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HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY (α, m)-CONVEX FUNCTIONS BY USING FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish some fractional Hermite-Hadamard type inequalities for harmonically (α, m)-convex functions. Also, we give some applications to special means of positive real numbers by using obtained inequalities.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $a, b \in I$ with $a < b$. Then $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, if and only if following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In the literature, the inequality (1.1) is well known qua Hermite-Hadamard integral inequality for convex functions. Also, it is known that f is a concave function, if and only if the inequality (1.1) hold in the reversed direction.

In [2], İşcan gave the definition of harmonically convex function, he established Hermite-Hadamard inequality for harmonically convex functions, and he obtained some Hermite-Hadamard type inequalities for harmonically convex functions as follows:

Definition 1.1. [2] Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

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Theorem 1.1. [2] *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 1.2. [2] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \tag{1.3}$$

$$\leq \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q\right]^{1/q}$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$

$$\lambda_2 = \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$

$$\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right) = \lambda_1 - \lambda_2.$$

Theorem 1.3. [2] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \tag{1.4}$$

$$\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left[\mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q\right]^{1/q}$$

where

$$\mu_1 = \frac{\left[a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]\right]}{2(b-a)^2(1-q)(1-2q)},$$

$$\mu_2 = \frac{\left[b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]\right]}{2(b-a)^2(1-q)(1-2q)}.$$

In [9], Mihaşen gave the definition of (α, m)-convex functions as follows:

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If one choose $(\alpha, m) \in \{(0,0), (\alpha, 0), (1,0), (1,m), (1,1), (\alpha, 1)\}$, then one has the following classes of functions respectively: increasing, α-starshaped, starshaped, m-convex, convex, α-convex.

For recent results and generalizations concerning (α, m)-convex functions we refer to the readers the recent papers [1, 3, 4, 6, 7, 10, 11] and references therein.

In [4], İşcan gave the definition of harmonically (α, m)-convex function as follows:

Definition 1.3. The function $f : (0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be harmonically (α, m)-convex, where $\alpha \in [0,1]$ and $m \in (0,1]$, if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y) \tag{1.5}$$

for all $x, y \in (0, b^*]$ and $t \in [0,1]$. If the inequality in (1.5) is reversed, then f is said to be harmonically (α, m)-concave.

Note that, if one choose $(\alpha, m) \in \{(1,m), (1,1), (\alpha, 1)\}$, then one has the following classes of functions respectively: harmonically m-convex, harmonically convex, harmonically α-convex (or harmonically s-convex in the first sense, if one choose s instead of α).

The following Lemma is used for various Theorems in this paper.

Lemma 1.1. [12]. *For $0 < \theta \leq 1$ and $0 \leq a < b$ we have*

$$\left|a^{\theta} - b^{\theta}\right| \leq (b-a)^{\theta}.$$

We recall the following special functions which are known as beta and hypergeometric function respectively.

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x, y > 0,$$

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

$$c > b > 0, \quad |z| < 1 \text{ (see [8])}.$$

Following definitions and mathematical preliminaries of fractional calculus theory are used throughout this paper.

Definition 1.4. [8]. Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\theta} f$ and $J_{b-}^{\theta} f$ of order $\theta > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\theta} f(x) = \frac{1}{\Gamma(\theta)} \int_a^x (x-t)^{\theta-1} f(t)\,dt, \; x > a$$

and

$$J_{b-}^{\theta} f(x) = \frac{1}{\Gamma(\theta)} \int_x^b (t-x)^{\theta-1} f(t)\,dt, \; x < b$$

respectively, where Γ is the Euler Gamma function defined by $\Gamma(\theta) = \int_0^{\infty} e^{-t} t^{\theta-1} dt$.

In [5], the authors presented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 1.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\theta+1)}{2}\left(\frac{ab}{b-a}\right)^{\theta}\left\{\begin{array}{c} J^{\theta}_{1/a-}(f\circ g)(1/b) \\ +J^{\theta}_{1/b+}(f\circ g)(1/a) \end{array}\right\} \leq \frac{f(a)+f(b)}{2}$$

with $\theta > 0$.

Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, throughout this paper we will take

$$I_f(g;\theta,a,b) = \frac{f(a)+f(b)}{2} - \frac{\Gamma(\theta+1)}{2}\left(\frac{ab}{b-a}\right)^{\theta}\left\{\begin{array}{c} J^{\theta}_{1/a-}(f\circ g)(1/b) \\ +J^{\theta}_{1/b+}(f\circ g)(1/a) \end{array}\right\}$$

where $a,b \in I$ with $a < b$, $\theta > 0$, $g(x) = 1/x$.

In [5], the authors gave the following identity for differentiable functions.

Lemma 1.2. [5] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. Then the following equality for fractional integrals holds:*

$$I_f(g;\theta,a,b) = \frac{ab(b-a)}{2}\int_0^1 \frac{t^{\theta}-(1-t)^{\theta}}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt. \tag{1.6}$$

Remark 1.1. The identity (1.6) is equal the following one

$$I_f(g;\theta,a,b) = \frac{ab(b-a)}{2}\int_0^1 \frac{(1-t)^{\theta}-t^{\theta}}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt. \tag{1.7}$$

In this paper, we aim to establish some Hermite-Hadamard type inequalities for harmonically (α,m)-convex functions by using fractional integrals. Our results have some relations with [2]. Also, we aim to give some applications to some special mean of real numbers.

2. MAIN RESULTS

Theorem 2.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^{\circ}$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α,m)-convex on $[a,b/m]$ for some fixed $q \geq 1$, with $\alpha \in [0,1]$, then*

$$|I_f(g;\theta,a,b)| \tag{2.1}$$

$$\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\theta;a,b)\left[\begin{array}{c} C_2(\theta;\alpha;a,b)|f'(a)|^q \\ +mC_3(\theta;\alpha;a,b)|f'(b/m)|^q \end{array}\right]^{1/q}$$

where

$$C_1(\theta;a,b) = \frac{b^{-2}}{\theta+1}\left[\begin{array}{c} {}_2F_1\left(2,\theta+1;\theta+2;1-\frac{a}{b}\right) \\ +\ {}_2F_1\left(2,1;\theta+2;1-\frac{a}{b}\right) \end{array}\right],$$

$$C_2(\theta;\alpha;a,b) = \left[\begin{array}{c} \frac{\beta(\theta+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ +\frac{b^{-2}}{\theta+\alpha+1}\ {}_2F_1\left(2,1;\theta+\alpha+2;1-\frac{a}{b}\right) \end{array}\right],$$

$$C_3(\theta;\alpha;a,b) = C_1(\theta;a,b) - C_2(\theta;\alpha;a,b).$$

Proof. Let $A_t = tb + (1-t)\,a,\ B_u = ua + (1-u)\,b$. By using (1.5) and the harmonically (α, m)-convexity of $|f'|^q$, we have

$$(2.2)\qquad \left|f'\left(\frac{ab}{A_t}\right)\right|^q = \left|f'\left(\frac{ab}{tb+(1-t)\,a}\right)\right|^q = \left|f'\left(\frac{ma\,(b/m)}{mt\,(b/m)+(1-t)\,a}\right)\right|^q$$
$$\leq t^\alpha\,|f'(a)|^q + m\,(1-t^\alpha)\,|f'(b/m)|^q .$$

A combination of (1.7), (2.2), the property of the modulus, and the power mean inequality, we have

$$(2.3)\qquad |I_f(g;\theta,a,b)| \leq \frac{ab\,(b-a)}{2}\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\leq \frac{ab\,(b-a)}{2}\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\leq \frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}dt\right)^{1-1/q}$$
$$\times\left(\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$
$$\leq \frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}dt\right)^{1-1/q}$$
$$\times\left(\begin{array}{c}\left(\int_0^1 \frac{(1-t)^\theta+t^\theta}{A_t^2}t^\alpha dt\right)|f'(a)|^q \\ +m\left(\int_0^1 \frac{(1-t)^\theta+t^\theta}{A_t^2}(1-t^\alpha)\,dt\right)|f'(b/m)|^q\end{array}\right)^{1/q}.$$

Calculating the appearing integrals with hypergeometric function, we have

$$(2.4)\qquad \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}dt = \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_u^2}du$$
$$= \frac{b^{-2}}{\theta+1}\left[\begin{array}{c} {}_2F_1\left(2,\theta+1;\theta+2;1-\frac{a}{b}\right) \\ +\ {}_2F_1\left(2,1;\theta+2;1-\frac{a}{b}\right)\end{array}\right] = C_1(\theta;a,b)\,,$$

$$(2.5)\qquad \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}t^\alpha dt = \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_u^2}(1-u)^\alpha du$$
$$= \left[\begin{array}{c} \frac{\beta(\theta+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ +\frac{b^{-2}}{\theta+\alpha+1}\ {}_2F_1\left(2,1;\theta+\alpha+2;1-\frac{a}{b}\right)\end{array}\right] = C_2(\theta;\alpha;a,b)\,,$$

$$(2.6)\qquad \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}(1-t^\alpha)\,dt = \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}dt - \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2}t^\alpha dt$$
$$= C_1(\theta;a,b) - C_2(\theta;\alpha;a,b) = C_3(\theta;\alpha;a,b)\,.$$

Finally, by using (2.4)-(2.6) in (2.3) then we have (2.1). This completes the proof. □

Corollary 2.1. *In Theorem 2.1, If we choose $\theta = 1$, then we have the following trapezoid type inequality for harmonically (α, m)-convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(1;a,b)\begin{bmatrix} C_2(1;\alpha;a,b)\left|f'(a)\right|^q \\ +mC_3(1;\alpha;a,b)\left|f'(b/m)\right|^q \end{bmatrix}^{1/q}.$$

If we choose $0 < \theta \leq 1$, by using Lemma 1.1 we obtain another result for harmonically (α, m)-convex functions as follows:

Theorem 2.2. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (α, m)-convex on $[a, b/m]$ for some fixed $q \geq 1$, with $\alpha \in [0, 1]$, then*

$$|I_f(g;\theta,a,b)| \leq \frac{ab(b-a)}{2} C_4^{1-1/q}(\theta;a,b)\begin{bmatrix} C_5(\theta;\alpha;a,b)\left|f'(a)\right|^q \\ +mC_6(\theta;\alpha;a,b)\left|f'(b/m)\right|^q \end{bmatrix}^{1/q} \tag{2.7}$$

where $0 < \theta \leq 1$ and

$$C_4(\theta;a,b) = \begin{bmatrix} \frac{b^{-2}}{\theta+1}\ {}_2F_1\left(2,1;\theta+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\theta+1}\ {}_2F_1\left(2,\theta+1;\theta+2;1-\frac{a}{b}\right) \\ +\left(\frac{a+b}{2}\right)^{-2}\frac{1}{\theta+1}\ {}_2F_1\left(2,\theta+1;\theta+2;\frac{b-a}{b+a}\right) \end{bmatrix},$$

$$C_5(\theta;\alpha;a,b) = \begin{bmatrix} \frac{b^{-2}}{\theta+\alpha+1}\ {}_2F_1\left(2,1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\theta+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ +\frac{\beta(\theta+1,\alpha+1)}{(a+b)^2 2^{\alpha-2}}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;\frac{b-a}{b+a}\right) \end{bmatrix},$$

$$C_6(\theta;\alpha;a,b) = C_4(\theta;a,b) - C_5(\theta;\alpha;a,b).$$

Proof. Let $A_t = tb + (1-t)a$, $B_u = ua + (1-u)b$. A combination of (1.7), (2.2), the property of the modulus, and the power mean inequality, we have

$$|I_f(g;\theta,a,b)| \leq \frac{ab(b-a)}{2}\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt \tag{2.8}$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}dt\right)^{1-1/q}$$
$$\times\left(\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}dt\right)^{1-1/q}$$
$$\times\left(\begin{array}{c} \left(\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}t^\alpha dt\right)\left|f'(a)\right|^q \\ +m\left(\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}(1-t^\alpha)dt\right)\left|f'(b/m)\right|^q \end{array}\right)^{1/q}.$$

Calculating appearing integrals with Lemma 1.1 and with hypergeometric function, we have

$$
\begin{aligned}
(2.9)\quad & \int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}dt = \int_0^{1/2} \frac{(1-t)^\theta - t^\theta}{A_t^2}dt + \int_{1/2}^1 \frac{t^\theta - (1-t)^\theta}{A_t^2}dt \\
&= \int_0^1 \frac{t^\theta - (1-t)^\theta}{A_t^2}dt + 2\int_0^{1/2} \frac{(1-t)^\theta - t^\theta}{A_t^2}dt \\
&\le \int_0^1 \frac{t^\theta}{A_t^2}dt - \int_0^1 \frac{(1-t)^\theta}{A_t^2}dt + 2\int_0^{1/2} \frac{(1-2t)^\theta}{A_t^2}dt \\
&= \int_0^1 \frac{(1-u)^\theta}{B_u^2}du - \int_0^1 \frac{u^\theta}{B_u^2}du + \int_0^1 \frac{(1-u)^\theta}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^2}du \\
&= \int_0^1 \frac{(1-u)^\theta}{B_u^2}du - \int_0^1 \frac{u^\theta}{B_u^2}du + \int_0^1 v^\theta \left(\frac{a+b}{2}\right)^{-2}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv \\
&= \left[\begin{array}{c} \frac{b^{-2}}{\theta+1}\ {}_2F_1\left(2,1;\theta+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\theta+1}\ {}_2F_1\left(2,\theta+1;\theta+2;1-\frac{a}{b}\right) \\ +\left(\frac{a+b}{2}\right)^{-2}\frac{1}{\theta+1}\ {}_2F_1\left(2,\theta+1;\theta+2;\frac{b-a}{b+a}\right) \end{array}\right] = C_4(\theta;a,b),
\end{aligned}
$$

$$
\begin{aligned}
(2.10)\quad & \int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}t^\alpha dt \\
&\le \int_0^1 \frac{t^{\theta+\alpha}}{A_t^2}dt - \int_0^1 \frac{(1-t)^\theta t^\alpha}{A_t^2}dt + 2\int_0^{1/2} \frac{(1-2t)^\theta t^\alpha}{A_t^2}dt \\
&= \int_0^1 \frac{(1-u)^{\theta+\alpha}}{B_u^2}du - \int_0^1 \frac{u^\theta (1-u)^\alpha}{B_u^2}du + \int_0^1 \frac{(1-u)^\theta \left(\frac{u}{2}\right)^\alpha}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^2}du \\
&= \int_0^1 \frac{(1-u)^{\theta+\alpha}}{B_u^2}du - \int_0^1 \frac{u^\theta (1-u)^\alpha}{B_u^2}du \\
&+ \frac{\left(\frac{a+b}{2}\right)^{-2}}{2^\alpha}\int_0^1 v^\theta (1-v)^\alpha \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv \\
&= \left[\begin{array}{c} \frac{b^{-2}}{\theta+\alpha+1}\ {}_2F_1\left(2,1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\theta+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;1-\frac{a}{b}\right) \\ +\frac{\beta(\theta+1,\alpha+1)}{(a+b)^2 2^{\alpha-2}}\ {}_2F_1\left(2,\theta+1;\theta+\alpha+2;\frac{b-a}{b+a}\right) \end{array}\right] = C_5(\theta;\alpha;a,b),
\end{aligned}
$$

$$
\begin{aligned}
(2.11)\quad & \int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}(1-t^\alpha)dt \\
&= \int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}dt - \int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}t^\alpha dt \\
&= C_4(\theta;a,b) - C_5(\theta;\alpha;a,b) = C_6(\theta;\alpha;a,b).
\end{aligned}
$$

Finally, by using (2.9)-(2.11) in (2.8), then we have (2.7). This completes the proof. □

Remark 2.1. If we choose $\theta = 1$, $\alpha = 1$, $m = 1$ in Theorem 2.2, then the inequality (2.7) becomes the inequality (1.3) of Theorem 1.2.

Corollary 2.2. *In Theorem 2.2, If we take $\theta = 1$, then we have the following trapezoid type inequality for harmonically (α, m)-convex functions:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$

$$\leq \frac{ab(b-a)}{2} C_4^{1-1/q}(1; a, b) \begin{bmatrix} C_5(1; \alpha; a, b) |f'(a)|^q \\ +mC_6(1; \alpha; a, b) |f'(b/m)|^q \end{bmatrix}^{1/q}.$$

Theorem 2.3. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically (α, m)-convex on $[a, b/m]$ for some fixed $q > 1$, with $\alpha \in [0, 1]$, then*

$$(2.12) \quad |I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left(\frac{1}{\theta p + 1} \right)^{1/p}$$

$$\times \left(\frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q} \begin{bmatrix} {}_2F_1^{1/p}\left(2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b}\right) \\ + {}_2F_1^{1/p}\left(2p, 1; \theta p + 2; 1 - \frac{a}{b}\right) \end{bmatrix}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $A_t = tb + (1-t)a$, $B_u = ua + (1-u)b$. A combination of (1.7), (2.2), the property of the modulus, and the Hölder inequality, we have

$$(2.13) \quad |I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} \begin{bmatrix} \int_0^1 \frac{(1-t)^\theta}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \\ + \int_0^1 \frac{t^\theta}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \end{bmatrix}$$

$$\leq \frac{ab(b-a)}{2} \begin{bmatrix} \left(\int_0^1 \frac{(1-t)^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{1/q} \\ + \left(\int_0^1 \frac{t^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \left(\int_0^1 \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{1/q} \end{bmatrix}$$

$$\leq \frac{ab(b-a)}{2} \left(\left(\int_0^1 \frac{u^{\theta p}}{B_u^{2p}} du \right)^{1/p} + \left(\int_0^1 \frac{(1-u)^{\theta p}}{B_u^{2p}} du \right)^{1/p} \right)$$

$$\times \left(\int_0^1 t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q dt \right)^{1/q}$$

$$= \frac{ab(b-a)}{2} \left(K_1^{1/p} + K_2^{1/p} \right) \left(\frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q}.$$

Calculating K_1 and K_2 with hypergeometric function, we have

$$(2.14) \quad K_1 = \int_0^1 \frac{u^{\theta p}}{B_u^{2p}} du = \frac{b^{-2p}}{\theta p + 1} \; {}_2F_1\left(2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b}\right),$$

$$(2.15) \quad K_1 = \int_0^1 \frac{(1-u)^{\theta p}}{B_u^{2p}} du = \frac{b^{-2p}}{\theta p + 1} \; {}_2F_1\left(2p, 1; \theta p + 2; 1 - \frac{a}{b}\right).$$

Finally, by using (2.14) and(2.15) in (2.13), then we have (2.12). This completes the proof. □

Corollary 2.3. *In Theorem 2.3, If we take $\theta = 1$, then we have the following trapezoid type inequality for harmonically (α, m)-convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{1/p}$$
$$\times\left(\frac{|f'(a)|^q+m\alpha|f'(b/m)|^q}{\alpha+1}\right)^{1/q}\left[\begin{array}{c} {}_2F_1^{1/p}\left(2p,p+1;p+2;1-\frac{a}{b}\right) \\ +\ {}_2F_1^{1/p}\left(2p,1;p+2;1-\frac{a}{b}\right)\end{array}\right].$$

Theorem 2.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α, m)-convex on $[a, b/m]$ for some fixed $q > 1$, with $\alpha \in [0,1]$, then*

$$(2.16)\quad |I_f(g;\theta,a,b)| \le \frac{a(b-a)}{2b}\left(\frac{1}{\theta p+1}\right)^{1/p}\left(\frac{1}{\alpha+1}\right)^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\ |f'(a)|^q \\ +m\left[\begin{array}{c}(\alpha+1)\ {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right) \\ -\ {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\end{array}\right]|f'(b/m)|^q\end{array}\right]^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $A_t = tb + (1-t)a$, $B_u = ua + (1-u)b$. A combination of (1.7), (2.2), the property of the modulus, the Hölder inequality, and Lemma 1.1, we have

$$(2.17)\quad |I_f(g;\theta,a,b)| \le \frac{ab(b-a)}{2}\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1 \left|(1-t)^\theta - t^\theta\right|^p dt\right)^{1/p}\left(\int_0^1 \frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^{\theta p}dt\right)^{1/p}$$
$$\times\left(\int_0^1 \frac{1}{A_t^{2q}}\left[t^\alpha|f'(a)|^q + m(1-t^\alpha)|f'(b/m)|^q\right]dt\right)^{1/q}.$$

Calculating appearing integrals with hypergeometric functions, we have

$$(2.18)\quad \int_0^1 |1-2t|^{\theta p}dt = \frac{1}{\theta p+1},$$

$$(2.19)\quad \int_0^1 \frac{t^\alpha}{A_t^{2q}}dt = \int_0^1 \frac{(1-u)^\alpha}{B_u^{2q}}dt = \frac{b^{-2q}}{\alpha+1}\ {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right),$$

$$(2.20)\quad \int_0^1 \frac{1-t^\alpha}{A_t^{2q}}dt = \int_0^1 \frac{1-(1-u)^\alpha}{B_u^{2q}}dt$$
$$= \left[\begin{array}{c} b^{-2q}\ {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right) \\ -\frac{b^{-2q}}{\alpha+1}\ {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\end{array}\right].$$

Finally, if we use (2.18)-(2.20) in (2.17), then we have (2.16). This completes the proof. □

Remark 2.2. If we take $\theta = 1$, $\alpha = 1$, $m = 1$ in Theorem 2.4, then inequality (2.16) becomes inequality (1.4) of Theorem 1.3.

Corollary 2.4. *In Theorem 2.4, If we take $\theta = 1$, then we have the following trapezoid type inequality for harmonically (α, m)-convex functions:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{\alpha+1}\right)^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\ |f'(a)|^q \\ +m\left[\begin{array}{c}(\alpha+1)\ {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right) \\ -\ {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\end{array}\right]|f'(b/m)|^q\end{array}\right]^{1/q}.$$

Theorem 2.5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b/m \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (α, m)-convex on $[a, b/m]$ for some fixed $q > 1$, with $\alpha \in [0,1]$, then*

$$|I_f(g;\theta,a,b)| \leq \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2}\left(\frac{1}{\theta p+1}\right)^{1/p} \tag{2.21}$$
$$\times\left(\frac{|f'(a)|^q+m\alpha|f'(b/m)|^q}{\alpha+1}\right)^{1/q}\left[\begin{array}{c}{}_2F_1\left(2p,\theta p+1;\theta p+2;\frac{b-a}{b+a}\right) \\ +\ {}_2F_1\left(2p,\theta p+1;\theta p+2;\frac{a-b}{b+a}\right)\end{array}\right]^{1/p}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $A_t = tb + (1-t)a$. A combination of (1.7), (2.2), the property of the modulus, the Hölder inequality, and Lemma 1.1, we have

$$|I_f(g;\theta,a,b)| \leq \frac{ab(b-a)}{2}\int_0^1 \frac{\left|(1-t)^\theta - t^\theta\right|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt \tag{2.22}$$
$$\leq \frac{ab(b-a)}{2}\int_0^1 \frac{|1-2t|^\theta}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|1-2t|^{\theta p}}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|1-2t|^{\theta p}}{A_t^{2p}}dt\right)^{1/p}$$
$$\times\left(\int_0^1 t^\alpha|f'(a)|^q + m(1-t^\alpha)|f'(b/m)|^q dt\right)^{1/q}$$
$$= \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{(1-2t)^{\theta p}}{A_t^{2p}}dt + \int_{1/2}^1 \frac{(2t-1)^{\theta p}}{A_t^{2p}}dt\right]^{1/p}$$
$$\times\left(\frac{|f'(a)|^q+m\alpha|f'(b/m)|^q}{\alpha+1}\right)^{1/q}.$$

Calculating appearing integrals, we have

$$\int_0^{1/2}\frac{(1-2t)^{\theta p}}{A_t^{2p}}dt = \frac{1}{2}\int_0^1 \frac{(1-u)^{\theta p}}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^{2p}}du \tag{2.23}$$
$$= \frac{(a+b)^{-2p}}{2^{1-2p}}\int_0^1 v^{\theta p}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2p}dv$$

$$= \frac{(a+b)^{-2p}}{2^{1-2p}(\theta p+1)}\ {}_2F_1\left(2p, \theta p+1; \theta p+2; \tfrac{b-a}{b+a}\right),$$

$$\text{(2.24)}\quad \int_{1/2}^{1} \frac{(2t-1)^{\theta p}}{A_t^{2p}} dt = \frac{1}{2}\int_1^2 \frac{(u-1)^{\theta p}}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^{2p}} du$$

$$= \frac{(a+b)^{-2p}}{2^{1-2p}} \int_0^1 v^{\theta p}\left(1 - v\left(\frac{a-b}{b+a}\right)\right)^{-2p} dv$$

$$= \frac{(a+b)^{-2p}}{2^{1-2p}(\theta p+1)}\ {}_2F_1\left(2p, \theta p+1; \theta p+2; \tfrac{a-b}{b+a}\right).$$

Finally, if we use (2.23),(2.24) in (2.22), then we have (2.21). This completes the proof. □

Corollary 2.5. *In Theorem 9, If we take $\theta = 1$, then we have the following trapezoid type inequality for harmonically (α, m)-convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2}\left(\frac{1}{p+1}\right)^{1/p}$$

$$\times\left(\frac{|f'(a)|^q + m\alpha\,|f'(b/m)|^q}{\alpha+1}\right)^{1/q}\left[\begin{array}{c} {}_2F_1\left(2p, p+1; p+2; \frac{b-a}{b+a}\right) \\ +\ {}_2F_1\left(2p, p+1; p+2; \frac{a-b}{b+a}\right)\end{array}\right]^{1/p}.$$

Remark 2.3. The inequalities in Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5 obviously holds for harmonically m-convex functions if we take $\alpha = 1$, harmonically α-convex functions if we take $m = 1$, and harmonically convex functions if we take $\alpha = 1$, $m = 1$.

3. Some applications to special means

Let us recall the following special means of positive numbers a, b with $a < b$.

(1) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}.$$

(2) The geometric mean:

$$G = G(a,b) := \sqrt{ab}.$$

(3) The n-logarithmic mean:

$$L_n = L_n(a,b) := \left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}}.$$

Proposition 3.1. *For $0 < a < b$, we have the following inequality*

$$\left|A\left(a^{\frac{\alpha}{q}+1}, b^{\frac{\alpha}{q}+1}\right) - G^2 L_{\frac{\alpha}{q}-1}^{\frac{\alpha}{q}-1}\right| \le \min\{K_1, K_2, K_3, K_4, K_5\}$$

where

$$K_1 = \frac{ab(b-a)}{2}\left(\frac{\alpha}{q}+1\right) C_1^{1-1/q}(1; a, b)\left[\begin{array}{c} C_2(1; \alpha; a, b)\, a^{\alpha} \\ +C_3(1; \alpha; a, b)\, b^{\alpha}\end{array}\right]^{1/q},$$

$$K_2 = \frac{ab(b-a)}{2}\left(\frac{\alpha}{q}+1\right) C_4^{1-1/q}(1; a, b)\left[\begin{array}{c} C_5(1; \alpha; a, b)\, a^{\alpha} \\ +C_6(1; \alpha; a, b)\, b^{\alpha}\end{array}\right]^{1/q},$$

$$K_3=\frac{a(b-a)}{2b}\left(\frac{\alpha}{q}+1\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{a^\alpha+\alpha b^\alpha}{\alpha+1}\right)^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1^{1/p}\left(2p,p+1;p+2;1-\frac{a}{b}\right) \\ +\ {}_2F_1^{1/p}\left(2p,1;p+2;1-\frac{a}{b}\right)\end{array}\right],$$
$$K_4=\frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{\alpha+1}\right)^{1/q}\left(\frac{\alpha}{q}+1\right)$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\ a^\alpha \\ +\left[\begin{array}{c}(\alpha+1)\ {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right) \\ -\ {}_2F_1\left(2q,1;\alpha+2;1-\frac{a}{b}\right)\end{array}\right]b^\alpha\end{array}\right]^{1/q},$$
$$K_5=\frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{\alpha}{q}+1\right)\left(\frac{a^\alpha+\alpha b^\alpha}{\alpha+1}\right)^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2p,p+1;p+2;\frac{b-a}{b+a}\right) \\ +\ {}_2F_1\left(2p,p+1;p+2;\frac{a-b}{b+a}\right)\end{array}\right]^{1/p}.$$

Proof. The proof follows by Corollary 2.1, Corollary 2.2, Corollary 2.3, Corollary 2.4 and Corollary 2.5 respectively, applied for the function $f:(0,+\infty)\to\mathbb{R}$, $f(x)=x^{\frac{\alpha}{q}+1}$, $q>1$, $\alpha\in(0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$. □

Proposition 3.2. *For $0<a<b$, we have the following inequality*

$$\left|A\left(a^{\frac{1}{q}+1},b^{\frac{1}{q}+1}\right)-G^2L_{\frac{1}{q}-1}^{\frac{1}{q}-1}\right|\le\min\{K_6,K_7,K_8,K_9,K_{10}\}$$

where

$$K_6=\frac{ab(b-a)}{2}\left(\frac{1}{q}+1\right)C_1^{1-1/q}(1;a,b)\left[\begin{array}{c}C_2(1;1;a,b)\,a \\ +C_3(1;1;a,b)\,b\end{array}\right]^{1/q},$$
$$K_7=\frac{ab(b-a)}{2}\left(\frac{1}{q}+1\right)C_4^{1-1/q}(1;a,b)\left[\begin{array}{c}C_5(1;1;a,b)\,a \\ +C_6(1;1;a,b)\,b\end{array}\right]^{1/q},$$
$$K_8=\frac{a(b-a)}{2b}\left(\frac{1}{q}+1\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{a+b}{2}\right)^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1^{1/p}\left(2p,p+1;p+2;1-\frac{a}{b}\right) \\ +\ {}_2F_1^{1/p}\left(2p,1;p+2;1-\frac{a}{b}\right)\end{array}\right],$$
$$K_9=\frac{a(b-a)}{2b}\left(\frac{1}{q}+1\right)\left(\frac{1}{p+1}\right)^{1/p}\frac{1}{2}^{1/q}$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)\ a \\ +\left[\begin{array}{c}2\ {}_2F_1\left(2q,1;2;1-\frac{a}{b}\right) \\ -\ {}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)\end{array}\right]b\end{array}\right]^{1/q},$$
$$K_{10}=\frac{ab(b-a)}{(a+b)^{2-\frac{1}{q}}}\left(\frac{1}{q}+1\right)\left(\frac{1}{p+1}\right)^{1/p}$$
$$\times\left[\begin{array}{c} {}_2F_1\left(2p,p+1;p+2;\frac{b-a}{b+a}\right) \\ +\ {}_2F_1\left(2p,p+1;p+2;\frac{a-b}{b+a}\right)\end{array}\right]^{1/p}.$$

Proof. The proof follows by Corollary 2.1, Corollary 2.2, Corollary 2.3, Corollary 2.4 and Corollary 2.5 respectively, applied for the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \frac{x^{\frac{1}{q}+1}}{\left(\frac{1}{q}+1\right)m^{\frac{1}{q}}}$, $q > 1$, $m \in (0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$. □

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