm of $l_p^{\lambda}(G^m)$ can not be obtained from an inner product. Thus the space $l_p^{\lambda}(G^m)$ is not a Hilbert space whenever $p \in [1,\infty)\backslash\{2\}$. This step completes the proof.

3. SOME INCLUSION RELATIONS

In this section, we examine some inclusion relations related to the sequence spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$, where $0 < p < \infty$.

Theorem 3.1 *The inclusion $l_p^\lambda(G^m) \subset l_q^\lambda(G^m)$ strictly holds in the meantime $0 < p < q < \infty$.*

Proof Given an arbitrary sequence $x = (x_k) \in l_p^\lambda(G^m)$. In case of $0 < p < q < \infty$, we know that the inclusion $l_p \subset l_q$ holds. If $x \in l_p^\lambda(G^m)$, then $T^{m\lambda}x \in l_p$. By considering these two results, we conclude that $T^{m\lambda}x \in l_q$, namely $x \in l_q^\lambda(G^m)$. So, we have $l_p^\lambda(G^m) \subset l_q^\lambda(G^m)$.

Now, we define a sequence $u = (u_k)$ as follows:

$$u_k = \frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+k-j-2}{m-2} \left(-\frac{s}{r}\right)^{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i (i+1)^{-\frac{1}{p}}}{\lambda_j - \lambda_{j-1}}$$

for all $k \in \mathbb{N}$. Then we obtain $T^{m\lambda}u = \left(\frac{1}{(k+1)^{\frac{1}{p}}}\right) \in l_q \backslash l_p$, that is $u \in l_q^\lambda(G^m) \backslash l_p^\lambda(G^m)$. As a consequence, the inclusion $l_p^\lambda(G^m) \subset l_q^\lambda(G^m)$ is strict. This step completes the proof.

Theorem 3.2 *The inclusions $l_p^\lambda(G^m) \subset c_0^\lambda(G^m) \subset c^\lambda(G^m) \subset l_\infty^\lambda(G^m)$ are strict, where $0 < p < \infty$ and $c_0^\lambda(G^m) = (c_0)_{T^{m\lambda}}$ and $c^\lambda(G^m) = c_{T^{m\lambda}}$ are defined in* [15].

Proof We know the fact that the inclusions $l_p \subset c_0 \subset c \subset l_\infty$ hold. By considering a similar way as used in the proof of Theorem 3.1, one can easily obtain that the inclusions $l_p^\lambda(G^m) \subset c_0^\lambda(G^m) \subset c^\lambda(G^m) \subset l_\infty^\lambda(G^m)$ hold.

Now, we define three sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ as follows:

$$x_k = \frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+k-j-2}{m-2} \left(-\frac{s}{r}\right)^{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i (i+1)^{-\frac{1}{p}}}{\lambda_j - \lambda_{j-1}}$$

$$y_k = \frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+j-2}{m-2} \left(-\frac{s}{r}\right)^{j}$$

and

$$z_k = \frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+k-j-2}{m-2} \left(-\frac{s}{r}\right)^{k-j} \sum_{i=j-1}^{j} (-1)^{j} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}}$$

for all $k \in \mathbb{N}$. Then we obtain $T^{m\lambda}x = \left(\frac{1}{(k+1)^{\frac{1}{p}}}\right) \in c_0 \backslash l_p$, $T^{m\lambda}y = (1,1,1,\dots) \in c \backslash c_0$ and $T^{m\lambda}z = \left((-1)^k\right) \in l_\infty \backslash c$, that is $x \in c_0^\lambda(G^m) \backslash l_p^\lambda(G^m)$, $y \in c^\lambda(G^m) \backslash c_0^\lambda(G^m)$ and $z \in l_\infty^\lambda(G^m) \backslash c^\lambda(G^m)$. Hence the inclusions $l_p^\lambda(G^m) \subset c_0^\lambda(G^m) \subset c^\lambda(G^m) \subset l_\infty^\lambda(G^m)$ strictly hold. This step completes the proof.

Theorem 3.3 *The inclusion $l_\infty \subset l_\infty^\lambda(G^m)$ is strict.*

Proof For a given arbitrary sequence $x = (x_k) \in l_\infty$, we write

$$\begin{aligned} \|x\|_{l_\infty^\lambda(G^m)} &= \sup_{k \in \mathbb{N}} \left| \left(T^{m\lambda}x\right)_k \right| \\ &= \sup_{k \in \mathbb{N}} \left| \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1}) \sum_{\vartheta=0}^{m-1} \binom{m-1}{\vartheta} r^{m-\vartheta-1} s^{\vartheta} x_{j-\vartheta} \right| \end{aligned}$$

$$\leq \sup_{k\in\mathbb{N}} \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1}) \sum_{\vartheta=0}^{m-1} \binom{m-1}{\vartheta} |r^{m-\vartheta-1} s^{\vartheta}| |x_{j-\vartheta}|$$

$$\leq \left(\sum_{\vartheta=0}^{m-1} \binom{m-1}{\vartheta} |r^{m-\vartheta-1} s^{\vartheta}| \right) \|x\|_\infty \sup_{k\in\mathbb{N}} \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1})$$

$$= \left(\sum_{\vartheta=0}^{m-1} \binom{m-1}{\vartheta} |r^{m-\vartheta-1} s^{\vartheta}| \right) \|x\|_\infty$$

$$< \infty$$

This shows that $x = (x_k) \in l_\infty^\lambda(G^m)$, namely the inclusion $l_\infty \subset l_\infty^\lambda(G^m)$ holds.

Let us define a sequence $u = (u_k)$ as follows:

$$u_k = \frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+j-2}{m-2} \left(-\frac{s}{r}\right)^j$$

for all $k \in \mathbb{N}$ with $\left|\frac{s}{r}\right| \geq 1$. It is obvious that $u = (u_k) \notin l_\infty$. But $T^{m\lambda} u = (1, 1, 1, \ldots) \in l_\infty$, that is $u = (u_k) \in l_\infty^\lambda(G^m)$. Thus the inclusion $l_\infty \subset l_\infty^\lambda(G^m)$ strictly holds. This step completes the proof.

Theorem 3.4 *If the inclusion* $l_p \subset l_p^\lambda(G^m)$ *holds, then the sequence* $\left(\frac{1}{\lambda_k}\right) \in l_p$ *, where* $0 < p < \infty$.

Proof We assume that the inclusion $l_p \subset l_p^\lambda(G^m)$ holds for $0 < p < \infty$. It is clear that $e^{(0)} = (1, 0, 0, \ldots) \in l_p$. Then, by assumption, we conclude that $e^{(0)} \in l_p^\lambda(G^m)$, that is $T^{m\lambda} e^{(0)} \in l_p$. This shows that

$$\sum_k \left| \left(T^{m\lambda} e^{(0)}\right)_k \right|^p = |r^{m-1} \lambda_0| \sum_k \left(\frac{1}{\lambda_k}\right)^p < \infty$$

namely, $\left(\frac{1}{\lambda_k}\right) \in l_p$, where $0 < p < \infty$. This step completes the proof.

4. SCHAUDER BASIS AND α-, β- AND γ-DUALS

In this section, we give the Schauder basis for the sequence space $l_p^\lambda(G^m)$. Also, we determine α-, β- and γ-duals of the sequence spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$.

Let $(X, \|.\|_X)$ be a normed space. A set $\{x_k : x_k \in X, k \in \mathbb{N}\}$ is called a Schauder basis for X if for every $x \in X$ there exist unique scalars μ_k , $k \in \mathbb{N}$, such that $x = \sum_k \mu_k x_k$; i.e.,

$$\left\| x - \sum_{k=0}^{n} \mu_k x_k \right\|_X \longrightarrow 0$$

as $n \to \infty$.

We know that the sequence $\{e^{(k)}\}$ is a Schauder basis for l_p , where $e^{(k)}$ is a sequence with 1 in k-th place and zeros elsewhere. Because of the transformation L defined in the proof of Theorem 2.2 is an isomorphism; the inverse image of $\{e^{(k)}\}$ is a Schauder basis for $l_p^\lambda(G^m)$.

So, we can give the following theorem.

Theorem 4.1 *Let* $\sigma_k = \{T^{m\lambda} x\}_k$ *for all* $k \in \mathbb{N}$. *Define a sequence* $h_{(k)}^{m\lambda}(r,s) = \{h_{n(k)}^{m\lambda}(r,s)\}_{n\in\mathbb{N}}$ *as following:*

$$h_{n(k)}^{m\lambda}(r,s)=\begin{cases}\frac{1}{r^{m-2}}\left(-\frac{s}{r}\right)^{n-k}\left[\frac{\binom{m+n-k-2}{m-2}\lambda_k}{r(\lambda_k-\lambda_{k-1})}+\frac{\binom{m+n-k-3}{m-2}\lambda_k}{s(\lambda_{k+1}-\lambda_k)}\right] & , \quad k<n\\ \frac{\lambda_k}{r^{m-1}(\lambda_k-\lambda_{k-1})} & , \quad k=n\\ 0 & , \quad k>n\end{cases}$$

for all fixed $k\in\mathbb{N}$. Then the sequence $\{h_{(k)}^{m\lambda}(r,s)\}_{k\in\mathbb{N}}$ is a Schauder basis for the space $l_p^\lambda(G^m)$ and every $x\in l_p^\lambda(G^m)$ has a unique representation of the form

$$x=\sum_k \sigma_k h_{(k)}^{m\lambda}(r,s)$$

If we consider the results of Theorem 2.1 (b) and Theorem 4.1, we can give next result.

Corollary 4.2 *The sequence space $l_p^\lambda(G^m)$ is separable for $1\le p<\infty$.*

Given arbitrary sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y)=\{y=y_k\in w : xy=(x_ky_k)\in Y \; for \; all \; x=(x_k)\in X\} \tag{4.1}$$

is called the multiplier space of X and Y. For a sequence space Z with $Y\subset Z\subset X$, one can easily observe that $M(X,Y)\subset M(Z,Y)$ and $M(X,Y)\subset M(X,Z)$ hold, respectively.

By using the sequence spaces l_1, cs and bs and the notation (4.1), the α-, β- and γ-duals of a sequence space X are defined by

$$X^\alpha=M(X,l_1)\,,X^\beta=M(X,cs) \text{ and } X^\gamma=M(X,bs)$$

respectively.

Now we write some properties which will be needed in the next lemma.

$$\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}a_{nk}\right|^q<\infty \tag{4.2}$$

$$\sup_{k\in\mathbb{N}}\sum_n|a_{nk}|<\infty \tag{4.3}$$

$$\lim_{n\to\infty}a_{nk} \quad exists \; for \; all \; k\in\mathbb{N} \tag{4.4}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|^q<\infty \tag{4.5}$$

$$\sup_{k,n\in\mathbb{N}}|a_{nk}|<\infty \tag{4.6}$$

$$\lim_{n\to\infty}\sum_k\left|a_{nk}-\lim_{n\to\infty}a_{nk}\right|=0 \tag{4.7}$$

where $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4.3 (see [16]) *Given an infinite matrix $A=(a_{nk})$, the following hold:*

(i) $A=(a_{nk})\in(l_p:l_1)$ *for* $1<p\le\infty\Leftrightarrow$ (4.2) *holds,*

(ii) $A=(a_{nk})\in(l_1:l_1)\Leftrightarrow$ (4.3) *holds,*

(iii) $A=(a_{nk})\in(l_p:c)$ *for* $1<p<\infty\Leftrightarrow$ (4.4) *and* (4.5) *hold,*

(iv) *$A=(a_{nk})\in(l_1:c)\Leftrightarrow(4.4)$ and (4.6) hold,*

(v) *$A=(a_{nk})\in(l_\infty:c)\Leftrightarrow(4.4)$, (4.5) and (4.7) hold with $q=1$,*

(vi) *$A=(a_{nk})\in\left(l_p:l_\infty\right)$ for $1<p\le\infty\Leftrightarrow(4.5)$ holds,*

(vii) *$A=(a_{nk})\in(l_1:l_\infty)\Leftrightarrow(4.6)$ holds.*

Theorem 4.4 *Define the sets $v_1^{m\lambda}(r,s)$ and $v_2^{m\lambda}(r,s)$ as follows:*

$$v_1^{m\lambda}(r,s)=\left\{a=(a_k)\in w:\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}d_{nk}^{m\lambda}\right|^q<\infty\right\}$$

and

$$v_2^{m\lambda}(r,s)=\left\{a=(a_k)\in w:\sup_{k\in\mathbb{N}}\sum_n\left|d_{nk}^{m\lambda}\right|<\infty\right\}$$

where the matrix $D^{m\lambda}=\left(d_{nk}^{m\lambda}(r,s)\right)$ is defined via the sequence $a=(a_n)$ by

$$d_{nk}^{m\lambda}(r,s)=\begin{cases}\dfrac{1}{r^{m-2}}\left(-\dfrac{s}{r}\right)^{n-k}\left[\dfrac{\binom{m+n-k-2}{m-2}\lambda_k}{r(\lambda_k-\lambda_{k-1})}+\dfrac{\binom{m+n-k-3}{m-2}\lambda_k}{s(\lambda_{k+1}-\lambda_k)}\right]a_n & ,\quad k<n\\ \dfrac{\lambda_n}{r^{m-1}(\lambda_n-\lambda_{n-1})}a_n & ,\quad k=n\\ 0 & ,\quad k>n\end{cases}$$

for all $n,k\in\mathbb{N}$ and $m\in\mathbb{N}_2$. Then, $\left\{l_p^\lambda(G^m)\right\}^\alpha=v_1^{m\lambda}(r,s)$ for $1<p\le\infty$ and $\left\{l_1^\lambda(G^m)\right\}^\alpha=v_2^{m\lambda}(r,s)$.

Proof Given $a=(a_n)\in w$, we consider the sequence $x=(x_n)$ defined by

$$x_n=\frac{1}{r^{m-1}}\sum_{k=0}^{n}\binom{m+n-k-2}{m-2}\left(-\frac{s}{r}\right)^{n-k}\sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k-\lambda_{k-1}}y_i \tag{4.8}$$

for all $n\in\mathbb{N}$ and $m\in\mathbb{N}_2$. Then, we obtain

$$\begin{aligned}a_nx_n&=\frac{1}{r^{m-1}}\sum_{k=0}^{n}\binom{m+n-k-2}{m-2}\left(-\frac{s}{r}\right)^{n-k}\sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k-\lambda_{k-1}}a_ny_i\\&=D_n^{m\lambda}(y)\end{aligned}$$

for all $n\in\mathbb{N}$ and $m\in\mathbb{N}_2$. Hence, we conclude that $ax=(a_nx_n)\in l_1$ whenever $x=(x_k)\in l_p^\lambda(G^m)$ if and only if $D^{m\lambda}y\in l_1$ whenever $y=(y_k)\in l_p$, that is $a=(a_k)\in\left\{l_p^\lambda(G^m)\right\}^\alpha$ if and only if $D^{m\lambda}\in\left(l_p:l_1\right)$. If we consider this and Theorem 4.3 (i), we deduce that $\left\{l_p^\lambda(G^m)\right\}^\alpha=v_1^{m\lambda}(r,s)$ for $1<p\le\infty$. By using a similar way, we obtain that $a=(a_k)\in\left\{l_1^\lambda(G^m)\right\}^\alpha$ if and only if $D^{m\lambda}\in(l_1:l_1)$. If we consider this and Theorem 4.3 (ii), we deduce that $\left\{l_1^\lambda(G^m)\right\}^\alpha=v_2^{m\lambda}(r,s)$. This step completes the proof.

Theorem 4.5 *Define the sets $v_3^{m\lambda}(r,s)$, $v_4^{m\lambda}(r,s)$, $v_5^{m\lambda}(r,s)$, $v_6^{m\lambda}(r,s)$ and $v_7^{m\lambda}(r,s)$ as follows:*

$$v_3^{m\lambda}(r,s)=\left\{a=(a_k)\in w:\sum_{j=k}^{\infty}\binom{m+n-j-2}{m-2}\left(-\frac{s}{r}\right)^{n-j}a_j\ \ exists\ \ \forall k\in\mathbb{N}\right\}$$

$$v_4^{m\lambda}(r,s)=\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n-1}\left|b_k^{m\lambda}(n)\right|^q<\infty\right\}$$

$$v_5^{m\lambda}(r,s) = \left\{a = (a_k) \in w : \sup_{n,k\in\mathbb{N}} \left|b_k^{m\lambda}(n)\right| < \infty\right\}$$

$$v_6^{m\lambda}(r,s) = \left\{a = (a_k) \in w : \lim_{n\to\infty} \sum_k \left|b_k^{m\lambda}(n)\right| = \sum_k \left|b_k^{m\lambda}\right|\right\}$$

and

$$v_7^{m\lambda}(r,s) = \left\{a = (a_k) \in w : \sup_{n\in\mathbb{N}} \left|\frac{\lambda_n}{r^{m-1}(\lambda_n - \lambda_{n-1})} a_n\right|^q < \infty\right\}$$

where

$$b_k^{m\lambda}(n) = \lambda_k \left[\frac{1}{r^{m-2}} \sum_{j=k+1}^{n} \left(-\frac{s}{r}\right)^{n-j} \left(\frac{\binom{m+n-j-2}{m-2}}{r(\lambda_k - \lambda_{k-1})} + \frac{\binom{m+n-j-3}{m-2}}{s(\lambda_{k+1} - \lambda_k)}\right) a_j\right]$$

$$+\lambda_k \left[\frac{a_k}{r^{m-1}(\lambda_k - \lambda_{k-1})}\right]$$

for all $k < n$ and

$$b_k^{m\lambda} = \lim_{n\to\infty} b_k^{m\lambda}(n).$$

Then, the following hold:

(a) $\{l_p^\lambda(G^m)\}^\beta = v_3^{m\lambda}(r,s) \cap v_4^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$, *for* $1 < p < \infty$,

(b) $\{l_1^\lambda(G^m)\}^\beta = v_3^{m\lambda}(r,s) \cap v_5^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$ *with* $q = 1$,

(c) $\{l_\infty^\lambda(G^m)\}^\beta = v_3^{m\lambda}(r,s) \cap v_4^{m\lambda}(r,s) \cap v_6^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$ *with* $q = 1$,

(d) $\{l_p^\lambda(G^m)\}^\gamma = v_4^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$, *for* $1 < p \le \infty$,

(e) $\{l_1^\lambda(G^m)\}^\gamma = v_5^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$ *with* $q = 1$.

Proof For an arbitrary sequence $a = (a_k) \in w$, by taking into account the sequence $x = (x_k)$ that is defined with the relation (4.8), we obtain

$$\begin{aligned} z_n &= \sum_{k=0}^{n} a_k x_k \\ &= \sum_{k=0}^{n} \left\{\frac{1}{r^{m-1}} \sum_{j=0}^{k} \binom{m+k-j-2}{m-2} \left(-\frac{s}{r}\right)^{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i\right\} a_k \\ &= \sum_{k=0}^{n-1} b_k^{m\lambda}(n) y_k + \frac{\lambda_n}{r^{m-1}(\lambda_n - \lambda_{n-1})} a_n y_n \\ &= U_n^{m\lambda}(y) \end{aligned}$$

for all n ∈ ℕ, where the matrix $U^{m\lambda} = \left(u_{nk}^{m\lambda}(r,s)\right)$ is defined as follows:

$$u_{nk}^{m\lambda}(r,s) = \begin{cases} b_k^{m\lambda}(n) & , \quad k < n \\ \dfrac{\lambda_n}{r^{m-1}(\lambda_n - \lambda_{n-1})} a_n & , \quad k = n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$ and $m \in \mathbb{N}_2$. Then,

(a) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in l_p^\lambda(G^m)$ if and only if $U^{m\lambda} y \in c$ whenever $y = (y_k) \in l_p$, that is $a = (a_k) \in \{l_p^\lambda(G^m)\}^\beta$ if and only if $U^{m\lambda} \in (l_p : c)$. If we combine this fact and Theorem 4.3 (iii), we obtain

$$\sum_{j=k}^{\infty} \binom{m+n-j-2}{m-2} \left(-\frac{s}{r}\right)^{n-j} a_j \ \ exists \ \forall k \in \mathbb{N}$$

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} \left|b_k^{m\lambda}(n)\right|^q < \infty$$

and

$$\sup_{n\in\mathbb{N}} \left|\frac{\lambda_n}{r^{m-1}(\lambda_n - \lambda_{n-1})} a_n\right|^q < \infty$$

As a consequence, these three results show that

$$\{l_p^\lambda(G^m)\}^\beta = v_3^{m\lambda}(r,s) \cap v_4^{m\lambda}(r,s) \cap v_7^{m\lambda}(r,s)$$

for $1 < p < \infty$.

(b), (c), (d) and **(e)** can be proven by using a similar way. So, to avoid the repetition of similar statements, we omit the details. This step completes the proof.

5. CONCLUSION

By considering the definitions of m-th order generalized difference matrix and the lambda matrix, one can observe that $G^2(r,s) = B(b_1, b_2)$, $G^3(r,s) = B(b_1, b_2, b_3)$, $G^4(r,s) = B(b_1, b_2, b_3, b_4)$, … where $B(b_1, b_2)$, $B(b_1, b_2, b_3)$, $B(b_1, b_2, b_3, b_4)$, … are double band(generalized difference), triple band, quadruple band, …matrix, respectively. Moreover, $G^m(1,-1) = \Delta^m$, $G^3(1,-1) = \Delta^2$ and $G^2(1,-1) = \Delta$. Furthermore, if we take $\lambda_n = n+1$ and $\lambda_n = P_n$ in the definition of the lambda matrix, we obtain the Cesàro mean of order one and the Riesz mean matrix which are defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases} \quad \text{and} \quad r_{nk}^p = \begin{cases} \frac{p_k}{P_n} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

respectively, where $p_0 > 0, p_n \ge 0 \ (n \ge 1)$ and $P_n = \sum_{k=0}^n p_k$. So, the results obtained from the matrix domain of the composition of m-th order generalized difference matrix and lambda matrix are more general and more comprehensive than the others that we have mentioned above.

As we finalize our work, we would like to mention that in the next one, we will focus on geometric properties of the space $l_p^\lambda(G^m)$ and matrix classes related to the spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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