INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 196–200 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1239222

A Note on Yamabe Solitons on 3-dimensional Almost Kenmotsu Manifolds with $Q\phi = \phi Q$

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In the present paper, we prove that if the metric of a three dimensional almost Kenmotsu manifold with $\mathbf{Q}\phi = \phi\mathbf{Q}$ whose scalar curvature remains invariant under the chracterstic vector field ζ and the divergence of the scalar curvature vanishes, admits a Yamabe soliton, then either the soliton is trivial or the manifold is of constant sectional curvature.

Keywords: *Yamabe solitons, almost Kenmotsu manifolds, sectional Curvature.*

AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C25.*

1. Introduction

In a Riemannian manifold $\mathcal{N}^{2m+1}$, the metric g is a Yamabe soliton if it allows a smooth vector field W such that

$$\pounds_W g = (\lambda - r)g, \tag{1.1}$$

where λ is a real constant and r represents the scalar curvature of g and $\pounds$ indicates the Lie-derivative operator.

The Ricci flow and Yamabe flow both were first introduced by Hamilton [11]. A given manifold is deformed due to alternation in its metric as per the equation $\frac{\partial}{\partial t}g(t) = -r(t)g(t)$, where $r(t)$ stands for the scalar curvature of the metric $g(t)$. Yamabe solitons are equivalent to the Yamabe flow's self-similar solutions. The Ricci flow, which is described by $\frac{\partial}{\partial t}g(t) = -2\mathbf{S}(t)$, is comparable to the Yamabe flow in two dimensions, where **S** indicates the Ricci tensor. Moreover, the Yamabe and Ricci flows disagree in the dimension of greater than 2.

Equation (1.1) becomes

$$Hess f = \frac{1}{2}(\lambda - r)g, \tag{1.2}$$

for a Yamabe soliton if a smooth function f satisfies $W = Df$, where the Hessian of f is denoted by $Hess\ f$ and D is the gradient operator of g. In this instance, g is referred to as a gradient Yamabe soliton and f is referred to as the potential function of the Yamabe soliton. We call a Yamabe soliton or a gradient Yamabe soliton to be trivial when W is Killing or f is constant respectively. Numerous authors, including Blaga [2], Calvaruso [5], Sharma [14], Chen and Deshmukh ([3], [4]), Wang ([16], [19]), Suh and De [15] and many more, have examined Yamabe solitons.

In 2016, Wang [16] researched Yamabe solitons on a three-dimensional Kenmotsu manifold. Recently, Wang [19] have been characterized Yamabe solitons in $(k, \mu)'$- almost Kenmotsu manifolds and proved that if the metric g of a $(k, \mu)'$- almost Kenmotsu manifold represents a Yamabe soliton, then either the manifold is locally isometric to the product space $\mathbf{H}^{n+1}(-4) \times \mathbb{R}^n$ or ζ is a infinitesimal contact transformation. The aforementioned investigations served as an inspiration for the current paper, which examines Yamabe solitons on a 3-dimensional almost Kenmotsu manifold with $\phi\mathbf{Q} = \mathbf{Q}\phi$, $\mathbf{Q}$ is the Ricci operator defined by $g(\mathbf{Q}T, U) = \mathbf{S}(T, U)$, where $\mathbf{S}$ is the Ricci tensor of type $(0, 2)$.

The present paper is set up as follows : After preliminaries in Section 3 we prove the main Theorem of the paper. We specifically demonstrate the following :

Received : *19–01–2023,* ***Accepted :*** *04–03–2023*
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Theorem 1.1. *Let $\mathcal{N}$ be a 3-dimensional almost Kenmotsu manifold with $\mathbf{Q}\phi = \phi\mathbf{Q}$, the scalar curvature remains invariant under the characterstic vector field ζ and divergence of r vanishes. Then either the soliton is trivial or the manifold is of constant sectional curvature.*

2. Preliminaries

Suppose $\mathcal{N}$ be a differentiable manifold of dimension $(2m+1)$. Assume that (ϕ, ζ, π, g) is an almost contact metric structure on $\mathcal{N}$. This means that (ϕ, ζ, π, g) is a quadruple made up of a $(1,1)$-tensor field ϕ, an associated vector field ζ, a 1-form π and a Riemannian metric g on $\mathcal{N}$ satisfying the following requirements

$$\phi^2(T) = -T + \pi(T)\zeta, \quad \pi(\zeta) = 1, \quad g(\phi T, \phi U) = g(T, U) - \pi(T)\pi(U), \tag{2.1}$$

where T, U are smooth vector fields on $\mathcal{N}$. In addition, we have

$$\phi\zeta = 0, \ \pi(\phi T) = 0, \ g(T, \zeta) = \pi(T), \ g(\phi T, U) = -g(T, \phi U). \tag{2.2}$$

An almost contact structure with a suitable Riemannian metric is referred to as a "almost contact metric structure." Furthermore, an almost contact metric manifold is one that has an almost contact metric structure. $\Phi(T, U) = g(T, \Phi U)$ for any smooth vector fields T, U defines the fundamental-2 form Φ on a almost contact metric manifold. The vanishing of the $(1,2)$-type torsion tensor N_ϕ, which is defined as $N_\phi = [\phi, \phi] + 2d\pi \otimes \zeta$, is the prerequisite for an almost contact mtetric manifold to be considered normal. In ([8], [9]), the authors have studied almost contact metric manifolds, also known as almost Kenmotsu manifolds, when π is closed and $d\Phi = 2\pi \wedge \Phi$. A normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be described by $(\nabla_T \phi)U = g(\phi T, U)\zeta - \pi(T)\phi U$, for any vector fields T, U. Kenmotsu manifolds was invented by Kenmotsu [13]. Later, the concept of almost Kenmotsu manifolds was first put up by Janssens and Vanhecke [12] as a generalization of Kenmotsu manifolds. Researchers such as Dileo and pastore ([8], [9]), De et el. ([6], [7]), Wang et el. [18], Wang [20] and many others examined almost Kenmotsu manifold.

Let $\mathcal{N}^{2m+1}$ be an almost Kenmotsu manifold. We denote the two symmetric operator h and l such that $h = \frac{1}{2}\pounds_\zeta \phi$ and $l = R(\cdot, \zeta)\zeta$ on $\mathcal{N}^{2m+1}$. The operators h and l satisfy the following relations [8]:

$$h\zeta = 0, \ l\zeta = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla_T \zeta = -\phi^2 T - \phi h T (\Rightarrow \nabla_\zeta \zeta = 0), \tag{2.4}$$

$$trl = \mathbf{S}(\zeta, \zeta) = g(\mathbf{Q}\zeta, \zeta) = -2n - trh^2, \tag{2.5}$$

where "tr" indicates trace.

In ([21]), the authors deduce the expression of the Ricci operator in a 3-dimensional almost Kenmotsu manifold with $\phi\mathbf{Q} = \mathbf{Q}\phi$ which is given by

$$\mathbf{Q}T = \frac{r - trl}{2}T + \frac{3trl - r}{2}\pi(T)\zeta. \tag{2.6}$$

3. Yamabe solitons on 3-dimensional almost Kenmotsu manifolds with $\mathbf{Q}\phi = \phi\mathbf{Q}$

In this section, we charaterize the Yamabe solitons in 3-dimensional almost Kenmtsu manifolds with $\mathbf{Q}\phi = \phi\mathbf{Q}$. The potential vector field W for Yamabe solitons is a conformal vector field, where $\pounds_W g = 2\alpha g$ and α is called the conformal coefficient. When α is a conformal coefficient then from (1.1), we acquire $\alpha = \frac{\lambda - r}{2}$. A conformal vector field in particular reduces to a Killing vector field when the conformal coefficient vanishes.

Here we first state the following Lemma:

Lemma 3.1. *[22] Suppose $(\mathcal{N}, g)$ be an (2m+1)-dimensional Riemannian manifold endowed with a conformal vector field W, then we have*

$$(\pounds_W \mathbf{S})(T, V) = -(2m-1)g(\nabla_T D\alpha, V) + (\Delta\alpha)g(T, V), \tag{3.1}$$

$$\pounds_W r = -2\alpha r + 4m\Delta\alpha, \tag{3.2}$$

for any vector fields T, V, where D represents the gradient operator and $\Delta = -divD$ indicates the Laplacian operator of g.

Now we prove the next Lemma as follows :

Lemma 3.2. *Suppose $\mathcal{N}$ be an almost Kenmotsu manifold of dimension 3 with $\mathbf{Q}\phi = \phi\mathbf{Q}$. If the metric g is a Yamabe soliton, then*

$$\pi(\pounds_W\zeta) = -(\pounds_W\pi)\zeta = \frac{r-\lambda}{2}. \tag{3.3}$$

Proof. We are aware that $g(\zeta,\zeta) = 1$. Applying the Lie-derivative of this relation along the vector field W and using (1.1) and (2.4), we obtain

$$\pi(\pounds_W\zeta) = -(\pounds_W\pi)\zeta = \frac{r-\lambda}{2}. \tag{3.4}$$

This completes the Lemma's proof. □

Now we prove our main theorem.

Proof of the main Theorem 1.1. :

Making use of $\alpha = \frac{\lambda-r}{2}$ and $m = 1$ in (3.1) and (3.2), we get

$$(\pounds_W\mathbf{S})(T,U) = \frac{1}{2}g(\nabla_T Dr, U) - \frac{1}{2}\Delta r g(T,U), \tag{3.5}$$

and

$$\pounds_W r = r(r-\lambda) - 2\Delta r. \tag{3.6}$$

Taking Lie derivative of (2.6) along the vector field W, we get

$$\begin{aligned}(\pounds_W\mathbf{S})(T,U) = {} & \frac{1}{2}\pounds_W r[g(T,U) - \pi(T)\pi(U)] + \frac{1}{2}(r - trl)(\pounds_W g)(T,U) \\ & + \frac{1}{2}(3trl - r)[((\pounds_W\pi)T)\pi(U) + ((\pounds_W\pi)U)\pi(T)] \\ & + \frac{1}{2}(\pounds_W trl)[-g(T,U) + 3\pi(T)\pi(U)].\end{aligned} \tag{3.7}$$

In view of (3.5) and (3.7), we have

$$\begin{aligned}& \frac{1}{2}g(\nabla_T Dr, U) - \frac{1}{2}\Delta r g(T,U) \\ & = \frac{1}{2}(\pounds_W r)[g(T,U) - \pi(T)\pi(U)] + \frac{1}{2}(r - trl)(\pounds_W g)(T,U) \\ & + \frac{1}{2}(3trl - r)[((\pounds_W\pi)T)\pi(U) + ((\pounds_W\pi)U)\pi(T)] \\ & + \frac{1}{2}(\pounds_W trl)[-g(T,U) + 3\pi(T)\pi(U)].\end{aligned} \tag{3.8}$$

Making use of (1.1) and (3.6) in (3.8) yields

$$\begin{aligned}& g(\nabla_T Dr, U) - \Delta r g(T,U) = [r(r-\lambda) - 2\Delta r][g(T,U) - \pi(T)\pi(U)] \\ & + (r - trl)(\lambda - r)g(T,U) + (3trl - r)[((\pounds_W\pi)T)\pi(U) + ((\pounds_W\pi)U)\pi(T)] \\ & + (\pounds_W trl)[-g(T,U) + 3\pi(T)\pi(U)].\end{aligned} \tag{3.9}$$

Replacing T and U both by ζ in the last equation we infer

$$\zeta(\zeta(r)) = \Delta r + (r - trl)(\lambda - r) + (3trl - r)(r - \lambda) + 2(\pounds_W trl). \tag{3.10}$$

Let us assume that the scalar curvature r is invariant along the Reeb vector field ζ and the divergence of r vanishes, that is, $\zeta(r) = 0$ and $\Delta r = 0$. Then from (3.10), it follows that

$$\pounds_W(trl) = (r-\lambda)(r - 2trl). \tag{3.11}$$

Utilizing (3.11) in (3.10) yields

$$\begin{aligned}& g(\nabla_T Dr, U) = r(r-\lambda)[g(T,U) - \pi(T)\pi(U)] \\ & + (r - trl)(\lambda - r)g(T,U) + (3trl - r)[((\pounds_W\pi)T)\pi(U) \\ & + (r-\lambda)(r - 2trl)[-g(T,U) + 3\pi(T)\pi(U)].\end{aligned} \tag{3.12}$$

Considering a local orthonormal basis $\{b_i : i = 1, 2, 3\}$ of tangent space at each point of the manifold $\mathcal{N}^3$. Substituting $T = U = b_i$ in (3.12) and taking summation over $i : 1 \leq i \leq 3$, we acquire

$$0 = 2r(r - \lambda) + 3(r - trl)(\lambda - r) + (3trl - r)(r - \lambda),$$

which implies,

$$0 = (r - \lambda)(3trl - r). \tag{3.13}$$

Therefore, either $r = \lambda$ or $r = 3trl$. If $r = \lambda$ then from (1.1) we get W is Killing, here the soliton is trivial. When $r = 3trl$, then from (2.6) the manifold becomes Einstein one. Since the manifold is of 3-dimensional, hence the manifold is of constant sectional curvature.

This finishes the proof of Theorem 1.1.

Acknowledgements

The author is extremely grateful to the referees and editors for reviewing the paper carefully and their valuable comments to improve the quality of the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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