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# The Stability Problem of Certain Anti-Invariant Submanifolds in Golden Riemannian Manifolds

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

**Abstract**

**In this study, we discuss the stability of some anti-invariant submanifolds of golden Riemannian manifolds under certain conditions in terms of the Ricci curvature tensors of the ambient manifold and the submanifold.**

***Keywords:*** *stability; metallic Riemannian manifold; anti-invariant submanifold.*

***AMS Subject Classification (2020):*** *Primary: 53C15; 53C25; 53C40 ; Secondary: 14F40.*

## 1. Introduction

Based on the polynomial structure [13], the notion of a golden structure on differentiable manifolds was introduced and studied by M. C. Crâşmăreanu and C. E. Hreţcanu [20] for investigating the impact of the golden mean on differential geometry. In [27], complete and horizontal lifts of golden structures in tangent bundles were examined by M. Özkan. In [3], A. M. Blaga defined and investigated the conjugate connections determined by a golden structure, called golden conjugate connections. Also, golden structures were generalized as metallic structures in [22].

The study of certain types of submanifolds, such as invariant, anti-invariant, non-invariant, semi-invariant, slant, semi-slant, hemi-slant submanifolds was considered in manifolds admitting compatible Riemannian metrics with respect to golden (or metallic) structures, i.e., golden (or metallic) Riemannian manifolds (see, e.g., [1, 5, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22]). Also, golden shaped hypersurfaces were defined and the full classification of such type of hypersurfaces was obtained by M. C. Crâşmăreanu *et a*l [11] in real space forms. Additionally, in the setting of the geometry of metallic manifolds, the full classification for metallic shaped hypersurfaces was given by C. Özgür and N. Y. Özgür in real space forms [25] and Lorentzian space forms [26].

On the other hand, the stability of minimal Lagrangian submanifolds was studied in Kaehler manifolds (see, e.g., [8, 24]). In [28], G. Pitiş obtained algebraic conditions for the stability of an anti-invariant submanifold of a locally product manifold in the case that the dimension of the submanifold is equal to half that of the ambient manifold. Also, the stability problem for slant and semi-slant submanifolds was considered in Sasakian manifolds by M. Cîrnu [6].

The main objective of this paper is to analyze the stability of a special type of anti-invariant submanifolds, namely Lagrangian-like submanifolds, in locally decomposable golden Riemannian manifolds under certain conditions. The paper consists of three sections and is prepared as follows: Section 1 is introduction including a brief literature review and the organization of the study. Section 2 is devoted to a short background to clarify the other ones. In section 3, the second variational formulas are first established for an arbitrary Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold. Later, three theorems are given for Lagrangian-like submanifolds to be stable or unstable.

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## 2. Preliminaries

We consider an isometrically immersed submanifold $M$ of a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ and use the same notation $\widetilde{g}$ for the induced metric on $M$. Denoting the Riemannian connection on $\widetilde{M}$ by $\widetilde{\nabla}$, the Gauss and Weingarten formulas are given, respectively, by [2, 31]

$$\widetilde{\nabla}_X Y = \nabla_X Y + h\left(X, Y\right) \tag{2.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.2}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where $\nabla$ is the induced connection on $M$, $h$ is the second fundamental form, $A_N$ is the Weingarten map with respect to $N$ and $\nabla^{\perp}$ is the normal connection. Furthermore,

$$\widetilde{g}\left(h\left(X, Y\right), N\right) = \widetilde{g}\left(A_N X, Y\right) \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. In addition, we have

$$\widetilde{g}\left(\widetilde{R}\left(X, Y\right) Z, W\right) = \widetilde{g}\left(R\left(X, Y\right) Z, W\right) - \widetilde{g}\left(h\left(X, W\right), h\left(Y, Z\right)\right) + \widetilde{g}\left(h\left(Y, W\right), h\left(X, Z\right)\right) \tag{2.4}$$

for any vector fields $X, Y, Z, W \in \Gamma\left(TM\right)$, where $\widetilde{R}$ and $R$ are Riemannian curvature tensors of $\widetilde{M}$ and $M$, respectively, known as Gauss equation [2, 31].

Let $M$ be an $m$-dimensional compact submanifold of a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$. The second normal variation of $M$ induced by a normal vector field $N \in \Gamma(TM^{\perp})$ is given by the formula [7, 23]

$$V''\left(N\right) = \int_M \left\{ \left\|\nabla^{\perp}_X N\right\|^2 - \left\|A_N\right\|^2 + m^2 \widetilde{g}^2\left(H, N\right) - m\widetilde{g}\left(H, \widetilde{\nabla}_N N\right) - \sum_{i=1}^{m} \widetilde{K}\left(E_i, N, N, E_i\right) \right\} dv \tag{2.5}$$

for any vector field $X \in \Gamma\left(TM\right)$, where $H$ is the mean curvature vector of $M$, $\widetilde{K}$ is the Riemann-Christoffel curvature tensor of $\widetilde{M}$, $\{E_1, \ldots, E_m\}$ is a local orthonormal frame of the tangent bundle $TM$ and $dv$ is the volume form of $M$.

A compact submanifold $M$ of a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ is called *stable* if $V''\left(N\right) \geq 0$ for each normal vector field $N \in \Gamma\left(TM^{\perp}\right)$; otherwise, $M$ is said to be *unstable*.

A *golden structure* [10, 20] on a differentiable manifold $\widetilde{M}$ is a polynomial structure $\widetilde{\psi}$ of degree 2 verifying

$$\widetilde{\psi}^2 = \widetilde{\psi} + I, \tag{2.6}$$

where $I$ is the Kronecker tensor field on $T\widetilde{M}$. Also, the pair $\left(\widetilde{M}, \widetilde{\psi}\right)$ is named a *golden manifold*. A Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ equipped with a golden structure $\widetilde{\psi}$ is said to be a *golden Riemannian manifold* provided that the Riemannian metric $\widetilde{g}$ is $\widetilde{\psi}$-compatible, i.e.,

$$\widetilde{g}\left(\widetilde{\psi}X, Y\right) = \widetilde{g}\left(X, \widetilde{\psi}Y\right), \tag{2.7}$$

or equivalently

$$\widetilde{g}\left(\widetilde{\psi}X, \widetilde{\psi}Y\right) = \widetilde{g}\left(\widetilde{\psi}X, Y\right) + \widetilde{g}\left(X, Y\right) \tag{2.8}$$

for any vector fields $X, Y \in \Gamma\left(T\widetilde{M}\right)$ and it is denoted by $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. In particular, a golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$ is called a *locally decomposable golden Riemannian manifold* if $\widetilde{\nabla}\widetilde{\psi} = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection on $\widetilde{M}$.

We recall from [4, Proposition 2.3] that the Riemannian curvature tensor $\widetilde{R}$ of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$ satisfies the following properties:

$$\widetilde{R}\left(X, Y\right) \widetilde{\psi}Z = \widetilde{\psi}\widetilde{R}\left(X, Y\right) Z, \tag{2.9}$$

$$\widetilde{R}\left(\widetilde{\psi}X, Y\right) = \widetilde{R}\left(X, \widetilde{\psi}Y\right), \tag{2.10}$$

$$\widetilde{R}\left(\widetilde{\psi}X, \widetilde{\psi}Y\right) = \widetilde{R}\left(\widetilde{\psi}X, Y\right) + \widetilde{R}(X, Y) \tag{2.11}$$

and

$$\widetilde{R}\left(\widetilde{\psi}^{n+1}X, Y\right) = f_{n+1}\widetilde{R}\left(\widetilde{\psi}X, Y\right) + f_n\widetilde{R}(X, Y) \tag{2.12}$$

for any vector fields $X, Y \in \Gamma\left(T\widetilde{M}\right)$, where $(f_n)$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$ with $f_0 = f_1 = 1$. Moreover, it can be easily seen from (2.7), (2.9) and (2.10) that the following relations are verified:

$$\widetilde{K}\left(\widetilde{\psi}X, Y, Z, W\right) = \widetilde{K}\left(X, \widetilde{\psi}Y, Z, W\right) \tag{2.13}$$

and

$$\widetilde{K}\left(X, Y, \widetilde{\psi}Z, W\right) = \widetilde{K}\left(X, Y, Z, \widetilde{\psi}W\right) \tag{2.14}$$

for any vector fields $X, Y, Z, W \in \Gamma\left(T\widetilde{M}\right)$, where $\widetilde{K}$ stands for the Riemann-Christoffel curvature tensor of $\widetilde{M}$.

An isometrically immersed submanifold $M$ of a golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$ is called *anti-invariant* [9, 20] if

$$\widetilde{\psi}(T_pM) \subseteq T_pM^{\perp} \tag{2.15}$$

for any point $p \in M$. Particularly, we say that $M$ is a *Lagrangian-like submanifold* if $\dim \widetilde{M} = 2\dim M$.

Let $M$ be an anti-invariant submanifold of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. In this case, by means of (2.1) and (2.2), it follows from the locally decomposability of the ambient manifold $\widetilde{M}$ that [1, Proposition 3.3]

$$A_{\widetilde{\psi}X}Y = 0 \tag{2.16}$$

and

$$\nabla^{\perp}_X \widetilde{\psi}Y = \widetilde{\psi}\nabla_X Y \tag{2.17}$$

for any vector fields $X, Y \in \Gamma(TM)$. Furthermore, the normal bundle $TM^{\perp}$ has the following orthogonal decomposition:

$$TM^{\perp} = \widetilde{\psi}(TM) \oplus \mu,$$

where $\mu$ denotes the maximal invariant subbundle of $TM^{\perp}$. Hence, it follows that $M$ is a Lagrangian-like submanifold if and only if $\mu = \{0\}$.

## 3. Main results

We start with a proposition that shows the importance of this study.

**Proposition 3.1.** *Any Lagrangian-like submanifold $M$ of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$ is totally geodesic.*

*Proof.* From the fact that $\dim \widetilde{M} = 2\dim M$, $\left\{\widetilde{\psi}X_p : X_p \in T_pM\right\}$ is a generating set for the normal space $T_pM^{\perp}$ at a point $p \in M$. Also, in virtue of (2.3), we have

$$\widetilde{g}\left(h(X, Z), \widetilde{\psi}Y\right) = \widetilde{g}\left(A_{\widetilde{\psi}Y}X, Z\right)$$

for any vector fields $Y, Z \in \Gamma(TM)$. Thus, it seems from (2.16) that $h = 0$, that is, $M$ is a totally geodesic submanifold. □

**Proposition 3.2.** *Let $M$ be an $m$-dimensional Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. Then the Riemann-Christoffel curvature tensor $\widetilde{K}$ of $\widetilde{M}$ verifies the following equation:*

$$\sum_{i=1}^{m} \widetilde{K}\left(E_i, \widetilde{\psi}X, \widetilde{\psi}X, E_i\right) = \widetilde{S}\left(X, X\right) - S\left(X, X\right)$$

*for any vector field $X \in \Gamma\left(TM\right)$, where $\{E_1, \ldots, E_m\}$ is a local orthonormal frame of the tangent bundle $TM$, $\widetilde{S}$ and $S$ are the Ricci curvature tensors of $\widetilde{M}$ and $M$, respectively.*

*Proof.* By means of (2.8), it is easy to check that the set $\left\{\widetilde{\psi}E_1, \ldots, \widetilde{\psi}E_m\right\}$ is a local orthonormal frame of the normal bundle $TM^{\perp}$. Thus, the set $\left\{E_1, \ldots, E_m, \widetilde{\psi}E_1, \ldots, \widetilde{\psi}E_m\right\}$ creates a local orthonormal frame of the ambient tangent bundle $T\widetilde{M}$. Hence, by means of (2.13) and (2.14), we derive from Gauss equation that

$$\sum_{i=1}^{m} \widetilde{K}\left(E_i, \widetilde{\psi}X, \widetilde{\psi}X, E_i\right) = \widetilde{S}\left(X, X\right) - S\left(X, X\right) + mHh\left(X, X\right) - \sum_{i=1}^{m} \left\|h\left(X, E_i\right)\right\|^2 \tag{3.1}$$

for any vector field $X \in \Gamma\left(TM\right)$, where $H$ is the mean curvature vector of $M$. As a result, by Proposition 3.1, (3.1) gives us

$$\sum_{i=1}^{m} \widetilde{K}\left(E_i, \widetilde{\psi}X, \widetilde{\psi}X, E_i\right) = \widetilde{S}\left(X, X\right) - S\left(X, X\right),$$

which completes the proof. □

**Proposition 3.3.** *Let $M$ be an $m$-dimensional compact Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. Then the second normal variation of $M$ determined by a normal vector field $N$ is given by the formula*

$$V''\left(N\right) = \int_M \left\{\left\|\nabla X\right\|^2 + S\left(X, X\right) - \widetilde{S}\left(X, X\right)\right\} dv \tag{3.2}$$

*for any vector field $X \in \Gamma\left(TM\right)$, where $N = \widetilde{\psi}X$.*

*Proof.* Taking account of (2.16) and (2.17), by Propositions 3.1 and 3.2, a simple calculation gives us the requested equality. □

**Theorem 3.1.** *Let $M$ be an $m$-dimensional compact Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. Then the following assertions are equivalent to each other:*

**(a)** *$M$ is stable (or unstable).*

**(b)** *$\left\|\nabla X\right\|^2 + S\left(X, X\right) - i^*\widetilde{S}\left(X, X\right)$ is non-negative (or negative) for any vector field $X \in \Gamma\left(TM\right)$, where $i : M \to \widetilde{M}$ is the inclusion map.*

*Proof.* The proof follows immediately from Proposition 3.3. □

**Theorem 3.2.** *Let $M$ be an $m$-dimensional compact Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{\psi}\right)$. Then the following assertions are true:*

**(a)** *If $i^*\widetilde{S} \leq S$ is satisfied, then $M$ is stable,*

**(b)** *If $i^*\widetilde{S} > 2S$ is satisfied and $M$ has a non zero Killing vector field, then $M$ is unstable.*

*Proof.* The assertion (a) is an immediate consequence of Theorem 3.1. The relation (3.6) can be also written in the following form [30, page 41]

$$\int_M \left\{S\left(X, X\right) + \frac{1}{2}\left\|\mathcal{L}_X\widetilde{g}\right\|^2 - \left\|\nabla X\right\|^2 - \left(\delta X\right)^2\right\} dv = 0 \tag{3.3}$$

for any vector field $X \in \Gamma(TM)$, where $\mathcal{L}_X$ is the Lie differentiation with respect to $X$ and $\delta$ is the co-differential operator. In particular, we have

$$\int_M \left\{ S(X,X) - \|\nabla X\|^2 - (\delta X)^2 \right\} dv = 0 \tag{3.4}$$

for a non zero Killing vector field $X \in \Gamma(TM)$. Thus, combining (3.2) and (3.4), we obtain

$$V''(N) = \int_M \left\{ 2S(X,X) - \widetilde{S}(X,X) - (\delta X)^2 \right\} dv,$$

which means that if $i^*\widetilde{S} > 2S$, then $M$ is unstable. □

**Proposition 3.4.** *Let $M$ be an $m$-dimensional compact Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\psi})$. In this case, for any vector field $X \in \Gamma(TM)$, the second normal variation of $M$ induced by a normal vector field $N = \widetilde{\psi}X$ is given by the formula*

$$V''(N) = \int_M \left\{ \frac{1}{2}\|d\eta\|^2 + (\delta\eta)^2 - \widetilde{S}(X,X) \right\} dv, \tag{3.5}$$

*where $\delta$ is the co-differential operator and $\eta$ is a 1-form associated with the vector field $X \in \Gamma(TM)$.*

*Proof.* We recall the following well known integral formula [30, page 41]

$$\int_M \left\{ S(X,X) + \|\nabla X\|^2 - \frac{1}{2}\|d\eta\|^2 - (\delta\eta)^2 \right\} dv = 0 \tag{3.6}$$

for any vector field $X \in \Gamma(TM)$. Thus, taking into consideration (3.6), Propositions 3.2 and 3.3 lead us to obtain the required equality

$$V''(N) = \int_M \left\{ \frac{1}{2}\|d\eta\|^2 + (\delta\eta)^2 - \widetilde{S}(X,X) \right\} dv,$$

where $N = \widetilde{\psi}X$. □

**Theorem 3.3.** *Let $M$ be an $m$-dimensional compact boundaryless Lagrangian-like submanifold of a locally decomposable golden Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\psi})$. Then the following assertions are true:*

**(a)** *If $\widetilde{S}$ is negative definite, then $M$ is stable,*

**(b)** *If $\widetilde{S}$ is positive definite and $H^1_{dR}M \neq 0$, then $M$ is unstable, where $H^1_{dR}M$ denotes the 1th de Rham cohomology group of $M$.*

*Proof.* (a) follows directly from Proposition 3.4. Now, we assume that the ambient manifold $\widetilde{M}$ has the positive Ricci tensor $\widetilde{S}$. We recall from Hodge theorem [29, Theorem 8.12] that every closed form on a compact, orientable, boundaryless Riemannian manifold has a unique harmonic cohomology class representative. Thus, if $H^1_{dR}M \neq 0$, then we have an harmonic 1-form $\eta$ on $M$, that is, $d\eta = \delta\eta = 0$. In this case, from (3.5), we obtain $V''(N) < 0$, where $N = \widetilde{\psi}X$ and $X$ is the vector field associated with the 1-form $\eta$. Consequently, $M$ is unstable. □

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