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Research Article

Composition-differentiation operators acting on certain Hilbert spaces of analytic functions

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Abstract

We study composition-differentiation operators acting on the Bergman and Dirichlet space of the open unit disk. We first characterize the compactness of composition-differentiation operator on weighted Bergman spaces. We shall then prove that for an analytic self-map φ on the open unit disk $\mathbb{D}$, the induced composition-differentiation operator is bounded with dense range if and only if φ is univalent and the polynomials are dense in the Bergman space on $\Omega := \varphi(\mathbb{D})$.

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1. Introduction

Let $\mathcal{H}$ denote a particular functional Hilbert space of analytic functions on the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane. For instance, we may assume that $\mathcal{H}$ is the classical Hardy space, the Bergman space, the Dirichlet space, and so on. For an analytic self-mapping φ on the open unit disk $\mathbb{D}$, the *composition operator* $C_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known [4, Corollary 3.7] that the composition operator is bounded on the Hardy space H^2 and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\|_{H^2} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

For a function $\psi \in \mathcal{H}$, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ is given by

$$C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi).$$

Another operator which is closely related to the composition operator is the so-called differentiation operator $D(f) = f'$ provided that f' belongs to $\mathcal{H}$ as well. In the context of analytic functions, it is easy to verify that the differentiation operator is not bounded

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on the Hardy space H^2; since $\{z^n\}_{n\geq 1}$ is a sequence of unit vectors in the Hardy space satisfying $\|D(z^n)\| = n$. Nevertheless, for many analytic self-mappings φ on the open unit disk, the operator $D_\varphi : H^2 \to H^2$ defined by

$$D_\varphi(f) = f' \circ \varphi$$

is bounded. Following Fatehi and Hammond (see [6], [7]), we call D_φ a *composition-differentiation operator*. Some authors consider D_φ as the product of two successive operators C_φ and D and write $D_\varphi = C_\varphi D$. This operator was already studied by several authors, among them, S. Ohno [9] characterized its boundedness and compactness in terms of Carleson measures; see also [6], [7] for a recent study of this operator from the perspective of spectral theory. In [1], the authors discussed the conditions to ensure that the composition-differentiation operator is Hilbert-Schmidt.

Before going any further in reviewing the results on this operator, let us recall the most common concrete underlying spaces of functional Hilbert spaces of analytic functions in the open unit disk. Let f be an analytic function in the open unit disk $\mathbb{D}$. The function f is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

It is easy to see that, for an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$, the norm of f in H^2 is given by

$$\|f\|^2 = \sum_{n=0}^\infty |a_n|^2.$$

Another functional Hilbert space on the open unit disk is the weighted Bergman space A^2_α consisting of all analytic functions f in the open unit disk for which the integral

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \mathrm{d}A(z)$$

is finite. Here α is a real parameter larger than -1, and $\mathrm{d}A(z) = \pi^{-1}\mathrm{d}x\mathrm{d}y$ is the normalized area measure in the open unit disk. The norm of f is defined by

$$\|f\|^2_{A^2_\alpha} = (\alpha+1) \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \mathrm{d}A(z).$$

A computation reveals that for $f(z) = \sum_{n=0}^\infty a_n z^n$, we have

$$\|f\|^2_{A^2_\alpha} = \sum_{n=0}^\infty \frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} |a_n|^2,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

The last Hilbert space of analytic functions we discuss is the Dirichlet space $\mathfrak{D}$, which consists of analytic functions f in the open unit disk for which the integral

$$\int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z)$$

is finite. The norm of f in the Dirichlet space is defined by

$$\|f\|^2_{\mathfrak{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z).$$

For a function $f(z) = \sum_{n=0}^\infty a_n z^n$ in the Dirichlet space, it is easy to see that the norm of $f \in \mathfrak{D}$ is given by

$$\|f\|^2_{\mathfrak{D}} = \sum_{n=0}^\infty (n+1)|a_n|^2.$$

In the year 2006 Shûichi Ohno proved that if

$$\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} < 1,$$

then D_φ is a Hilbert-Schmidt operator on H^2, and hence bounded and compact; see [9, Theorem 3.3]. According to [9, Corollary 3.2], for a univalent self-map φ of the open unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, the operator D_φ on H^2 is compact if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Assuming that the symbol function φ is analytic and maps the open unit disk into itself, we intend to study the operator D_φ on the Bergman and on the Dirichlet space. In the next section we shall provide a condition to guarantee the compactness of D_φ on the weighted Bergman space A^2_α. We shall see that a necessary condition for the compactness of $D_\varphi : A^2_\alpha \to A^2_\alpha$ is

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\alpha+2}}{(1-|\varphi(z)|^2)^{\alpha+4}} = 0.$$

Moreover, we prove that for $\alpha \geq 1$, the operator $D_\varphi : A^2_\alpha \to A^2_\alpha$ is compact if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^2} = 0.$$

In the last section we take up the Dirichlet space. We aim to study the boundedness of D_φ in terms of some properties of the symbol function φ. Letting $\Omega = \varphi(\mathbb{D})$, we prove that D_φ is bounded with dense range if and only if φ is univalent and the polynomials are dense in the Bergman space $A^2(\Omega)$. This result generalizes a recent result of G. Cao, L. He, and Kehe Zhu established for the composition operator C_φ (see [2]).

We close this section by mentioning that the study of composition operators has a long history; it has been started by the seminal paper of E. Nordgren [8], and was publicized by the authors of the books [4] and [12]. In general, the results obtained in the complex plane can be carried over to higher dimensions; see, for instance [11] and the references therein, for a discussion on the boundedness and compactness of the composition operators on the polydisk in $\mathbb{C}^n$.

2. Compactness on weighted Bergman spaces

In this section we aim to characterize the compactness of composition-differentiation operator D_φ on the weighted Bergman space A^2_α. For this reason, we begin by computing the adjoint of the composition-differentiation operator $D_{\psi,\varphi}$. Recall that the Hardy, and the Bergman space are reproducing kernel Hilbert spaces. The reproducing kernel of H^2 is the function

$$\mathcal{K}_w(z) = \frac{1}{1-z\overline{w}}, \quad z, w \in \mathbb{D}.$$

This kernel function has the property that for every $f \in H^2$,

$$f(w) = \langle f, \mathcal{K}_w \rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{f^*(e^{i\theta})}{1-e^{-i\theta}w} \mathrm{d}\theta,$$

where

$$f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$$

is the boundary function of f (the above limit exists for almost every point on the boundary of the open unit disk (see [5]) or [10]). Let us denote the first derivative of the kernel function by

$$\mathcal{K}^{(1)}_w(z) = \frac{z}{(1-\overline{w}z)^2}, \quad (z, w) \in \mathbb{D}\times\mathbb{D}.$$

It is well-known that the functional $w \mapsto f'(w)$ is bounded on H^2 (see [4, Theorem 2.16]), so that by Riesz' representation theorem the kernel function $\mathcal{K}_w^{(1)}$ exists, and satisfies

$$f'(w) = \langle f, \mathcal{K}_w^{(1)} \rangle, \quad f \in H^2, w \in \mathbb{D}.$$

The following lemma describes the adjoint of the operator $D_{\psi,\varphi}$ on the Hardy space.

Lemma 2.1 ([7])**.** . *Let φ be an analytic self map on $\mathbb{D}$, and let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $D_{\psi,\varphi}$ is bounded on H^2. Then $D^*_{\psi,\varphi}(\mathcal{K}_w) = \overline{\psi(w)}\mathcal{K}^{(1)}_{\varphi(w)}$.*

It is well-known that the reproducing kernel for the weighted Bergman space A^2_α is

$$K_w^\alpha(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \quad (z,w) \in \mathbb{D} \times \mathbb{D}.$$

This means that for each $f \in A^2_\alpha$ we have

$$f(w) = \langle f, K_w^\alpha \rangle = \int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}w)^{\alpha+2}} \mathrm{d}A_\alpha(z),$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z).$$

It now follows that

$$f'(w) = \int_{\mathbb{D}} f(z) \frac{(\alpha+2)\overline{z}}{(1-\overline{z}w)^{\alpha+3}} \mathrm{d}A_\alpha(z).$$

Now, the uniqueness of the kernel function implies that

$$K^{(1)}_{\alpha,w}(z) := \frac{(\alpha+2)z}{(1-\overline{w}z)^{\alpha+3}}, \quad (z,w) \in \mathbb{D} \times \mathbb{D}$$

is the reproducing kernel corresponding to the functional $f \mapsto f'(w)$ defined on the weighted Bergman space A^2_α. In other words, for each $f \in A^2_\alpha$ we have

$$f'(w) = \langle f, K^{(1)}_{\alpha,w} \rangle, \quad w \in \mathbb{D}.$$

Lemma 2.2. *Let φ be an analytic self map on $\mathbb{D}$, and let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $D_{\psi,\varphi}$ is bounded on A^2_α. Then $D^*_{\psi,\varphi}(K^\alpha_w) = \overline{\psi(w)}K^{(1)}_{\alpha,\varphi(w)}$.*

Proof. Let $w \in \mathbb{D}$ be fixed. Then for each $f \in A^2_\alpha$ we have

$$\begin{aligned}
\langle f, D^*_{\psi,\varphi}(K^\alpha_w) \rangle &= \langle D_{\psi,\varphi}(f), K^\alpha_w \rangle \\
&= \langle \psi \cdot (f' \circ \varphi), K^\alpha_w \rangle \\
&= \psi(w) \langle f' \circ \varphi, K^\alpha_w \rangle \\
&= \psi(w) \langle f, K^{(1)}_{\alpha,\varphi(w)} \rangle \\
&= \langle f, \overline{\psi(w)} K^{(1)}_{\alpha,\varphi(w)} \rangle,
\end{aligned}$$

from which the result follows. □

Theorem 2.3. *Let φ be an analytic self map of the unit disk.*
(a). A necessary condition for the compactness of the operator $D_\varphi : A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ is

$$\lim_{|w| \to 1^-} \frac{(1-|w|^2)^{\alpha+2}}{(1-|\varphi(w)|^2)^{\alpha+4}} = 0. \tag{2.1}$$

(b). For $\alpha \geq 1$, a sufficient condition for the compactness of D_φ is

$$\lim_{|w| \to 1^-} \frac{1-|w|^2}{(1-|\varphi(w)|^2)^2} = 0. \tag{2.2}$$

Proof. For part (a), let D_φ be compact and let

$$k_w^\alpha(z) = \sqrt{(1-|w|^2)^{\alpha+2}}K_w^\alpha(z) = \frac{\sqrt{(1-|w|^2)^{\alpha+2}}}{(1-\overline{w}z)^{\alpha+2}}$$

be the normalized reproducing kernel of $A^2_\alpha(\mathbb{D})$. Let

$$K^{(1)}_{\alpha,w}(z) = \frac{(\alpha+2)z}{(1-\overline{w}z)^{\alpha+3}}$$

be the reproducing kernel corresponding to the functional $f \mapsto f'(w)$ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$. By Lemma 2.2, $D^*_\varphi(K^\alpha_w) = K^{(1)}_{\alpha,\varphi(w)}$. Therefore,

$$\begin{aligned}\|D^*_\varphi(k^\alpha_w)\|^2 &= (1-|w|^2)^{\alpha+2}\|K^{(1)}_{\alpha,\varphi(w)}\|^2\\ &= (1-|w|^2)^{\alpha+2}\frac{(\alpha+2)\left[1+(\alpha+2)|\varphi(w)|^2\right]}{(1-|\varphi(w)|^2)^{\alpha+4}}\\ &= (\alpha+2)\left[1+(\alpha+2)|\varphi(w)|^2\right]\frac{(1-|w|^2)^{\alpha+2}}{(1-|\varphi(w)|^2)^{\alpha+4}}.\end{aligned}$$

But $k^\alpha_w \to 0$ weakly as $|w| \to 1^-$ (see [4, Theorem 2.17]), so that the compactness of D^*_φ implies that $\|D^*_\varphi k^\alpha_w\| \to 0$ as $|w| \to 1^-$. This yields (2.1).

For part (b), we assume that φ satisfies the condition (2.2). Let (f_n) be a (norm) bounded sequence that converges weakly to zero. For simplicity, we may assume that $\|f_n\|_{A^2_\alpha(\mathbb{D})} \le 1$. It follows that f_n and all its derivatives converge to zero on compact subsets of the unit disk. We proceed to show that $\|D_\varphi f_n\|_{A^2_\alpha(\mathbb{D})} \to 0$ as $n \to \infty$. By the assumption (2.2), given $\epsilon > 0$, there is a $0 < \delta < 1$ such that

$$\delta < |z| < 1 \implies 1-|z|^2 < \epsilon(1-|\varphi(z)|^2)^2. \tag{2.3}$$

Note first that there is $C_1 > 0$ such that

$$\|D_\varphi f_n\|^2 \le C_1\left[|f'_n(\varphi(0))|^2 + \int_{\mathbb{D}}|f''_n(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\mathrm{d}A(z)\right].$$

We already mentioned that $f'_n(\varphi(0)) \to 0$ as $n \to \infty$. We now write the second term on the right as the sum of

$$I_n = \int_{|z|\le\delta}|f''_n(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\mathrm{d}A(z),$$

and

$$J_n = \int_{\delta<|z|<1}|f''_n(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\mathrm{d}A(z).$$

Since $f''_n \to 0$ uniformly on the compact set $\{z : |z| \le \delta\}$, and φ' is bounded on this set, we conclude that $I_n \to 0$ as $n \to \infty$. It now follows from (2.3) and the well-known inequality

$$1-|z|^2 \le 2\log\frac{1}{|z|}, \quad |z| \le 1$$

that

$$J_n \le 2\epsilon^{\alpha+1}\int_{\delta<|z|<1}|f''_n(\varphi(z))|^2|\varphi'(z)|^2(1-|\varphi(z)|^2)^{2(\alpha+1)}\log\frac{1}{|z|}\mathrm{d}A(z).$$

Recall the Nevanlinna counting function of φ defined by

$$N_\varphi(z) = \sum_{\varphi(w)=z}\log\frac{1}{|w|},$$

and make a change of variables to obtain

$$J_n \le 2\epsilon^{\alpha+1}\int_{\mathbb{D}}|f''_n(z)|^2(1-|z|^2)^{2(\alpha+1)}N_\varphi(z)\mathrm{d}A(z).$$

By Littlewood's inequality (see [4, Theorem 2.29]), we have

$$N_\varphi(z) \le \log\left|\frac{1-\overline{\varphi(0)}z}{\varphi(0)-z}\right| \asymp 1-\left|\frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}\right|^2 = \frac{(1-|\varphi(0)|^2)(1-|z|^2)}{|1-\overline{\varphi(0)}z|^2}.$$

Thus, there is a constant $C_2 > 0$ such that

$$J_n \le C_2\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\epsilon^{\alpha+1}\int_{\mathbb{D}}|f_n''(z)|^2(1-|z|^2)^{2\alpha+3}\mathrm{d}A(z). \tag{2.4}$$

Since $\alpha \ge 1$, by ignoring the coefficient $\alpha+1$ in front of $\mathrm{d}A_\alpha(z)$, we have

$$\begin{aligned}\int_{\mathbb{D}}|f_n''(z)|^2(1-|z|^2)^{2\alpha+3}\mathrm{d}A(z) &= \int_{\mathbb{D}}|f_n''(z)|^2(1-|z|^2)^{\alpha+3}\mathrm{d}A_\alpha(z)\\ &\le \int_{\mathbb{D}}|f_n''(z)|^2(1-|z|^2)^{4}\mathrm{d}A_\alpha(z).\end{aligned} \tag{2.5}$$

A computation shows that for an analytic function $f(z)=\sum_{n=0}^{\infty}a_nz^n$, using the identity

$$n\,\Gamma(n)=\Gamma(n+1),$$

we have

$$\begin{aligned}\int_{\mathbb{D}}|f''(z)|^2(1-|z|^2)^4\mathrm{d}A_\alpha(z) &= \sum_{n=2}^{\infty}\frac{n^2(n-1)^2\Gamma(n-1)\Gamma(\alpha+5)}{\Gamma(n+\alpha+4)}|a_n|^2\\ &= \sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(n+1)\Gamma(\alpha+5)}{\Gamma(n+\alpha+4)}|a_n|^2\\ &= \frac{\Gamma(\alpha+5)}{\Gamma(\alpha+2)}\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(n+1)\Gamma(\alpha+2)}{(n+\alpha+2)(n+\alpha+3)\Gamma(n+\alpha+2)}|a_n|^2\\ &\le \frac{\Gamma(\alpha+5)}{\Gamma(\alpha+2)}\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}|a_n|^2\\ &\le \frac{\Gamma(\alpha+5)}{\Gamma(\alpha+2)}\int_{\mathbb{D}}|f(z)|^2\mathrm{d}A_\alpha(z)\\ &= \frac{\Gamma(\alpha+5)}{\Gamma(\alpha+2)}\|f\|^2_{A^2_\alpha}.\end{aligned}$$

This, together with (2.4) and (2.5), implies that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}J_n &\le C_3\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\epsilon^{\alpha+1}\int_{\mathbb{D}}|f_n(z)|^2\mathrm{d}A_\alpha(z)\\ &\le C_3\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\epsilon^{\alpha+1}.\end{aligned}$$

Thus $J_n \to 0$ as $n \to \infty$. This completes the proof. □

Note that $D_{\psi,\varphi}(f) = M_\psi D_\varphi(f)$ where M_ψ is the operator of multiplication by ψ, hence bounded on the weighted Bergman space if $\psi \in H^\infty(\mathbb{D})$. Therefore, we get the following corollary.

Corollary 2.4. *Let φ be an analytic self map of the unit disk, $\alpha \ge 1$, and let ψ be an injective self map of the unit disk. Then the operator $D_{\psi,\varphi} : A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ is compact if*

$$\lim_{|z|\to 1^-}\frac{1-|z|^2}{(1-|\varphi(z)|^2)^2}=0.$$

3. Dense range operators on Dirichlet space

The Dirichlet space, denoted by $\mathfrak{D}$, consists of analytic functions on the open unit disk for which the integral

$$\int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z)$$

is finite. The norm of a function $f \in \mathfrak{D}$ is defined by

$$\|f\|_{\mathfrak{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z),$$

where $\mathrm{d}A(z)$ is the normalized area measure in the open unit disk. The inner product in $\mathfrak{D}$ is defined by

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \mathrm{d}A(z).$$

We denote by $\mathrm{Hol}(\mathbb{D})$ the space of all holomorphic functions on the open unit disk. Let

$$\mathfrak{D}' = \{f \in \mathrm{Hol}(\mathbb{D}) : f' \in \mathfrak{D}\}.$$

The (standard) Bergman space A^2 consists of all $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z) < \infty.$$

The Bergman space on a bounded simply connected domain $\Omega \subset \mathbb{C}$, denoted by $A^2(\Omega)$, is defined similarly; we just replace the unit disk by Ω:

$$\|f\|_{A^2(\Omega)}^2 = \int_{\Omega} |f(z)|^2 \mathrm{d}A(z) < \infty.$$

It is clear from the definition of norm in the Dirichlet space that $f \in \mathfrak{D}$ if and only if $f' \in A^2$. Using the Taylor series for the norm of a given function in these spaces, we see that $\mathfrak{D} \subset A^2$. Now, let $f \in \mathfrak{D}'$. It follows that $f' \in \mathfrak{D} \subset A^2$, implying that $f \in \mathfrak{D}$. Therefore, $\mathfrak{D}' \subset \mathfrak{D}$. It is easy to see that $\mathfrak{D}'$, equipped with the norm of $\mathfrak{D}$, is a closed subspace of $\mathfrak{D}$. Indeed, we could define

$$\mathfrak{D}' = \{f \in \mathfrak{D} : f' \in \mathfrak{D}\}.$$

Let φ be an analytic self-map of $\mathbb{D}$. In this section, we consider the composition-differentiation operator

$$D_\varphi : \mathfrak{D}' \to \mathfrak{D}$$

given by

$$D_\varphi(f) = f' \circ \varphi.$$

In the following theorem we address the boundedness of the composition-differentiation operator $D_\varphi : \mathfrak{D}' \to \mathfrak{D}$.

Theorem 3.1. *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $\Omega = \varphi(\mathbb{D})$. Then $D_\varphi : \mathfrak{D}' \to \mathfrak{D}$ is bounded with dense range if and only if φ is univalent and the polynomials are dense in $A^2(\Omega)$.*

Proof. Suppose D_φ is bounded and has dense range in $\mathfrak{D}$. To see that φ is univalent, assume that z_1 and z_2 are two points in the unit disk such that $\varphi(z_1) = \varphi(z_2)$. This implies that for each $f \in \mathfrak{D}'$ we have $(D_\varphi f)(z_1) = (D_\varphi f)(z_2)$. By assumption, there exists a sequence (f_n) in $\mathfrak{D}'$ such that $D_\varphi f_n$ converges to $f(z) = z$ in the norm topology of $\mathfrak{D}$; and hence pointwise. But, $(D_\varphi f_n)(z_1) = (D_\varphi f_n)(z_2)$, from which it follows that $z_1 = z_2$. Hence, φ is univalent.

If the polynomials are not dense in $A^2(\Omega)$, we can find a non-identically zero function $g \in A^2(\Omega)$ in such a way that g annihilates all monomials $\{z^k\}_{k\geq 0}$:

$$\int_{\Omega} g(z)\overline{z^k} \mathrm{d}A(z) = 0, \quad k = 0, 1, 2, \cdots .$$

Note that $(g \circ \varphi)\varphi'$ is an analytic function on the unit disk, so it has an anti-derivative; say f. We may assume that $f(0) = 0$. This f is not identically zero, since otherwise $f' = (g \circ \varphi)\varphi' = 0$ from which it follows that either g is identically zero (which is not possible by its choice), or φ is a constant, contradicting it is univalent. A computation shows that

$$\begin{aligned}\|f\|_{\mathfrak{D}}^2 &= \int_{\mathbb{D}} |f'|^2 \mathrm{d}A(z) \\ &= \int_{\mathbb{D}} |g(\varphi(z))|^2 |\varphi'(z)|^2 \mathrm{d}A(z) \\ &= \int_{\Omega} |g(w)|^2 \mathrm{d}A(w) \\ &= \|g\|_{A^2(\Omega)}^2 < \infty. \end{aligned} \tag{3.1}$$

Thus $f \in \mathfrak{D}$. Furthermore, $\langle f, \varphi^0 \rangle = f(0) = 0$, and for each $n \geq 1$ we have

$$\begin{aligned}\langle f, \varphi^n \rangle_{\mathfrak{D}} &= n \int_{\mathbb{D}} f'(z) \left(\overline{\varphi^{n-1}(z)\varphi'(z)}\right) \mathrm{d}A(z) \\ &= n \int_{\mathbb{D}} g(\varphi(z))\, \overline{\varphi^{n-1}(z)}\, |\varphi'(z)|^2 \mathrm{d}A(z) \\ &= n \int_{\Omega} g(w) \overline{w^{n-1}} \mathrm{d}A(w) = 0. \end{aligned} \tag{3.2}$$

This shows that for every polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 \cdots + a_k z^k,$$

the function f is orthogonal to

$$D_\varphi p(z) = p'(\varphi(z)) = a_1 + 2a_2 \varphi(z) + \cdots + k a_k \varphi^{k-1}(z).$$

Since D_φ is bounded, it follows that for each $g \in \mathfrak{D}'$, the function f is orthogonal to $D_\varphi g$. On the other hand, the range of D_φ is dense in the Dirichlet space, so that f annihilates the Dirichlet space, and hence f must be zero, a contradiction. Thus the polynomials are dense in $A^2(\Omega)$.

Conversely, we assume that φ is univalent, and that the polynomials are dense in $A^2(\Omega)$. To prove the boundedness of D_φ, we assume $f \in \mathfrak{D}'$. This means that $f'' \in A^2$ and hence

$$\begin{aligned}\|D_\varphi f\|_{\mathfrak{D}}^2 &= |f'(\varphi(0))|^2 + \int_{\mathbb{D}} |f''(\varphi(z))|^2 |\varphi'(z)|^2 \mathrm{d}A(z) \\ &= |f'(\varphi(0))|^2 + \int_{\Omega} |f''(w)|^2 \mathrm{d}A(w) \\ &= |f'(\varphi(0))|^2 + \int_{\mathbb{D}} |f''(w)|^2 \mathrm{d}A(w) \\ &< \infty. \end{aligned}$$

Thus, D_φ maps $\mathfrak{D}'$ into $\mathfrak{D}$. Now, an application of the Closed Graph Theorem shows that D_φ is bounded. To see that the range of D_φ is dense in the Dirichlet space, suppose that f is a function in $\mathfrak{D}$ that annihilates

$$\{D_\varphi p : p \text{ is a polynomial}\}.$$

Such a function f cannot be a nonzero constant, since otherwise all polynomials p must satisfy $p'(\varphi(0)) = 0$. We now consider the function

$$g = (f'/\varphi') \circ \varphi^{-1}$$

on Ω. It is easy to see that $f' = (g \circ \varphi)\varphi'$. It now follows from (3.1) that $\|g\|_{A^2(\Omega)} = \|f\|_{\mathfrak{D}}$, that is $g \in A^2(\Omega)$. Since $f \perp D_\varphi p$ for each polynomial p, we conclude that $f \perp \varphi^n$, for $n \geq 0$. It now follows from (3.2) that g annihilates each polynomial p in $A^2(\Omega)$. By our assumption, the polynomials are dense in the Bergman space $A^2(\Omega)$, so that $g = 0$. This, in turn, implies that $f' = 0$ or $f = 0$. Thus the only function in $\mathfrak{D}$ that is orthogonal to $\{D_\varphi p : p \text{ is a polynomial}\}$ is the zero function; meaning that the range of D_φ is dense in $\mathfrak{D}$. □

Among other results, it was observed in [3] that if $C_\varphi : \mathfrak{D} \to \mathfrak{D}$ has dense range, then so does $C_{r\varphi}$ for any $0 < r < 1$. The following corollary generalizes this result to composition-differentiation operators.

Corollary 3.2. *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $\Omega = \varphi(\mathbb{D})$. If $D_\varphi : \mathfrak{D}' \to \mathfrak{D}$ is bounded and has dense range, then so does $D_{r\varphi}$ for any $0 < r < 1$.*

Proof. We note that if the polynomials are dense in the Bergman space $A^2(\Omega)$, then the same is true for the Bergman space $A^2(r\Omega)$ for any $0 < r < 1$. The result now follows from the preceding theorem. □

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