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The Geometry of Bézier Curves in Minkowski $3-$Space

Ayşe Yılmaz Ceylan[1]

[1] *Department of Mathematics, Faculty of Science, Akdeniz University, 07058 Antalya, Türkiye*

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Abstract

The scope of this paper is to look at some aspects of the differential geometry of Bézier curves in Minkowski space. For that purpose, we firstly introduce Frenet Bézier curve in Minkowski 3-space. Especially, we investigate the Serret-Frenet frame, curvature and torsion of the Frenet Bézier curves at all points. Moreover, we give the Frenet apparatus of these curves at the end points.

1. Introduction and Background

Let E_1^3 be the three dimensional Minkowski space with the metric $< dx, dx >= dx_1^2 + dx_2^2 - dx_3^2$ where x_1, x_2, x_3 denotes the canonical coordinates in E^3. An arbitrary vector x is said to be spacelike if $< x, x >> 0$ or $x = 0$, timelike if $< x, x >< 0$ and lightlike or null if $< x, x >= 0$. The norm is defined by $\|x\| = \sqrt{|< x, x >|}$ for $x \in E_1^3$. A regular curve in E_1^3 is called locally spacelike, timelike or null, if all its velocity vectors are spacelike, timelike or null, respectively [1]. For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3, the inner product is the real number $< x, y >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $x \wedge_{IL} y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$. See for more information on Minkowski space in [1, 2].

Bézier curves are represented by Pierre Bézier in 1968. Bézier curves are essential among the curves since they are applicable to computer graphics and related areas. See for more detailed information in [3, 4]. Recently, the geometry of Bézier curves have been investigated by many researechers due to the fact that they have several important properties. Incesu and GÃ¼rsoy studied the curvatures and principal form of the Bézier curve in [5]. Georgiev worked on the shapes of planar and cubic Bézier curve in [6, 7].

In the theory of curves in the Minkowski space, one of the interesting problem is the characterization of a regular curve. In [8], Georgiev studied on the geometry of the spacelike Bézier curve. He also examined the spacelike Bézier surfaces in Minkowski $3-$space in [9]. Chalmoviansky, Pokorna studied quadratic and planar cubic spacelike Bézier curves in Minkowski $3-$space in [10, 11]. In [12], Ugail, Marquaez and Yılmaz handled the conditions of timelike and spacelike Bézier surfaces. The Serret-Frenet frames, curvatures and torsion of the timelike and spacelike Bézier curves were calculated at the end points in [13–16]. Our aim in this paper is to investigate the timelike and spacelike Bézier curve of degree m at all points.

A classical Bézier curve of degree m with control points p_j is defined as

$$b(t) = \sum_{j=0}^{m} p_j B_j^m(t), t \in [0,1] \tag{1.1}$$

where

$$B_{j,m}(t) = \begin{cases} \frac{m!}{(m-j)!j!}(1-t)^{m-j}t^j, & \text{if } 0 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

are called the Bernstein basis functions of degree m. The polygon formed by joining the control points $p_0, p_1, ..., p_m$ in the specified order is called the Bézier control polygon.

If a curve is differentiable at its each point in an open interval, in this case a set of orthogonal unit vectors can be obtained. And these unit vectors are called Frenet frame. The rates of these frame vectors along the curve define curvatures of the curves. The set of these vectors and curvatures of a curve, is called Frenet apparatus of the curve.

Email address and ORCID number: **ayilmazceylan@akdeniz.edu.tr, 0000-0002-8051-2879 (A. Yılmaz Ceylan)**
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Theorem 1.1 ([14]). *Let $\vec{u},\vec{v}$ and $\vec{w}$ vectors in E_1^3. Then*

(i) $< u\wedge_{IL} v, w >= -\det(u,v,w)$,
(ii) $(u\wedge_{IL} v)\wedge_{IL} w = -< u,w > v + < v,w > u$,
(iii) $< u\wedge_{IL} v, u >= 0$ *and* $< u\wedge_{IL} v, v >= 0$,
(iv) $< u\wedge_{IL} v, u\wedge_{IL} v >= -< u,u >< v,v > +(< u,v >)^2$.

Let β be a curve in E_1^3. Then β is called timelike (resp. spacelike, null) at t, if the tangent vector $\beta'(t)$ is a timelike (resp.cspacelike, null) vector.

Theorem 1.2 ([17]). *For a regular curve β with speed $v = \frac{ds}{dt}$, and curvature $\kappa > 0$,*

(i) β is spacelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$\begin{aligned} T' &= v\kappa N, \\ N' &= v(-\delta\kappa T + \tau B), \\ B' &= v\tau N. \end{aligned}$$

(ii) β is timelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$\begin{aligned} T' &= v\kappa N, \\ N' &= v(\kappa T + \tau B), \\ B' &= v\tau N. \end{aligned}$$

Theorem 1.3. *Let $\vec{u}$ and $\vec{v}$ be vectors in Minkowski 3−space.*

(i) If $\vec{u}$ and $\vec{v}$ are future pointing (or past pointing) timelike vectors, then $\vec{u}\wedge_{IL}\vec{v}$ is a spacelike vector, $<\vec{u},\vec{v}>= -\|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cosh\theta$ and $\|\vec{u}\wedge_{IL}\vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.
(ii) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|<\vec{u},\vec{v}>| < \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u}\wedge_{IL}\vec{v}$ is timelike vector, $<\vec{u},\vec{v}>= \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cos\theta$ and $\|\vec{u}\wedge_{IL}\vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.
(iii) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|<\vec{u},\vec{v}>| > \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u}\wedge_{IL}\vec{v}$ is timelike vector, $<\vec{u},\vec{v}>= -\|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cosh\theta$ and $\|\vec{u}\wedge_{IL}\vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.
(iv) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|<\vec{u},\vec{v}>| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u}\wedge_{IL}\vec{v}$ is lightlike.

See more [1, 2, 18, 19].

Theorem 1.4 ([8, 14]). *Let $b(t)$ be a Bézier curve. If all the vectors of the Bézier control polygon is spacelike (timelike), then $b(t)$ is spacelike (timelike) curve.*

Definition 1.5. *Timelike Bézier curves and spacelike Bézier curves with spacelike or timelike normal vectors are called Frenet Bézier curves.*

2. Main Results

2.1. Timelike Bézier curves

In this section, we give Serret-Frenet frame, curvature and torsion of timelike Bézier curves.

Theorem 2.1. *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ is given by*

$$T(t) = \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\left(-\sum_{j,i=0}^{m-1} B_j^{m-1}(t)B_i^{m-1}(t) < \triangle p_j, \triangle p_i >\right)^{\frac{1}{2}}}, \tag{2.1}$$

$$N(t) = -\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-1} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\wedge_{IL}\triangle p_k}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}\|\sum_{k=0}^{m-1} B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.2}$$

$$B(t) = \frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}, \tag{2.3}$$

$$\kappa(t) = \frac{m-1}{m}\frac{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.4}$$

$$\tau(t) = -\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)det(\triangle p_j, \triangle^2 p_i, \triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}^2}, \tag{2.5}$$

where $\triangle p_j$ are in the same cone, $\triangle p_j = p_{j+1} - p_j$, $\triangle^2 p_j = \triangle p_{j+1} - \triangle p_j$ and $\triangle^3 p_j = \triangle^2 p_{j+1} - \triangle^2 p_j$.

Proof. Since all the vectors $\triangle p_j$ are timelike vectors, the norm of $\triangle p_j$ is

$$\|\triangle p_j\|_{IL} = \sqrt{-<\triangle p_j, \triangle p_j>} \tag{2.6}$$

for $t \in [0,1]$. The tangent vector is calculated as:

$$\begin{aligned} T(t) &= \frac{b'(t)}{\|b'(t)\|_{IL}} \\ &= \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}}. \end{aligned} \tag{2.7}$$

From the equation (2.6) and (2.7), the equation (2.1) is handled.
The binormal vector is obtained by

$$\begin{aligned} B(t) &= \frac{b'(t)\wedge_{IL} b''(t)}{\|b'(t)\wedge_{IL} b''(t)\|_{IL}} \\ &= \frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)\|_{IL}}. \end{aligned}$$

Since the tangent **T** of the timelike Bézier curve is timelike, **N** and **B** are spacelike vectors, the principal normal vector **N** is provided by

$$\begin{aligned} N(t) &= -B(t)\wedge_{IL} T(t) \\ &= -\frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)\|_{IL}} \wedge_{IL} \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}}. \end{aligned}$$

The curvature of timelike Bézier curve is

$$\begin{aligned} \kappa(t) &= \frac{\|b'(t)\wedge_{IL} b''(t)\|_{IL}}{\|b'(t)\|_{IL}^3} \\ &= \frac{m-1}{m}\frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}^3}. \end{aligned}$$

and the torsion of timelike Bézier curve is

$$\begin{aligned} \tau(t) &= \frac{<b'(t)\wedge_{IL} b''(t), b'''(t)>}{\|b'(t)\wedge_{IL} b''(t)\|_{IL}} \\ &= \frac{m-2}{m}\frac{<\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j \wedge_{IL} \sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_j, \sum_{k=0}^{m-3} B_k^{m-3}(t)\triangle^3 p_k>}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle^2 p_i)\|_{IL}}. \end{aligned}$$

□

From the Theorem 1.3 and Theorem 2.1, the following results can be handled.

Corollary 2.2 ([16]). *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned}
T(0) &= \frac{\triangle p_0}{\sqrt{-<\triangle p_0, \triangle p_0>}},\\
N(0) &= \frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\coth\theta - \frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc h\theta,\\
B(0) &= \frac{\triangle p_0 \wedge_{IL} \triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sinh\theta},\\
\kappa(0) &= \frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2},\\
\tau(0) &= -\frac{m-2}{m}\frac{\det(\triangle p_0, \triangle p_1, \triangle p_2)}{\|\triangle p_0 \wedge_{IL} \triangle p_1\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.3 ([16]). *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ at $t=1$ is given by*

$$\begin{aligned}
T(1) &= \frac{\triangle p_{m-1}}{\sqrt{-<\triangle p_{m-1}, \triangle p_{m-1}>}},\\
N(1) &= \frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta - \frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,\\
B(1) &= -\frac{\triangle p_{m-1} \wedge_{IL} \triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},\\
\kappa(1) &= \frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1) &= \frac{m-2}{m}\frac{\det(\triangle p_{m-1}, \triangle p_{m-2}, \triangle p_{m-3})}{\|\triangle p_{m-1} \wedge_{IL} \triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_{m-2}$ and $\triangle p_{m-1}$.

2.2. Spacelike Bézier Curves

In this section, we calculate Serret-Frenet frame, curvature and torsion of spacelike Bézier curves with spacelike and timelike normals.

2.2.1. Spacelike Bézier Curves with Spacelike normal

In this subsection, we calculate Frenet apparatus of a spacelike Bézier curve with spacelike normal.

Theorem 2.4. *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ is given by*

$$T(t) = \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\left(\sum_{j,i=0}^{m-1} B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j, \triangle p_i>\right)^{\frac{1}{2}}}, \tag{2.8}$$

$$N(t) = -\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-1} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i) \wedge_{IL} \triangle p_k}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}\|\sum_{k=0}^{m-1} B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.9}$$

$$B(t) = \frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}, \tag{2.10}$$

$$\kappa(t) = \frac{m-1}{m}\frac{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.11}$$

$$\tau(t) = -\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)det(\triangle p_j, \triangle^2 p_i, \triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}^2}, \tag{2.12}$$

where $\triangle p_j$ are in the same cone.

Proof. Since all the vectors $\triangle p_j$ are spacelike vectors, the norm of $\triangle p_j$ is

$$\|\triangle p_j\|_{IL} = \sqrt{< \triangle p_j, \triangle p_j >}. \tag{2.13}$$

for $t \in [0,1]$. The tangent vector is calculated as:

$$\begin{aligned} T(t) &= \frac{b'(t)}{\|b'(t)\|_{IL}} \\ &= \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{(\sum_{j,i=0}^{m-1} B_j^{m-1}(t)B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}}}. \end{aligned} \tag{2.14}$$

From the equation (2.13) and (2.14), the equation (2.8) is handled.
Since the tangent **T, N** spacelike and **B** is timelike, **N** is given by the equation

$$\mathbf{N} = \mathbf{B} \wedge_{IL} \mathbf{T}.$$

The rest of the proof is similar to Theorem 2.1. □

From the Theorem 1.3 and Theorem 2.4, the following results can be seen easily.

Corollary 2.5 ([13]). *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The tangent vector $\boldsymbol{T}$ of $b(t)$ at $t=0$ is given by*

$$T(0) = \frac{\triangle p_0}{\sqrt{< \triangle p_0, \triangle p_0 >}},$$

*If the inequality $|< \triangle p_0, \triangle p_1 >|_{IL} < \|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, **N,B**, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned} N(0) &= \frac{\triangle p_1}{\|\triangle p_1\|_{IL}} \csc\theta - \frac{\triangle p_0}{\|\triangle p_0\|_{IL}} \cot\theta, \\ B(0) &= \frac{\triangle p_0 \wedge_{IL} \triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL} \sin\theta}, \\ \kappa(0) &= \frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sin\theta}{\|\triangle p_0\|_{IL}^2}, \\ \tau(0) &= -\frac{m-2}{m}\frac{\det(\triangle p_0, \triangle p_1, \triangle p_2)}{\|\triangle p_0 \wedge_{IL} \triangle p_1\|_{IL}^2}, \end{aligned}$$

*and if the inequality $|< \triangle p_0, \triangle p_1 >|_{IL} > \|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, **N,B**, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned} N(0) &= \frac{\triangle p_1}{\|\triangle p_1\|_{IL}} \csc h\theta + \frac{\triangle p_0}{\|\triangle p_0\|_{IL}} \coth\theta, \\ B(0) &= \frac{\triangle p_0 \wedge_{IL} \triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL} \sinh\theta}, \\ \kappa(0) &= \frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2}, \\ \tau(0) &= -\frac{m-2}{m}\frac{\det(\triangle p_0, \triangle p_1, \triangle p_2)}{\|\triangle p_0 \wedge_{IL} \triangle p_1\|_{IL}^2}, \end{aligned}$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.6 ([13]). *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The tangent vector $\boldsymbol{T}$ of $b(t)$ at $t=1$ is given by*

$$T(1) = \frac{\triangle p_{m-1}}{\sqrt{< \triangle p_{m-1}, \triangle p_{m-1} >}}.$$

If the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}<\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ *holds for* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*, **N,B**,* κ *and* τ *of* $b(t)$ *at* $t=1$ *is given by*

$$\begin{aligned}
N(1) &= -\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc\theta+\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\cot\theta,\\
B(1) &= -\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sin\theta},\\
\kappa(1) &= \frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sin\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1) &= \frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}>\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ *holds for* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*, **N,B**,* κ *and* τ *of* $b(t)$ *at* $t=1$ *is given by*

$$\begin{aligned}
N(1) &= -\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta-\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,\\
B(1) &= -\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},\\
\kappa(1) &= \frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1) &= \frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

where θ *is the angle between* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*.*

2.2.2. Spacelike Bézier curves with timelike normal

In this subsection, we calculate Frenet apparatus of a spacelike Bézier curve with timelike normal.

Theorem 2.7. *Let* $b(t)$ *be a spacelike Bézier curve with timelike normal and* p_j *are control points. The Serret-Frenet frame **T,N,B**, curvature* κ *and torsion* τ *of* $b(t)$ *is given by*

$$T(t)=\frac{\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j}{\left(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>\right)^{\frac{1}{2}}}, \tag{2.15}$$

$$N(t)=\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-1}B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\wedge_{IL}\triangle p_k}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}\|\sum_{k=0}^{m-1}B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.16}$$

$$B(t)=\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}, \tag{2.17}$$

$$\kappa(t)=\frac{m-1}{m}\frac{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}{\|\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.18}$$

$$\tau(t)=-\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3}B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)\det(\triangle p_j,\triangle^2 p_i,\triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}^2}, \tag{2.19}$$

where $\triangle p_i$ *and* $\triangle p_j$ *are in the same cone.*

Proof. Since the tangent **T, B** spacelike and **N** is timelike, **N** is given by the equation

$$\mathbf{N}=\mathbf{B}\wedge_{IL}\mathbf{T}.$$

The rest of the proof is similar to Theorem 2.4. □

From the Theorem 1.3 and Theorem 2.7, the following results can be obtained.

Corollary 2.8 ([15]). *Let $b(t)$ be a spacelike Bézier curve with timelike normal and p_j are control points. The tangent vector* ***T*** *of $b(t)$ at $t=0$ is given by*

$$T(0)=\frac{\triangle p_0}{\sqrt{<\triangle p_0,\triangle p_0>}}.$$

If the inequality $|<\triangle p_0,\triangle p_1>|_{IL}<\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, ***N,B****, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned}
N(0)&=-\frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc\theta+\frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\cot\theta,\\
B(0)&=\frac{\triangle p_0\wedge_{IL}\triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sin\theta},\\
\kappa(0)&=\frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sin\theta}{\|\triangle p_0\|_{IL}^2},\\
\tau(0)&=-\frac{m-2}{m}\frac{\det(\triangle p_0,\triangle p_1,\triangle p_2)}{\|\triangle p_0\wedge_{IL}\triangle p_1\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_0,\triangle p_1>|_{IL}>\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, ***N,B****, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned}
N(0)&=-\frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc h\theta-\frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\coth\theta,\\
B(0)&=\frac{\triangle p_0\wedge_{IL}\triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sinh\theta},\\
\kappa(0)&=\frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2},\\
\tau(0)&=-\frac{m-2}{m}\frac{\det(\triangle p_0,\triangle p_1,\triangle p_2)}{\|\triangle p_0\wedge_{IL}\triangle p_1\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.9 ([15]). *Let $b(t)$ be a spacelike Bézier curve with timelike normal and p_j are control points. The tangent vector* ***T*** *of $b(t)$ at $t=1$ is given by*

$$T(1)=\frac{\triangle p_{m-1}}{\sqrt{<\triangle p_{m-1},\triangle p_{m-1}>}}.$$

If the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}<\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ holds for $\triangle p_{m-2}$ and $\triangle p_{m-1}$, ***N,B****, κ and τ of $b(t)$ at $t=1$ is given by*

$$\begin{aligned}
N(1)&=\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc\theta-\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\cot\theta,\\
B(1)&=-\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sin\theta},\\
\kappa(1)&=\frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sin\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1)&=\frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}>\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ holds for $\triangle p_{m-2}$ and $\triangle p_{m-1}$, ***N,B****, κ and τ of $b(t)$ at $t=1$ is given by*

$$\begin{aligned}
N(1)&=\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta+\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,\\
B(1)&=-\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},\\
\kappa(1)&=\frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1)&=\frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_{m-2}$ and $\triangle p_{m-1}$.

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References

[1] B. O'Neill, *Semi-Riemann Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[2] R. Lopez, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom., **7**(1) (2014), 44-107.
[3] G. Farin, *History of Curves and Surfaces in CAGD, in: Handbook of CAGD*, 2002.
[4] D. Marsh, *Applied Geometry for Computer Graphics and CAD*, Springer- Verlag, Berlin, 2nd ed, 2005.
[5] M. İncesu, O. Gürsoy, *Bézier Eğrilerinde Esas Formlar ve Eğrilikler*, *XVII Ulusal Matematik Sempozyumu*, Bildiriler, Abant İzzet Baysal Üniversitesi, (2004), 146-157.
[6] G. H. Geoergiev, Shapes of Plane Bézier Curves in Curve and Surface Design, Avignon, edited by P. Chenin, T. Lyche and L. L Schumaker, Nashboro Prees, Brentwood, TN, (2006) 143-152.
[7] G. H. Georgiev, On the shape of the cubic Bézier Curve, Proc. Of International Congress Pure and Applied Differential Geometry- Brussels, edited by F. Dillen and I. Van de Woestyne, Shaker Verlag, Aachen,(2007), 98-106.
[8] G. H. Georgiev, Spacelike Bézier curves in the three-dimensional Minkowski space, Proceedings of AIP Conference, **1067**(1) (2008), 373-380.
[9] G. H. Georgiev, Constructions of spacelike Bézier surfaces in the three-dimensional Minkowski space, Proceedings of AIP Conference **1184** (1)(2009), 199-206.
[10] P. Chalmoviansky, B. Pokorna, Quadratic space-like Bézier curves in the three dimensional Minkowski space, Proceeding of Symposium on Computer Geometry, **20** (2011), 104-110.
[11] B. Pokorna, P. Chalmoviansky, Planar cubic spacelike Bézier curves in three dimensional Minkowski space, Proceeding of Symposium on Computer Geometry, **23** (2012), 93-98.
[12] H. Ugail, M. C. Marquez, A. Yilmaz, *On Bézier surfaces in three-dimensional Minkowski space*, Computers and Mathematics with Applications, **62** (2011), 2899-2912.
[13] H. Kuşak Samancı, S. Çelik, *The curvatures of the spacelike Bézier curve with a spacelike principal normal in Minkowski* $3-$*Space*, International Journal of Mathematics and Computation, **29**(4) (2018).
[14] S. Çelik, *Investigation of Bézier curves and surfaces from computer aided geometric design elements in Minkowski space*, Master Thesis, Bitlis Eren University, 2017.
[15] H. Kuşak Samancı, *Some geometric properties of the spacelike Bézier curve with a timelike principal normal in Minkowski* $3-$*space*, Cumhuriyet Science Journal, **39**(1) (2018), 71-79.
[16] H. Kuşak Samancı, Ö. Kalkan, S. Çelik, *The timelike Bézier spline in Minkowski* $3-$*space*, Journal of Science and Art,**47**(2) (2019), 357-374.
[17] Y. Ünlütürk, Z. Savcı, *On non-unit speed curves in Minkowski* $3-$*space*, Scientia Magna, **8**(4) (2012), 66-74.
[18] S. Birman, K. Nomuzi, *Trigonometry in Lorentzian geometry*, Am. Math. Monthly, **91** (1984).
[19] W. H. Greub, *Linear Algebra*, Academic Press, New York, 434 p., 1963.