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Research Article

A comparative study for the spectral properties of Toeplitz and Hankel operators

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Abstract

In this introductory review, we study Hankel and Toeplitz operators considering them as acting on certain spaces of analytic functions, namely Hardy spaces and compare their spectral properties such as their compactness criteria. In contrast to Toeplitz operators, the symbol of a Hankel operator is not uniquely determined by the operator. We also connect Toeplitz operators with Fredholm operators and give some of the most beautiful properties of Toeplitz operators such as the essential spectrum of Toeplitz operator with continuous symbol and the index of Toeplitz operator introducing Fredholm operators firstly.

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1. Introduction

The theory of Toeplitz and Hankel operators is a very wide area and even a huge monograph can deal with only some selected topics. The main purpose of this article is to provide an introduction to this theory. We introduce Hankel operators, one of the most important classes of operators on Hardy spaces and define them as operators having infinite Hankel matrices with entries depending only on the sum of the coordinates with respect to some orthonormal basis. We also introduce another very important class of operators on Hardy spaces, the class of Toeplitz operators and define them as operators having infinite Toeplitz matrices with entries depending only on the difference of the coordinates with respect to some orthonormal basis. Although Hankel and Toeplitz operators are closely related to each other, they have quite different properties such as their compactness criteria. While the symbol of a Toeplitz operator is uniquely determined by the operator, the symbol of a Hankel operator is not. This study mainly focuses on the book [13] and follows the proofs of some of its nice results and theorems in detail.

It is organised as follows. Section 2 deals with the spaces of analytic functions, Hardy spaces. We are only concerned with H^2, H^∞ and give some useful background information. The ideas in Section 2 are standard and can be found in [6, 8, 12, 13, 15, 16], for example. Having established the background knowledge, we will be able to introduce Hankel operators in Section 3 and Toeplitz operators in Section 4. The definitions in Section 3 and 4 are taken from [13]. The final section connects Toeplitz operators with

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Fredholm operators. Some of the most important properties of Toeplitz operators such as the essential spectrum of Toeplitz operator with continuous symbol and the index of Toeplitz operator are given in this section introducing Fredholm operators firstly.

2. Spaces of analytic functions: Hardy spaces

As will be seen in the next sections, the most fruitful way of looking at Hankel and Toeplitz operators is to consider them as acting on certain spaces of analytic functions, namely Hardy spaces. The classic Hardy spaces consist of analytic functions defined on the open unit disc. The discussion is for H^2, which is naturally regarded as a closed subspace of $L^2(\mathbb{T})$ and thus Hilbert space. After treating H^2, we consider H^∞, the isomorphism between L^2 and ℓ^2 and the orthogonal projections.

We start by reviewing the basic properties of $L^2(\mathbb{T})$.

Let $\mathbb{D}$ denote the open unit disc in the complex plane:

$$\mathbb{D} = \{z; |z| < 1\}$$

and let $\mathbb{T}$ denote the unit circle:

$$\mathbb{T} = \{z; |z| = 1\}.$$

Definition 2.1. $L^2(\mathbb{T})$ is the space of all measurable functions on the circle $\mathbb{T}$ with the norm

$$\|f\|_{L^2} = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid f(e^{i\theta}) \mid^2 d\theta\right\}^{\frac{1}{2}}$$

and each $f \in L^2(\mathbb{T})$ has Fourier coefficients

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta \quad (n = 0, 1, -1, 2, -2, ...).$$

Definition 2.2. $L^\infty(\mathbb{T})$ is the space of all essentially bounded measurable functions on the circle $\mathbb{T}$ with the norm

$$\|f(z)\|_\infty = \inf\{C \geq 0 : \mid f(z) \mid \leq C \text{ for almost every } z\}.$$

Definition 2.3. The Hardy space H^2 is the space of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^2} = \sup_{r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid f(re^{i\theta}) \mid^2 d\theta\right)^{\frac{1}{2}} < \infty.$$

The basic properties of H^2 are summarized in the following theorem.

Theorem 2.4. ([14], Theorem 17.10.) *A function f, of the form*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

is in H^2 if and only if $\sum \mid f_n \mid^2 < \infty$; in that case,

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} \mid f_n \mid^2.$$

Since the radial limits of a H^2 function converge almost everywhere on $\mathbb{T}$ and the resulting function is in $L^2(\mathbb{T})$, we can identify H^2 as a subspace of $L^2(\mathbb{T})$.

Moreover, the subspace H^2 and L^2 admits the following descriptions:

$$H^2 := \{f \in L^2 : f_n = 0 \text{ for } n < 0\},$$

$$H^2_- := \{f \in L^2 : f_n = 0 \text{ for } n \geq 0\},$$

where f_n is the nth Fourier coefficients of f. Besides, we know that

$$H^2_- \bigoplus H^2 = L^2.$$

Definition 2.5. The Hardy class H^∞ is the space of all bounded analytic functions in $\mathbb{D}$ with the norm

$$\|f(z)\|_\infty = \sup_{|z|<1} | f(z) | .$$

Theorem 2.6. ([10], Theorem 5.5.) *Let $f \in L^2(\mathbb{T})$. Then*

(i) $\sum | f_n |^2 = \frac{1}{2\pi} \int | f(t) |^2 dt$.

(ii) $f = \lim_{N\to\infty} \sum_{-N}^{N} f_n e^{int}$ *in the $L^2(\mathbb{T})$ norm.*

(iii) *For any square summable sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers, that is, such that $\sum | a_n |^2 < \infty$, there exists a unique $f \in L^2(\mathbb{T})$ such that $a_n = f_n$.*

(iv) *Let f and g are in $L^2(\mathbb{T})$.*

Then

$$\frac{1}{2\pi} \int f(t)\overline{g(t)}dt = \sum_{n=-\infty}^{\infty} f_n \overline{g_n}.$$

Theorem 2.6 amounts to the statement that the correspondense $f \to \{f_n\}$ is an isometry between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. Let denote this isomorphism as an operator

$$U : L^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z})$$

such that

$$\sum_{n\in\mathbb{Z}} f_n z^n \mapsto \{f_n\}_{n\in\mathbb{Z}}$$

to be able to say that Hankel operators defined on L^2 and $\ell^2(\mathbb{Z})$ are unitarily equivalent through this isomorphism (see, for example, [10]).
Additionally, we may define the operator

$$V : H^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z}_+)$$

such that

$$\sum_{n\in\mathbb{Z}_+} f_n z^n \mapsto \{f_n\}_{n\in\mathbb{Z}_+}$$

to be able to say that Toeplitz operators defined on H^2 and $\ell^2(\mathbb{Z}_+)$ are unitarily equivalent through this isomorphism (see, for example, [2], Section 2.6).

Moreover, we define the orthogonal projections which will be used in the next sections. On the space L^2 define the P_+ and P_- onto the subspaces H^2 and H^2_- by

$$P_+ f = \sum_{n\geq 0} f_n z^n, \quad P_- f = \sum_{n<0} f_n z^n.$$

Clearly, $P_- + P_+ = I$.

3. Hankel operators

3.1. Matrices $\{\alpha_{i+j}\}$ in $\ell^2(\mathbb{Z})$ and operators H_a

Definition 3.1. An infinite matrix is called a *Hankel matrix* if it has the form

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \dots \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{pmatrix},$$

where $\alpha = \{\alpha_j\}_{j\geq 0}$ is a sequence of complex numbers. In other words, Hankel matrices are the matrices whose entries depend only on the sum of the coordinates. If $\alpha \in \ell^2(\mathbb{Z})$, we can consider the operator $\Gamma : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ with matrix $\{\alpha_{j+k}\}_{j,k\geq 0}$ that is defined on the dense subset of finitely supported sequences. In other words, if $a = \{a_n\}_{n\geq 0}$ is a finitely supported sequence, then $\Gamma a = b \in \ell^2(\mathbb{Z})$, where $b = \{b_k\}_{k\geq 0}$ is defined by

$$b_k = \sum_{j\geq 0} \alpha_{j+k} a_j, \quad k \geq 0.$$

We call such operators *Hankel operators*.

So we can define Hankel operators as operators having infinite Hankel matrices with entries depending only on the sum of the coordinates with respect to some orthonormal basis. Furthermore, we say that the matrix of H_a is a Hankel matrix $\{a_{-j-k}\}_{j\geq 1,k\geq 0}$ with respect to the bases $\{z^n\}_{n\geq 0}$ for H^2 and $\{\overline{z}^m\}_{m\geq 0}$ for H^2_-: $(H_a z^k, \overline{z}^j) = a_{-j-k}$.

3.2. Non-uniqueness of symbol for Hankel operator

In this part, we are going to consider another realization of Hankel operators on the Hardy space H^2 of functions on the unit circle.

Definition 3.2. Let a be function in the space L^2 on the unit circle. We define the Hankel operator $H_a : H^2 \longrightarrow H^2_-$ on the dense subset of polynomials in H^2 by

$$H_a f = P_- a f,$$

where P_- is the orthogonal projection from L^2 onto H^2_-. Then the function a is called a *symbol* of the Hankel operator H_a.

As stated in [13] and we shall see below in detail, a Hankel operator has many different symbols. So, it is possible to have $a_1 \neq a$ but $H_{a_1} = H_a$. Indeed, let suppose $a_1 = z$, $a = z^2$ and f is analytic since $f \in H^2$. We have

$$H_{a_1} f = P_- a_1 f = P_-(\sum_{n=0}^{\infty} z f_n z^n) = P_-(\sum_{n=0}^{\infty} f_n z^{n+1}) = 0.$$

On the other hand, we have

$$H_a f = P_- a f = P_-(\sum_{n=0}^{\infty} z^2 f_n z^n) = P_-(\sum_{n=0}^{\infty} f_n z^{n+2}) = 0.$$

Therefore, we have $a_1 \neq a$ but $H_{a_1} = H_a$ as required.

The following theorem helps to find the class $\{a_1 \in L^\infty : H_{a_1} = H_a\}$ for fixed $a \in L^\infty$.

Theorem 3.3. *Fix $a \in L^\infty$. Then*

$$\{a_1 \in L^\infty : H_{a_1} = H_a\} = \{a_1 = a + f : f \in H^\infty\}.$$

Proof. Let $H_a : H^2 \longrightarrow H^2_-$, $H_a f = (I - P_+) a f = af - P_+ af$ and $\mathcal{B}(L^2(\mathbb{T}))$ be the space of bounded linear operators from $L^2(\mathbb{T})$ to itself.

For fix a, let describe the set a_1 such that $H_{a_1} = H_a$. Let $g : L^\infty(\mathbb{T}) \longrightarrow \mathcal{B}(L^2(\mathbb{T}))$ be an operator such that $a \longmapsto H_a$.

Then we need to find the *kernel* of g which is

$$\mathrm{Ker}(g) = \{a \in L^\infty(\mathbb{T}) : g(a) = H_a = 0\}.$$

As shown in [1], if a happens to be in H^∞, then it is clear that $P_+ af = af$ for all $f \in H^2$, so the resulting Hankel operator H_a is zero operator. On the other hand, if $H_a = 0$, then

$$H_a 1 = a - P_+ a = 0,$$

so $a = P_+ a \in H^\infty$. Therefore, $H_a = 0$ if and only if $a \in H^\infty$. Now, we may define

$$\mathrm{Ker}(g) = \{a \in H^\infty : g(a) = H_a = 0\}.$$

If $H_{a_1} = H_a$ then we have that $H_{a_1-a} = 0$ by the linearity of Hankel operators. If $H_{a_1-a} = 0$, then $a_1 - a \in \operatorname{Ker} g$, so $a_1 = a + f$, where $f \in H^\infty$. □

Furthermore, we may say that the definition of $\operatorname{Ker} g$ implies that g is not injective. For this reason, it is also easy to see that if $H_{a_1} = H_a$ it does not have to imply $a_1 = a$, which means that the symbol of Hankel operator is not unique.

3.3. Boundedness of Hankel operator

In this section we shall give the criteria for boundedness for Hankel operators before discussing compactness criteria of Hankel operators. The following theorem characterizing the bounded Hankel operators on $\ell^2(\mathbb{Z}_+)$ is due to Nehari.

Theorem 3.4. ([13], Theorem 1.1.1.) *The Hankel operator Γ with matrix $\{\alpha_{j+k}\}_{j,k\geq 0}$ is bounded on $\ell^2(\mathbb{Z}_+)$ if and only if there exists a function a in L^∞ on the unit circle $\mathbb{T}$ such that*

$$\alpha_k = a_k,\ \ k \geq 0.$$

In this case

$$\|\Gamma\| = \inf\{\|a\|_\infty : a_k = \alpha_k,\, k \geq 0\}.$$

Recall that a_k is the kth Fourier coefficient of a.
It follows from Theorem 3.4 that a Hankel operator H_a is bounded and $\|H_a\| \leq \|a\|_\infty$.

3.4. Compactness of Hankel operator with continuous symbol

In order to be ready to prove the main theorem, which is Theorem 3.8, in this section we give the following theorems.

Theorem 3.5. ([10], Weierstrass approximation theorem) *Every continuous 2π-periodic function can be approximated uniformly by trigonometric polynomials.*

Theorem 3.6. ([11], Theorem 8.1-4) *Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. Then*
(i) If T is bounded and $\dim T(X) < \infty$, the operator T is compact.
(ii) If $\dim X < \infty$, the operator T is compact.

Theorem 3.7. ([11], Theorem 8.1-5) *Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, say, $\|T_n - T\| \to 0$, then the limit operator T is compact.*

Theorem 3.8. *If symbol $a \in C(\mathbb{T})$ then $H_a = P_- a$ is compact.*

Proof. Suppose $a \in C(\mathbb{T})$ then there exists a sequence of polynomials (a_n) such that $\|a_n - a\|_{C(\mathbb{T})} \to 0$, as $n \to \infty$ by Theorem 3.5. Firstly, we check that $\|H_a - H_{a_n}\| \to 0$ as $n \to \infty$. We have that

$$\|H_a f - H_{a_n} f\| \leq \|(H_a - H_{a_n})f\| = \|H_{a-a_n} f\| \leq 2\pi \cdot C \cdot \|a - a_n\|_{C(\mathbb{T})} \|f\|.$$

Since $\|a_n - a\|_{C(\mathbb{T})} \to 0$, $\|H_a - H_{a_n}\| \to 0$ as $n \to \infty$. Secondly, we need to show that a sequence of operators (H_{a_n}) is compact. To begin with, let us prove that if $a(z) = \overline{z}^m$ $(m > 0)$, then $H_{\overline{z}^m}$ is compact. Now, suppose $a(z) = \overline{z}^m$. Then

$$P_-(a(z)f(z)) = P_-(\sum_{n=0}^{\infty} f_n \overline{z}^m z^n) = P_-(\sum_{n=0}^{\infty} f_n z^{(n-m)}).$$

We have that $P_-(a(z)f(z)) = f_0 z^{-m} + f_1 z^{1-m} + ... + f_{m-1} z^{-1}$. Since $H_{\overline{z}^m} f$ is always a polynomial of degree at most m, $H_{\overline{z}^m}$ has finite rank and so is compact by Theorem 3.6 (i).

Let us prove that if $a(z) = z^m (m \geq 0)$, then H_{z^m} is compact.

$$P_-(a(z)f(z)) = P_-(\sum_{n=0}^{\infty} f_n z^m z^n) = P_-(\sum_{n=0}^{\infty} f_n z^{(n+m)}) = 0.$$

Thus, $H_{z^m} = 0$ and H_{z^m} is compact. Now, let $a_n = \sum_{m=-M}^{M} \gamma_m z^m$. Then

$$H_{a_n} = \sum_{m=-M}^{M} \gamma_m H_{z^m}.$$

Then we have that

$$H_{a_n} = \sum_{m=-M}^{-1} \gamma_m H_{z^m} + \sum_{m=0}^{M} \gamma_m H_{z^m}.$$

Therefore, (H_{a_n}) is a sequence of compact operators since it can be represented as a linear combinations of compact operators. Consequently, by Theorem 3.7, H_a is compact due to the fact that (H_{a_n}) is a sequence of compact operators and $\|H_a - H_{a_n}\| \to 0$ as $n \to \infty$. □

4. Toeplitz operators

4.1. Matrices $\{\alpha_{i-j}\}$ in $\ell^2(\mathbb{Z}_+)$ and operators T_a

Definition 4.1. An infinite matrix is called a *Toeplitz matrix* if it has the form

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & \cdots \\ \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{pmatrix},$$

where $\alpha = \{\alpha_j\}_{j\geq 0}$ is a sequence of complex numbers. Toeplitz matrices are the matrices whose entries depend only on the difference of the coordinates.

We may define Toeplitz operators on the Hardy class H^2 as those which have Toeplitz matrices in the basis $\{z^n\}_{n\geq 0}$.

Definition 4.2. An operator $T : H^2 \longrightarrow H^2$ defined on the set of polynomials is called a *Toeplitz operator* if there is a two-sided sequence of complex numbers $\{t_n\}_{n\in Z}$ such that

$$(Tz^k, z^j) = t_{j-k}, \ j, k \in \mathbb{Z}_+. \tag{4.1}$$

4.2. Uniqueness of symbol for Toeplitz operator

Definition 4.3. Given $a \in L^\infty$ we define the Toeplitz operator T_a on H^2 by

$$T_a f = P_+ af, \quad f \in H^2,$$

where P_+ is the orthogonal projection from L^2 onto H^2. Then the function a is said to be a *symbol* of Toeplitz operator T_a.

We have given an isomorphism

$$V : H^2(\mathbb{T}) \to \ell^2(\mathbb{Z}_+)$$

in the first section. Let $T_a : H^2 \to H^2$ and $A : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be Toeplitz operators. Let denote $A = VT_aV^*$. Thus, we can connect the Toeplitz operator A with the Toeplitz operator T_a. As stated in [13], in contrast with Hankel operators the symbol of a Toeplitz operator is uniquely determined by the operator, which can be seen in the following theorem in detail.

Theorem 4.4. *If $T_a = T_b$ then $a = b$.*

Proof. To begin with, we need to check that if $T_a = 0$ then $a = 0$. Suppose $T_a = 0$, i.e. $T_a f = 0$ for $\forall f$. Firstly, let $f(z) = z, a(z) = \sum_{m=-\infty}^{\infty} a_m z^m$ and $T_a f = P_+ af = 0$. Then $P_+(a(z)f(z)) = P_+(z \sum_{m=-\infty}^{\infty} a_m z^m) = P_+(\sum_{m=-\infty}^{\infty} a_m z^{m+1}) = 0$.

$$T_a f = a_{-1} + a_0 z^1 + a_1 z^2 + ... = 0.$$

$$a_{-1} = a_0 = a_1 = ... = 0.$$

Secondly, let $f(z) = z^2$. Then $P_+(a(z)f(z)) = P_+(\sum_{m=-\infty}^{\infty} a_m z^{m+2}) = 0$.

$$T_a f = a_{-2} + a_{-1} z^1 + a_0 z^2 + a_1 z^3 + ... = 0.$$

$$a_{-2} = a_{-1} = a_0 = a_1 = ... = 0.$$

Finally, let $f(z) = z^n$.
Then $P_+(a(z)f(z)) = P_+(\sum_{m=-\infty}^{\infty} a_m z^{m+n}) = 0$.

$$T_a f = a_{-n} + a_{-n+1} z^1 + ... = 0.$$

$$a_{-n} = a_{-n+1} = ... = 0.$$

If we continue this type of calculation, then we will find $a = 0$ due to the fact that all coefficients of a are zero. Thus, if $T_a f = 0$ for $\forall f$, then $a = 0$. Now, if $T_a = T_b$, then $T_{a-b} = 0$ by the linearity of Toeplitz operator. If $T_{a-b} = 0$, then $a - b = 0$, so $a = b$. $\square$

4.3. Boundedness of Toeplitz operator

We now give the boundedness criteria for Toeplitz operators before considering non-compactness criteria of Toeplitz operators.

Theorem 4.5. ([13], Theorem 3.1.1.) *The Toeplitz operator T with matrix defined by (4.1) is bounded on H^2 if and only if there exists a bounded function a on the unit circle $\mathbb{T}$ whose Fourier coefficients coincide with the t_j:*

$$a_n = t_n.$$

In this case

$$\|T\| = \|a\|_\infty .$$

4.4. Non-compactness of Toeplitz operator with non-zero symbol

Some important definition and theorems are going to be given to be able to prove Theorem 4.9 which shows non-compactness of Toeplitz operator with non-zero symbol.

Definition 4.6. A sequence (x_n) in a Hilbert space $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$ if

$$\lim_{n\to\infty} (x_n, y) = (x, y) \quad \text{for all} \quad y \in \mathcal{H}.$$

Weak converges is usually written as

$$x_n \rightharpoonup x \quad \text{as} \quad n \to \infty$$

(see, for example, [9]).

Theorem 4.7. ([11], Theorem 8.1-7) *Let X and Y be normed spaces and $T : X \longrightarrow Y$ a compact linear operator. Suppose that (x_n) in X is weakly convergent, say, $x_n \longrightarrow x$. Then (Tx_n) is strongly convergent in Y and has the limit $y = Tx$.*

Theorem 4.8. *Let $f^{(n)} \in \ell^2(\mathbb{Z})$ be a sequence such that $f^{(n)} = (0, ..., 0, 1, 0, ..., 0)$, where 1 is on the nth position. Then $f^{(n)}$ is weakly convergent to 0.*

Proof. Suppose that $\mathcal{H} = \ell^2(\mathbb{Z})$. Let $f^{(n)}$ be a sequence whose nth term is 1 and whose other terms are 0. If $y = (y_1, y_2, y_3, ...) \in \ell^2$, then

$$(f^{(n)}, y) = \overline{y_n} \to 0 \quad \text{as} \quad n \to \infty,$$

since $\sum | y_n |^2$ converges. Hence $f^{(n)} \to 0$ as $n \to \infty$ (see, for example, [9], Example 8.38.). □

Now we are ready to prove the following theorem.

Theorem 4.9. *If $a \neq 0$ then $T_a = P_+ a$ is not compact.*

Proof. Suppose a is not zero. Then $\exists j$ such that $a_j \neq 0$. Using the information about the correspondence between $\ell^2(\mathbb{Z})$ and L^2, we may define $f^{(n)} = (0, ..., 0, 1, 0, ...)$, (where 1 is on the nth position) as $f^{(n)} = z^n$. Then

$$P_+(a(z)f(z)) = P_+(\sum_{k=-\infty}^{\infty} a_k z^k z^n) = P_+(\sum_{k=-\infty}^{\infty} a_k z^{k+n}) = \sum_{k=-n}^{\infty} a_k z^{k+n}.$$

We may define

$$T_a : H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T})$$
$$f^{(n)} = z^n \mapsto \sum_{k=-n}^{\infty} a_k z^{k+n}$$

and

$$T'_a : \ell^2(\mathbb{Z}_+) \longrightarrow \ell^2(\mathbb{Z}_+)$$
$$f^{(n)} = (0, ..., 0, 1, 0, ...) \mapsto (a_{-n}, a_{-(n-1)}, ..., a_0, a_1, ...)$$

As we stated in the first section there is an isomorphism

$$V : H^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z}_+).$$

In fact, V is the restriction of the isomorphism U. Let denote $T_a = V^* T'_a V$. Then we may say that Toeplitz operators defined on H^2 and $\ell^2(\mathbb{Z}_+)$ are unitarily equivalent through this isomorphism as well. After defining $T_a f^{(n)}$, we need to find the norm of $T_a f^{(n)}$.

$$\left\| T_a f^{(n)} \right\| = \left\| \sum_{k=-n}^{\infty} a_k z^{k+n} \right\| = \left(\sum_{k=-n}^{\infty} | a_k |^2 \right)^{\frac{1}{2}} > | a_j | > 0$$

since $a \neq 0$ and $-n < j$ for sufficiently large n. This means that $\left\| T_a f^{(n)} \right\| \not\longrightarrow 0$. Consequently, we have that there exists a sequence $f^{(n)} \to 0$ as $n \to \infty$, yet $\left\| T_a f^{(n)} \right\| \not\to 0$. This means that $T_a = P_+ a$ is not compact by Theorem 4.7. □

After having considered Theorem 4.9, we may easily say that the only compact Toeplitz operator is $T_0 = 0$.

5. Fredholm operators

This section connects Toeplitz operators with Fredholm operators and gives some of the most important properties of Toeplitz operators such as the essential spectrum of Toeplitz operator with continuous symbol and the index of Toeplitz operator introducing Fredholm operators firstly. The following definitions are taken from [5, 7, 9, 13] respectively.

Definition 5.1. A bounded linear operator T on a Hilbert space is said to be *Fredholm* if:
(i) $\operatorname{Ran} T$ is closed;
(ii) $\operatorname{Ker} T$ and $\operatorname{Ker} T^*$ are finite dimensional, i.e. $\dim \operatorname{Ker} T < \infty$ and $\dim \operatorname{Ker} T^* < \infty$.
The *index* of a Fredholm operator T, $\operatorname{index} T$, is the integer

$$\operatorname{index} T = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*.$$

For example, the identity operator on an infinite-dimensional Hilbert space is Fredholm operator with index zero owing to the fact that the kernel of the identity operator is 0-dimensional subspace consisting of $(0, 0, 0, ...)$ and the adjoint of the identity operator is again itself (see, for example, [9]).

Definition 5.2. The *essential spectrum* of T is the set of all complex numbers λ such that $T - \lambda I$ is not a Fredholm operator, i.e.

$$\mathrm{spec}_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm}\}.$$

Let a be a function in $C(\mathbb{T})$ that does not vanish on $\mathbb{T}$. We define the winding number $\mathrm{wind}\, a$ with respect to the origin in the following way.

Definition 5.3. Consider a continuous branch of the argument $\arg_a$ of the function $t \to a(e^{it})$, $t \in [0, 2\pi]$, i.e. $\arg_a \in C([0, 2\pi])$,

$$\exp(i\arg_a(t)) = \frac{a(e^{it})}{\mid a(e^{it}) \mid}, t \in [0, 2\pi].$$

Then $\mathrm{wind}\, a = \frac{1}{2\pi}(\arg_a(2\pi) - \arg_a(0))$.

Definition 5.4. Let $D(T)$ be a linear subspace of X and let $T : D(T) \to Y$ be linear. The map T is said to have a left approximate inverse if and only if there is a map $R_\ell \in \mathcal{B}(Y, X)$ such that $R_\ell \subset D(T)$ and $R_\ell T - I_X$ is compact. Similarly, T has a right approximate inverse if and only if there is a map $R_r \in \mathcal{B}(Y, X)$ such that $R_r(Y) \subset D(T)$ and $TR_r - I_Y$ is compact. The maps R_ℓ and R_r are called left and right approximate inverses of T respectively.

We shall refer to a map which is both a left and a right approximate inverse of a map T as an approximate inverse of T. Now, we shall give the some important theorems which will be used to prove Theorem 5.10. We will denote $\mathcal{F}(X, Y)$ as a space of all Fredholm operators which map from X to Y.

Theorem 5.5. ([5], Theorem 3.15.) *Let $T \in \mathcal{B}(X, Y)$. Then the following statements are equivalent:*
(i) $T \in \mathcal{F}(X, Y)$,
(ii) T has an approximate inverse.

Theorem 5.6. ([5], Theorem 3.16.) *Let X, Y, Z be Banach spaces. Let $S \in \mathcal{F}(X, Y)$ and suppose that $T \in \mathcal{F}(Y, Z)$. Then $TS \in \mathcal{F}(X, Z)$ and*

$$\mathrm{index}\, TS = \mathrm{index}\, T + \mathrm{index}\, S.$$

Theorem 5.7. ([5], Theorem 3.17.) *Let $T \in \mathcal{F}(X, Y)$ and suppose that S is compact. Then $T + S \in \mathcal{F}(X, Y)$ and* $\mathrm{index}(T + S) = \mathrm{index}\, T$.

As stated in [5], the index of Fredholm operator is unchanged by compact perturbations.

Theorem 5.8. ([11], Theorem 7.3-1) *Let $T \in \mathcal{B}(X, X)$, where X is a Banach space. If $\|T\| < 1$, then $(I - T)^{-1}$ exists as a bounded linear operator on the whole space X and*

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + ...$$

where the series on the right is convergent in the norm on $\mathcal{B}(X, X)$.

Theorem 5.9. *If $\|B\| < 1$ then $I + B$ is Fredholm and* $\mathrm{index}\,(I + B) = 0$.

Proof. We need to find that the image of $I + B$ is closed and

$$\dim \mathrm{Ker}(I + B) < \infty \quad \text{and} \quad \dim \mathrm{Ker}(I + B)^* < \infty.$$

To begin with, we need to check that if $\{\varphi_n\} \in \mathrm{Ran}(I+B)$ and $\varphi_n \to \varphi$ then $\varphi \in \mathrm{Ran}(I+B)$. Let $\varphi_n = (I+B)\psi_n$ and $\varphi = (I+B)\psi$. Then $\psi_n = (I+B)^{-1}\varphi_n$ and $\psi = (I+B)^{-1}\varphi$ since $(I+B)^{-1}$ exists by Theorem 5.8. We have that $\psi_n \to \psi$ as $n \to \infty$ since $\varphi_n \to \varphi$ as $n \to \infty$. Besides, we have $(I+B)\psi_n \to (I+B)\psi$, which means that $\varphi_n \to \varphi$ as $n \to \infty$ and $\varphi \in \mathrm{Ran}(I+B)$. Now, we need to find $\mathrm{Ker}(I+B)$ and $\mathrm{Ker}(I+B)^*$. Firstly, $(I+B)\psi = 0 \Leftrightarrow B\psi = -\psi$, which contradicts to $\|B\psi\| < \|\psi\|$. Then we have that $\psi = 0$. Thus,

$$\mathrm{Ker}(I+B) = \{0\} \quad \text{and} \quad \dim \mathrm{Ker}(I+B) = 0 < \infty.$$

Secondly, we know that if $\|B\| < 1$ then $\|B^*\| = \|B\| < 1$.

$$(I+B)^*\psi = 0 \Leftrightarrow (B^*+I)\psi = 0 \Leftrightarrow B^*\psi = -\psi,$$

which contradicts to $\|B^*\psi\| < \|\psi\|$. Hence, we have that $\psi = 0$.

$$\mathrm{Ker}(I+B)^* = \{0\} \quad \text{and} \quad \dim \mathrm{Ker}(I+B)^* = 0 < \infty.$$

Therefore, $I+B$ is Fredholm operator. Moreover,

$$\mathrm{index}(I+B) = \dim \mathrm{Ker}(I+B) - \dim \mathrm{Ker}(I+B)^* = 0.$$

□

Now, we are ready to give one of the main theorems in this section and closely follow its proof in [5] in detail.

Theorem 5.10. ([5], Theorem 3.18.) *Let $T \in \mathcal{F}(X,Y)$. Then there is a positive number δ such that if $S \in \mathcal{B}(X,Y)$ and $\|S\| < \delta$, then $T+S \in \mathcal{F}(X,Y)$ and* $\mathrm{index}(T+S) = \mathrm{index}\, T$.

Proof. Let $R \neq 0$ be an approximate inverse of T, so that there are compact maps K_1 and K_2 such that $RT = I_X + K_1$, $TR = I_Y + K_2$. Put $\delta = \|R\|^{-1}$. We show that δ has the desired properties. Let $S \in \mathcal{B}(X,Y)$ be such that $\|S\| < \delta$. Then $\|RS\| < 1$ and then by Theorem 5.8, $(I_X + RS)^{-1}$ exists and is in $\mathcal{B}(X)$. Thus,

$$R(T+S) = I_X + K_1 + RS = (I_X + RS)[I_X + (I_X + RS)^{-1}K_1] \tag{5.1}$$

and

$$(I_X + RS)^{-1}R(T+S) = I_X + (I_X + RS)^{-1}K_1.$$

Since $(I_X+RS)^{-1}K_1$ is compact this shows that $T+S$ has a left approximate inverse, which is $(I_X + RS)^{-1}R$. In the same way it follows that $R(I_Y + SR)^{-1}$ is a right approximate inverse of $T+S$. Now, we need to show that

$$(I+RS)^{-1}R = R(I+SR)^{-1}.$$

Let us start with the summation of R and RSR. We know that

$$R + RSR = R + RSR.$$

Then we have that

$$R(I+SR) = (I+RS)R. \tag{5.2}$$

If we apply $(I+SR)^{-1}$ to both right sides of the equation (5.2), then we have

$$R(I+SR)(I+SR)^{-1} = (I+RS)R(I+SR)^{-1}.$$

$$R = RI = (I+RS)R(I+SR)^{-1}. \tag{5.3}$$

If we apply $(I+RS)^{-1}$ to both left sides of equation (5.3), then we have

$$(I+RS)^{-1}R = (I+RS)^{-1}(I+RS)R(I+SR)^{-1}.$$

$$(I+RS)^{-1}R = IR(I+SR)^{-1} = R(I+SR)^{-1}.$$

In conclusion, we have

$$(I+RS)^{-1}R = R(I+SR)^{-1} \quad \text{as required.}$$

Hence $T+S \in \mathcal{F}(X,Y)$ by Theorem 5.5. By Theorem 5.6, Theorem 5.9, we have

$$\operatorname{index} R + \operatorname{index}(T+S) = \operatorname{index} R(T+S).$$

$$\operatorname{index} R(T+S) = \operatorname{index}(I_X + RS) + \operatorname{index}[I_X + (I_X+RS)^{-1}K_1] = 0$$

from equation (5.1). Therefore, $\operatorname{index}(T+S) = -\operatorname{index} R$. In addition to this, we know that $RT = I_X + K_1$, I_X is Fredholm with index 0 and K_1 is compact.

$$\text{By Theorem 5.7, } \operatorname{index}(RT) = \operatorname{index}(I_X + K_1) = \operatorname{index} I_X = 0.$$

Then we have that

$$0 = \operatorname{index}(RT) = \operatorname{index} R + \operatorname{index} T \quad \text{by Theorem 5.6.}$$

Thus, $\operatorname{index} T = -\operatorname{index} R$. Consequently, we have shown that $\operatorname{index}(T+S) = \operatorname{index} T$. □

In order to evaluate the second main theorem in this section, which is Theorem 5.13 about the essential spectrum of Toeplitz operator with continuous symbol we give the followings.

Theorem 5.11. ([13], Theorem 1.4.) *Let a be a nonzero function in L^∞. Then either*

$$\operatorname{Ker} T_a = \{0\} \quad \textit{or} \quad \operatorname{Ker} T_a^* = \{0\}.$$

Proof. Let $f \in \operatorname{Ker} T_a$ and $g \in \operatorname{Ker} T_a^*$. Then $af \in H^2_-$ and $\bar{a}g \in H^2_-$. Therefore, $af\bar{g} \in H^1_- = \{a \in L^1 : a_n = 0,\ n \geq 0\}$ and $\overline{af}g \in H^1_-$. Let $h = af\bar{g}$. Then both h and $\bar{h}$ belong to H^1_-, which means that $h_n = 0$ for any $n \in \mathbb{Z}$ and so $h = 0$. Since $a \neq 0$, it follows that either $f = 0$ or $g = 0$. □

Theorem 5.12. ([13], Theorem 1.7.) *Suppose* $\operatorname{Ker} T_a = \{0\}$. *Then there exists $\epsilon > 0$ such that*

$$\epsilon \|f\|_2 \leq \|T_a f\|_2 \leq \|af\|_2, \quad f \in H^2.$$

Furthermore, there is important relation between Hankel and Toeplitz operators and there is a useful formula that relate Hankel operators with Toeplitz ones. This formula that will be used is the following:

$$T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi, \quad \varphi, \psi \in L^\infty$$

(see [13]).

We now ready to give the following main theorem and closely follow its proof in [13] in detail.

Theorem 5.13. ([13], Theorem 3.3.) *Let $a \in C(\mathbb{T})$. Then* $\operatorname{spec}_e(T_a) = a(\mathbb{T})$.

Proof. Firstly, let us prove that $\operatorname{spec}_e(T_a) \subset a(\mathbb{T})$. Let $a \in C(\mathbb{T})$ and

$$\begin{aligned} a(\mathbb{T}) &= \{\lambda \in \mathbb{C} : a(e^{i\theta}) = \lambda \quad \text{for some} \quad \theta \in [0, 2\pi]\} \\ &= \{\lambda \in \mathbb{C} : a - \lambda \quad \text{vanishes on} \quad \mathbb{T}\}. \end{aligned}$$

Suppose that $\lambda \notin a(\mathbb{T})$. We need to show that $\lambda \notin \operatorname{spec}_e T_a$. We may say that $a - \lambda$ does not vanish on $\mathbb{T}$ since $\lambda \notin a(\mathbb{T})$. If $a - \lambda$ does not vanish on $\mathbb{T}$, then $b = \frac{1}{a-\lambda}$ is continuous. We need to show that T_b is approximate inverse of $T_{a-\lambda}$. We have

$$T_{(a-\lambda)b} - T_{a-\lambda} T_b = H^*_{\overline{(a-\lambda)}} H_b. \tag{5.4}$$

Using equation (5.4), we have $T_1 - T_{a-\lambda} T_b = H^*_{\overline{(a-\lambda)}} H_b$. Then we have

$$I - T_{a-\lambda} T_b = H^*_{\overline{(a-\lambda)}} H_b, \quad I - T_b T_{a-\lambda} = H^*_{\bar{b}} H_{a-\lambda}.$$

Both operators are compact by Theorem 3.8. This means that $T_{a-\lambda} = T_a - \lambda I$ is Fredholm by Theorem 5.5. Thus, $\lambda \notin \operatorname{spec}_e T_a$. Therefore, we have proven that $\operatorname{spec}_e T_a \subset a(\mathbb{T})$. Secondly, let us prove that $a(\mathbb{T}) \subset \operatorname{spec}_e T_a$. It suffices to show that if T_a is Fredholm, then

a is invertible in $C(\mathbb{T})$. By Theorem 5.11, either $\operatorname{Ker} T_a = \{0\}$ or $\operatorname{Ker} T_a^* = \{0\}$. Suppose that $\operatorname{Ker} T_a = \{0\}$. Then by Theorem 5.12, there exists $\epsilon > 0$ such that

$$\epsilon \left\| f \right\|_2 \leq \left\| T_a f \right\|_2 \leq \left\| af \right\|_2, \quad f \in H^2. \tag{5.5}$$

Therefore, we can extend (5.5) to

$$\epsilon \left\| \overline{z}^n f \right\|_2 \leq \left\| a \overline{z}^n f \right\|_2$$

to be able to study on L^2. The set $\{\overline{z}^n f : f \in H^2, \quad n \geq 0\}$ is dense in L^2. Indeed,

$$\overline{z}^n \sum_{k=0}^{\infty} f_k z^k = \sum_{k=0}^{\infty} f_k z^{k-n} = \sum_{k=-n}^{\infty} f_{k+n} z^k.$$

$$g \in L^2 : g(z) = \sum_{k=-\infty}^{\infty} g_k z^k \quad \text{and} \quad g^{(n)}(z) = \sum_{k=-n}^{\infty} g_k z^k.$$

So, $g^{(n)}$ is of the form $\overline{z}^n f$, $f \in H^2$ and $\left\| g^{(n)} - g \right\|_{L^2} \to 0$ as $n \to \infty$. That is,

$$\left\| g^{(n)} - g \right\|_{L^2} = \left\| \sum_{k=-\infty}^{-n-1} g_k z^k \right\|_{L^2} \leq \left\| \sum_{k=-\infty}^{\infty} g_k z^k \right\|_{L^2},$$

$$\left\| \sum_{k=-\infty}^{\infty} g_k z^k \right\| = \Big(\sum_{k=-\infty}^{\infty} \mid g_k \mid^2 \Big)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. Since the set $\{\overline{z}^n f : f \in H^2, \quad n \geq 0\}$ is dense in L^2, it follows that

$$\epsilon \left\| g \right\|_2 \leq \left\| ag \right\|_2$$

for any $g \in L^2$, which implies that $\frac{1}{a} \in C(\mathbb{T})$. □

The following definitions and theorem are taken from [4].

Definition 5.14. A homotopy between two continuous functions

$$f, g : X \to Y$$

is a family of continuous functions $h_t : X \to Y$ for $t \in [0, 1]$ such that $h_0 = f$, $h_1 = g$, and the map $t \to h_t$ is continuous functions $X \to Y$.

Moreover, if we continuously deform a loop without touching a fixed point, say the origin 0, the winding number around that point is constant during the entire deformation, even if we allow the basepoint to move during the deformation. The type of deformation we allow is called a free homotopy, which is just a homotopy through loops. More formally, the definition of free homotopy is given below.

Definition 5.15. A *free homotopy* between two loops α and β in $\mathbb{R}^2 \setminus 0$ is a function $h : [0, 1]^2 \to \mathbb{R}^2 \setminus 0$ such that $h(0, t) = \alpha(t)$ and $h(1, t) = \beta(t)$ for all t and $h(s, 0) = h(s, 1)$ for all s.

Theorem 5.16. *Two loops α and β are freely homotopic in $\mathbb{R}^2 \setminus 0$ if and only if* $\operatorname{wind} \alpha = \operatorname{wind} \beta$.

We now ready to give the following final main theorem and closely follow its proof in [13] in detail.

Theorem 5.17. ([13], Theorem 3.3.) *Let $a \in C(\mathbb{T})$. For $\lambda \notin \operatorname{spec}_e T_a$. We have*

$$\operatorname{index}(T_a - \lambda I) = -\operatorname{wind}(a - \lambda).$$

Proof. Suppose $\lambda \notin \mathrm{spec}_e T_a$. Without loss of generality we can assume that $\lambda = 0$. Since a does not vanish on $\mathbb{T}$, the curve $t \to a(e^{it})$, $t \in [0, 2\pi]$, is homotopic in $\mathbb{C} \setminus \{0\}$ to the curve $t \to e^{int}$ for some integer n. Let $t \to a_\gamma(e^{it})$ be this homotopy,

$$\gamma = 0 : a_\gamma = a \quad \text{and} \quad \gamma = 1 : a_\gamma = e^{itn} = z^n.$$

Theorem 5.10 establishes that the index is a homotopy invariant: if $T(\gamma)$ is a norm continuous family of Fredholm operators then $\operatorname{index} T(\gamma)$ does not depend on γ (see, for example, [3], Theorem 4.3.11.). Now, we need to check that T_{a_γ} is norm continuous in γ. That is, need to check $\left\| T_{a_{\gamma_1}} - T_{a_{\gamma_2}} \right\| \to 0$ as $\mid \gamma_1 - \gamma_2 \mid \to 0$. Let $\tilde{a} = a_{\gamma_1} - a_{\gamma_2}$. We know that $\left\| (T_{a_{\gamma_1}} - T_{a_{\gamma_2}})f \right\| = \left\| T_{a_{\gamma_1} - a_{\gamma_2}} f \right\| = \| P_+ (a_{\gamma_1} - a_{\gamma_2}) f \|$. $\| P_+ \tilde{a} f \|_{L^2} \leq C \| \tilde{a} f \|_{L^2} \leq 2\pi \cdot C \cdot \| \tilde{a} \|_{C(\mathbb{T})} \cdot \| f \|_{L^2}$. Then we have

$$\left\| (T_{a_{\gamma_1}} - T_{a_{\gamma_2}}) f \right\| \leq 2\pi \cdot C \cdot \| a_{\gamma_1} - a_{\gamma_2} \|_{C(\mathbb{T})} \cdot \| f \|_{L^2}.$$

Because of the definition of the homotopy, $\| a_{\gamma_1} - a_{\gamma_2} \| \to 0$ as $\mid \gamma_1 - \gamma_2 \mid \to 0$. Then we have that $\left\| T_{a_{\gamma_1}} - T_{a_{\gamma_2}} \right\| \to 0$ as $\mid \gamma_1 - \gamma_2 \mid \to 0$. Hence, we have that the index does not change under homotopy. Furthermore, Theorem 5.16 shows that the winding number is homotopy invariant. Now, we may find $\operatorname{wind} a$ and $\operatorname{index} T_a$ finding $\operatorname{wind} z^n$ and $\operatorname{index} T_{z^n}$. Firstly, $\operatorname{wind} a = \operatorname{wind} z^n = n$. Indeed, we know the formulas

$$\exp(i \arg_a(t)) = \frac{a(e^{it})}{\mid a(e^{it}) \mid} \quad \text{and} \quad \operatorname{wind} a = \frac{1}{2\pi} (\arg_a(2\pi) - \arg_a(0)).$$

We know that $\arg_a(t) = -i \log(\frac{a(e^{it})}{|a(e^{it})|})$. Then we find that

$$\arg_{z^n}(2\pi) = -i \log(\frac{e^{2\pi i n}}{\mid e^{2\pi i n} \mid}) = -i \log(e^{2\pi i n}) = 2\pi n,$$

$$\arg_{z^n}(0) = -i \log(\frac{e^0}{\mid e^0 \mid}) = -i.0 = 0.$$

Thus, $\operatorname{wind} a = \frac{1}{2\pi}(2\pi n - 0) = n$. Secondly, we know that if $a(z) = z^n$, we have the operator $T_{z^n} : H^2 \to H^2$ such that $T_{z^n} x = (0, 0, ..., 0, f_0, f_1, ...)$ (n zeros) where $x = (f_0, f_1, ...)$. Firstly, we need to find the kernel of T_{z^n}. The kernel of T_{z^n} is the 0-dimensional subspace consisting of vectors $(0, 0, 0, ...)$. Secondly, we need to find the kernel of $T^*_{z^n}$, which is $(\operatorname{Ran}(T_{z^n}))^\perp$.

$$\operatorname{Ran}(T_{z^n}) = (0, 0, ..., 0, f_0, f_1, ...)(n \, \text{zeros}) \quad \text{and} \quad (\operatorname{Ran}(T_{z^n}))^\perp = (f_0, f_1, ..., f_{n-1}, 0, 0, ...).$$

Thus, $\dim \operatorname{Ker}(T^*_{z^n}) = \dim(\operatorname{Ran}(T_{z^n}))^\perp = n$ and $\dim \operatorname{Ker}(T_{z^n}) = 0$. We have the formula

$$\operatorname{index} T_{z^n} = \dim \operatorname{Ker}(T_{z^n}) - \dim \operatorname{Ker}(T^*_{z^n}) = 0 - n = -n.$$

Consequently, we have shown that

$$\operatorname{index} T_a = \operatorname{index} T_{z^n} = -n = -\operatorname{wind} z^n = -\operatorname{wind} a.$$

$\square$

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