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μ-Symmetries and μ-Conservation Laws for the Nonlinear Dispersive Modified Benjamin-Bona-Mahony Equation

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Abstract

This work discusses the μ-symmetry and conservation law of μ procedure for the nonlinear dispersive modified Benjamin-Bona-Mahony equation (NDMBBME). This equation models an approximation for surface long waves in nonlinear dispersive media. It can also describe the hydromagnetic waves in a cold plasma, acoustic waves in inharmonic crystals, and acoustic gravity waves in compressible fluids. First and foremost, we offer some essential pieces of information about the μ-symmetry and the conservation law of μ concepts. In light of such information, μ-symmetries are found. Using characteristic equations, the NDMBBME is reduced to ordinary differential equations (ODEs). We obtained the exact invariant solutions by solving the nonlinear ODEs. Furthermore, employing the variational problem procedure, we get the Lagrangian and the μ-conservation laws. The exact solutions and conservation laws are new for the NDMBBME that are not reported by the other studies. We also demonstrate the properties with figures for these solutions.

1. Introduction

Nonlinear partial differential equations (NLPDEs) play a paramount role in the investigation of considerable problems in physics and geometry. The struggle to discover exact solutions to nonlinear equations is crucial for understanding most nonlinear physical phenomena. Nonlinear wave phenomena arise in diverse scientific and engineering specializations, such as solid-state physics, chemical physics, and geometry.

Lately, influential and efficient procedures for discovering analytic solutions to nonlinear equations have lured considerable interest from various groups of scientists, such as Semi-inverse variational technique [1], New extended direct algebraic method (NEDAM) [2], Extended rational sine-cosine methods and sinh-cosh methods [3], Multiwave solutions [4], Generalized exponential rational function method (GERFM) [5], Lie symmetry analysis [6–10], Simplified Hirota technique [11], Extended simple equation method [12], Multiple exp-function method [13], Improved auxiliary equation approach [14], Modulation instability [15], Modified Jacobi elliptic expansion method [16], μ-symmetries method [17–20] and so on.

Lie symmetry analysis, which was first studied by S. Lie, is one of the most general and influential strategies for getting exact solutions for NLPDEs. A symmetry group of a differential equation means a transformation that maps (smooth) solutions to solutions. Lie utilized a continuous group of transformations to develop solution strategies for ODEs. ODEs with trivial Lie or no symmetries but possess λ-symmetries can be integrated using the λ-symmetry procedure. λ-symmetry was introduced by Muriel and Romero as a new kind of symmetry [21]. Morando and Gaeta viewed the case of PDEs and extended the λ-symmetries to the μ-symmetries [22–24]. In the event of the $\mu-$symmetries of the Lagrangian, the conservation law is referred to as the conservation law of μ.

The principal purpose of the current investigation is to scrutinize the $\mu-$symmetries, reductions, invariant solutions, and conservation law of μ for the NDMBBME.

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The study is assembled as follows. Section 2 offers the main concepts of the $\mu-$symmetry and $\mu-$conservation law procedure. We yield the μ-symmetries of the NDMBBME and build the invariant solutions of the model by employing the accepted μ-symmetries in Section 3. We obtain Lagrangian in potential form by using the variational problem method and the Frechet derivative in Section 4. For the NDMBBME, the conservation law of μ is investigated in Section 5. Lastly, in Section 6, conclusions are given.

2. The Principal Vision of the μ-Symmetry and Conservation Law of μ Procedure

2.1. μ-symmetry concept

Surmise that $\mu=\lambda_i dx_i$ be a semi basic one-form on first order jet space $(J^{(1)}\aleph,\pi,\aleph)$, which is compatible, namely, $\wp_j\lambda_i=\wp_i\lambda_j$ [17–20,24]. Here, $\wp_i$ and $\wp_j$ are total derivative with respect to x_i, and λ_i defines from $J^{(1)}\aleph$ to $\mathbb{R}$.
Think that Δ be the sth-order partial differential equation (PDE) as follows

$$\Delta:\bar{h}(x,w^{(s)})=0. \tag{2.1}$$

Here $w=w(x)=w(x_1,x_2,...,x_p)$ and $w^{(s)}$ symbolizes all sth order derivatives of w as to x.
Let Ω be a vector field on $J^{(s)}\aleph$. Then, we describe the Ω as

$$\Omega=\Upsilon+\sum_{|J|=1}^{s}\psi_J\partial w_J, \tag{2.2}$$

in which Υ is a vector field on $\aleph$ and defines as

$$\Upsilon=\xi^i(x,w)\frac{\partial}{\partial x^i}+\varphi(x,w)\frac{\partial}{\partial w}. \tag{2.3}$$

Here, (2.2) is the prolongation of μ of (2.3) if its coefficient provides the prolongation formula of μ

$$\psi_{J,i}=(\wp_i+\lambda_i)\psi_J-w_{J,m}(\wp_i+\lambda_i)\xi^m, \tag{2.4}$$

in which $\psi_0=\varphi$. Let $R\subset J^{(s)}\aleph$ be the solution manifold for Δ. If $\Omega:R\to TR$, it is said that, for Eq. (2.1), (2.3) is a $\mu-$symmetry. To get μ-symmetry of Eq. (2.1), then applies (2.2) to Eq. (2.1), and restrain the got outcomes to the solution manifold $R_\Delta\subset\aleph^{(s)}$ that will be up to ξ, φ, λ_i. If we deem the λ as functions on $\aleph^{(s)}$ and compatibility conditions between the λ_i, a system of all the dependence on w_J form the determining equations [24]. $V=\exp(\int\mu)\Upsilon$ is an exponential vector field if (2.3) is a vector field on $\aleph$.

Theorem 2.1. *Let sth-order PDE defines as $\Delta(x,w^s)$, (2.3) be a vector field on $\aleph$, with invariant surface condition $Q=\varphi-w_i\xi^i$, and Ω be the $\mu-$prolong of order s of Υ. In this case, for Δ, (2.3) is a $\mu-$symmetry, then $\Omega:R_\Upsilon\to TR_\Upsilon$, in which $R_\Upsilon\subset J^{(s)}\aleph$ is the solution manifold for Δ_Υ made of Δ and $\grave{E}_J:=\wp_JQ=0$, $\forall J$ with $|J|=0,1,...,s-1$ [17–20, 24].*

2.2. μ-conservation law

Surmise that $\mu=\lambda_i dx_i$ be a semi-basic one-form and with the compability condition $\wp_j\lambda_i=\wp_i\lambda_j$.
A conservation law of μ is

$$(\wp_i+\lambda_i)P^i=0.$$

Here, P^i is a conserved vector of μ and this vector is a matrix-valued $\aleph$-vector.
Surmise that $\mathscr{L}=\mathscr{L}(x,w^{(s)})$ depicts the sth order Lagrangian. For $\mathscr{L}$, (2.3) is a $\mu-$symmetry, namely, $\exists\aleph$-vector P^i such that $(\wp_i+\lambda_i)P^i=0$ where the necessary and sufficient condition is $\Omega[\mathscr{L}]=0$ [22].
Let second-order Lagrangian defines as $\mathscr{L}=\mathscr{L}(x,t,w,w_x,...,w_{tt})$ and for $\mathscr{L}$, $\Upsilon=\varphi(\frac{\partial}{\partial w})$ be a $\mu-$symmetry. $\aleph-$vector P^i is got as [22]

$$P^i:=\varphi\frac{\partial\mathscr{L}}{\partial w_i}+\left[(\wp_j+\lambda_j)\varphi\right]\frac{\partial\mathscr{L}}{\partial w_{ij}}-\varphi\wp_j(\frac{\partial\mathscr{L}}{\partial w_{ij}}). \tag{2.5}$$

Here, $\wp_j$ is the total derivative.
The Frechet derivative $\wp_\Delta$ is self adjoint, namely, $\wp_\Delta^*=\wp_\Delta$ is necessary and sufficient condition in which a system admits a variational formulation [17–20, 25].

Theorem 2.2. *Let $\Delta=0$ be a system of differential equations. For some variational problem $\pounds=\int L dx$, Δ is the Euler-Lagrange expression, i.e., $\wp_\Delta=\wp_\Delta^*$ if and only if $\Delta=\grave{E}(L)$. Then, by employing the homotopy formula $L[u]=\int_0^1 u\Delta[\lambda u]d\lambda$, a Lagrangian can be found for Δ.*

3. Application of the μ-Symmetry Procedure to NDMBBME

The NDMBBME can be represented as

$$\Delta_w: w_t+w_x-\delta w^2 w_x+w_{xxx}=0. \tag{3.1}$$

Here, δ is a nonzero and real constant, and $w=w(x,t)$.
The NDMBBME was first used to define an approximation for surface long waves in nonlinear dispersive media. It can also describe the hydromagnetic waves in a cold plasma, acoustic waves in inharmonic crystals, and acoustic gravity waves in compressible fluids [26–28].

Classical Lie symmetry analysis of Eq. (3.1) was also examined in [29] and 3-dimensional Lie algebra was obtained.
Assume that we have a semi-basic one-form $\mu = \lambda_1 dx + \lambda_2 dt$ such that $\wp_t \lambda_1 = \wp_x \lambda_2$ when $w_t + w_x - \delta w^2 w_x + w_{xxx} = 0$.
Let

$$\Upsilon = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial w} \tag{3.2}$$

be a vector field on $\aleph$, and ξ, τ, φ based on x, t, w. The third prolongation is given as

$$\Omega = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial w} + \psi^x \frac{\partial}{\partial w_x} + \psi^t \frac{\partial}{\partial w_t} + \psi^{xxx} \frac{\partial}{\partial w_{xxx}}.$$

Ω satisfies the following $\mu-$symmetry condition:

$$\psi^t + \psi^x - 2\beta \varphi w_x - \beta u^2 \psi^x + \psi^{xxx} \mid_{\Delta=0} = 0,$$

where

$$\begin{aligned}
\psi^x &= (\wp_x + \lambda_1)\varphi - w_x(\wp_x + \lambda_1)\xi - w_t(\wp_x + \lambda_1)\tau, \\
\psi^t &= (\wp_t + \lambda_2)\varphi - w_x(\wp_t + \lambda_2)\xi - w_t(\wp_t + \lambda_2)\tau, \\
\psi^{xx} &= (\wp_x + \lambda_1)\psi^x - w_{xx}(\wp_x + \lambda_1)\xi - w_{xt}(\wp_x + \lambda_1)\tau, \\
\psi^{xxx} &= (\wp_x + \lambda_1)\psi^{xx} - w_{xxx}(\wp_x + \lambda_1)\xi - w_{xxt}(\wp_x + \lambda_1)\tau,
\end{aligned}$$

and $\wp_t$, $\wp_i$ denote the total differentiations as to t and x^i:

$$\begin{aligned}
\wp_t &= \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_{tt} \frac{\partial}{\partial w_t} + w_{tx^k} \frac{\partial}{\partial w_{x^k}} + \dots, \\
\wp_i &= \frac{\partial}{\partial x^i} + w_{x^i} \frac{\partial}{\partial w} + w_{tx^i} \frac{\partial}{\partial w_t} + w_{x^i x^k} \frac{\partial}{\partial w_{x^k}} + \dots.
\end{aligned}$$

By applying Ω to Eq. (3.1) and substituting $-w_t - w_x + \delta w^2 w_x$ for w_{xxx}, we obtain an over-determined system for $\lambda_1, \lambda_2, \tau, \xi, \varphi$

$$-3\tau_{ww} = 0, \quad -6\xi_{ww} = 0,$$

$$-3\tau\lambda_1 - 3\tau_x = 0,$$

$$-6\tau_w\lambda_1 - 3\tau\lambda_{1w} - 6\tau_{xw} = 0,$$

$$-9\xi_w\lambda_1 - 4\xi\lambda_{1w} - 9\xi_{xw} + 3\varphi_{ww} = 0,$$

$$\vdots$$

$$-3\xi_w\lambda_{1w} - \xi\lambda_{1ww} - 3\lambda_1\xi_{ww} + \varphi_{www} - 3\xi_{wwx} = 0,$$

$$-6\tau_{xw}\lambda_1 - 3\tau_w\lambda_{1x} - 3\tau_x\lambda_{1w} - 2\tau\lambda_{1xw} - 3\tau_w\lambda_1^2 + 3\xi_w - 3\lambda_1\tau\lambda_{1w} - 3\tau_{xwx} = 0. \tag{3.3}$$

Surmise that $\lambda_1 = \wp_x[H] + y$ and $\lambda_2 = \wp_t[H] + z$, in which $H = H(x,t)$, $y = y(x)$ and $z = z(t)$ are arbitrary functions, and λ_1, λ_2 satisfy to $\wp_x\lambda_2 = \wp_t\lambda_1$ on solutions to Eq. (3.1).

Case 1: When $y = 0$, $z = 0$, and $H = -\ln(\Xi)$ in the functions of λ_1 and λ_2, then by substituting the functions

$$\lambda_1 = -\frac{\Xi_x}{\Xi}, \quad \lambda_2 = -\frac{\Xi_t}{\Xi}$$

into the system of (3.3) and solving them , we get

$$\xi = \Xi, \quad \tau = 0, \quad \varphi = 0.$$

Then, by inserting the ξ, τ, and φ into (3.2), we obtain

$$\Upsilon_1 = \Xi \frac{\partial}{\partial x}. \tag{3.4}$$

(3.4) is $\mu-$symmetry of Eq. (3.1). Also,

$$\begin{aligned} V &= \exp\left(\int \lambda_1 dx + \lambda_2 dt\right)\Upsilon \\ &= \exp\left(\int (-\frac{\Xi_x}{\Xi})dx + (-\frac{\Xi_t}{\Xi})dt\right)\Upsilon_1. \end{aligned}$$

Thanks to the Theorem 2.1, the order reduction of Eq. (3.1) is

$$\begin{aligned} Q &= \varphi - \xi w_x - \tau w_t \\ &= -\Xi w_x. \end{aligned} \tag{3.5}$$

Case 2: When $y = 0$, $z = 0$, and $H = -\ln(\Xi)$ in the functions of λ_1 and λ_2, then by placing the functions

$$\lambda_1 = -\frac{\Xi_x}{\Xi},\ \ \lambda_2 = -\frac{\Xi_t}{\Xi}$$

into the system of (3.3) and solving them, we attain

$$\xi = \frac{2}{3}\Xi, \quad \tau = \Xi, \quad \varphi = 0.$$

Then, by substituting the ξ, τ, and φ into (3.2), we reach

$$\Upsilon_2 = \Xi\left(\frac{2}{3}\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right). \tag{3.6}$$

(3.6) is $\mu-$symmetry of Eq. (3.1). Also,

$$V = \exp\left(\int (-\frac{\Xi_x}{\Xi})dx + (-\frac{\Xi_t}{\Xi})dt\right)\Upsilon_2.$$

By using the Theorem 2.1, the order reduction of Eq. (3.1) is

$$\begin{aligned} Q &= \varphi - \xi w_x - \tau w_t \\ &= -\Xi\left(\frac{2}{3}w_x + w_t\right). \end{aligned} \tag{3.7}$$

Case 3: When $y = 0$, $z = \frac{C_1}{C_1 t - 3}$, and $H = -\ln(\Xi)$ in the functions of λ_1 and λ_2, then by inserting the functions

$$\lambda_1 = -\frac{\Xi_x}{\Xi},\ \ \lambda_2 = \frac{C_1}{C_1 t - 3} - \frac{\Xi_t}{\Xi}$$

into the system of (3.3) and solving them, we get

$$\xi = \left(\frac{(2t+x)C_1 - C_2 - 6}{3C_1 t - 9}\right)\Xi, \quad \tau = \Xi, \quad \varphi = 0.$$

Then, by substituting the ξ, τ, and φ into the vector field, we obtain

$$\Upsilon_3 = \Xi\left(\left(\frac{(2t+x)C_1 - C_2 - 6}{3C_1 t - 9}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right). \tag{3.8}$$

(3.8) is $\mu-$symmetry of Eq. (3.1). Also,

$$V = \exp\left(\int \left(-\frac{\Xi_x}{\Xi}\right)dx + \left(\frac{C_1}{C_1 t - 3} - \frac{\Xi_t}{\Xi}\right)dt\right)\Upsilon_3.$$

By using the Theorem 2.1, the order reduction of Eq. (3.1) is

$$\begin{aligned} Q &= \varphi - \xi w_x - \tau w_t \\ &= -\Xi\left[\left(\frac{(2t+x)C_1 - C_2 - 6}{3C_1 t - 9}\right)w_x + w_t\right]. \end{aligned} \tag{3.9}$$

Here, $\Xi = \Xi(x,t)$ is an arbitrary positive function, C_1 and C_2 are arbitrary constants.

3.1. μ-invariant solutions for the NDMBBME

Thanks to the invariant surface condition, the characteristic equation forms are constructed. By solving the characteristic equation form, similarity variables are obtained. Then, thanks to the similarity variables and the original equation, a PDE can be converted to an ODE. Then, by solving the ODE, the invariant solution is obtained.
The characteristic equation corresponding to (3.5) is written as

$$\frac{dx}{-\Xi}=\frac{dt}{0}=\frac{dw}{0}. \tag{3.10}$$

By solving (3.10), we get similarity variables as indicated below

$$\sigma=t, \quad w=\Xi_1(\sigma).$$

After placing w into Eq. (3.1), Eq. (3.1) can be reduced to the ODE

$$\begin{aligned}\frac{d}{d\sigma}\Xi_1 &= 0,\\ \Xi_1(\sigma) &= C.\end{aligned}$$

Therefore, we have an invariant solution

$$w=C.$$

For (3.7), let us consider $\Xi\neq 0$. Then, we have $\frac{2}{3}w_x+w_t=0$. The characteristic equation corresponding to (3.7) is written as

$$\frac{dx}{\frac{2}{3}}=\frac{dt}{1}=\frac{dw}{0}. \tag{3.11}$$

By solving (3.11), we get similarity variables as indicated below

$$\varpi=t-\frac{3}{2}x, \quad w=\Xi_2(\varpi).$$

After placing w into Eq. (3.1), Eq. (3.1) can be reduced to the ODE as

$$12\zeta\Xi_2^2(\frac{d}{d\varpi}\Xi_2)-4(\frac{d}{d\varpi}\Xi_2)-27(\frac{d^3}{d\varpi^3}\Xi_2)=0.$$

Solving the above ODE, we get an integral form, specifically,

Solution Set-1: letting $C_1=C_3=0$, $C_2=1$, we obtain

$$w(x,t)=-\frac{9\ JacobiSN\left(\frac{1}{9}\sqrt{6+6\sqrt{1-27\delta}}(t-\frac{3}{2}x),\frac{1}{9}\sqrt{-\frac{3\left(27\delta-2+2\sqrt{1-27\delta}\right)}{\delta}}\right)}{\sqrt{3+3\sqrt{1-27\delta}}}. \tag{3.12}$$

Solution Set-2: Let $C_1=C_2=0$, $C_3=1$, we get

$$w(x,t)=\frac{\sqrt{2}\sqrt{\delta\left(\tan\left(\frac{2\sqrt{3}}{9}\left(t-\frac{3}{2}x+1\right)\right)^2+1\right)}}{\delta\tan\left(\frac{2\sqrt{3}}{9}(t-\frac{3}{2}x+1)\right)}. \tag{3.13}$$

Solution Set-3: If we choose $C_1=C_2=C_3=0$, we reach

$$w(x,t)=\frac{\sqrt{2}\sqrt{\delta\left(\tan\left(\frac{2\sqrt{3}}{9}\left(t-\frac{3}{2}x\right)\right)^2+1\right)}}{\delta\tan\left(\frac{2\sqrt{3}}{9}(t-\frac{3}{2}x)\right)}. \tag{3.14}$$

For (3.9), let $-\Xi\neq 0$. Then we have $\left(\frac{(2t+x)C_1-C_2-6}{3C_1t-9}\right)w_x+w_t=0$. The characteristic equation corresponding to (3.9) is written as

$$\frac{dx}{\frac{(2t+x)C_1-C_2-6}{3C_1t-9}}=\frac{dt}{1}=\frac{dw}{0}. \tag{3.15}$$

By solving (3.15), we obtain similarity variables as indicated below

$$\rho=-\frac{C_1(t-x)+C_2-3}{C_1(tC_1-3)^{\frac{1}{3}}},\quad w=\Xi_3(\rho).$$

After placing w into Eq. (3.1), Eq. (3.1) can be reduced to the ODE

$$-(\frac{d}{d\rho}\Xi_3)C_1\rho+3(\frac{d^3}{d\rho^3}\Xi_3)=0.$$

Solving the above equation, we have an invariant solution

$$w(x,t)=C_1+C_2\rho\left(\begin{array}{c}3\Gamma(\frac{2}{3})^2\rho(-C_1)^{\frac{1}{3}}hypergeom\left(\left[\frac{2}{3}\right],\left[\frac{4}{3},\frac{5}{3}\right],\frac{1}{27}C_1\rho^3\right)\\+4\pi\sqrt{3}hypergeom\left(\left[\frac{1}{3}\right],\left[\frac{2}{3},\frac{4}{3}\right],\frac{1}{27}C_1\rho^3\right)\end{array}\right)$$

$$+C_3\rho\left(\begin{array}{c}\sqrt{3}\Gamma(\frac{2}{3})^2\rho(-C_1)^{\frac{1}{3}}hypergeom\left(\left[\frac{2}{3}\right],\left[\frac{4}{3},\frac{5}{3}\right],\frac{1}{27}C_1\rho^3\right)\\-4hypergeom\left(\left[\frac{1}{3}\right],\left[\frac{2}{3},\frac{4}{3}\right],\frac{1}{27}C_1\rho^3\right)\pi\end{array}\right). \tag{3.16}$$

(3.16) holds the Eq. (3.1) when $\delta=0$. Here, $\rho=-\frac{C_1(t-x)+C_2-3}{C_1(tC_1-3)^{\frac{1}{3}}}$. Also, *hypergeom* is hypergeometric function. In particular, we deal with the following case:

$$\Upsilon_{1,2}=\Upsilon_2+\gamma_1\Upsilon_1.$$

Thus, we have

$$\Upsilon_{1,2}=\Xi\left(\left(\frac{2}{3}+\gamma_1\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right). \tag{3.17}$$

(3.17) is $\mu-$symmetry of Eq. (3.1). By using the Theorem 2.1, we have

$$\begin{aligned}Q&=\varphi-\xi w_x-\tau w_t.\\&=-\Xi\left[\left(\frac{2}{3}+\gamma_1\right)w_x+w_t\right].\end{aligned} \tag{3.18}$$

The characteristic equation corresponding to (3.18) is written as

$$\frac{dx}{(\frac{2}{3}+\gamma_1)}=\frac{dt}{1}=\frac{dw}{0}. \tag{3.19}$$

By solving (3.19), we get similarity variables as indicated below

$$\kappa=\frac{3t\gamma_1+2t-3x}{2+3\gamma_1},\quad w=\Xi_4(\kappa).$$

After placing w into Eq. (3.1), Eq. (3.1) can be reduced to the ODE

$$27(\frac{d}{d\kappa}\Xi_4)\Xi_4^2\delta\gamma_1^2+36(\frac{d}{d\kappa}\Xi_4)\Xi_4^2\delta\gamma_1+12\delta\Xi_4^2(\frac{d}{d\kappa}\Xi_4)$$

$$+27(\frac{d}{d\kappa}\Xi_4)\gamma_1^3+27(\frac{d}{d\kappa}\Xi_4)\gamma_1^2-4(\frac{d}{d\kappa}\Xi_4)-27(\frac{d^3}{d\kappa^3}\Xi_4)=0.$$

By solving the above equation, we get an integral form, especially, if we choose $C_1=C_3=0$, $C_2=1$, we attain

$$w(x,t)=-\frac{1}{\sqrt{\dfrac{-81\gamma_1^3-81\gamma_1^2+12+9\sqrt{\begin{array}{c}81\gamma_1^4+54\gamma_1^3-27\gamma_1^2-108\delta\\-12\gamma_1+4\end{array}}\gamma_1+}{6\sqrt{81\gamma_1^4+54\gamma_1^3-27\gamma_1^2-108\delta-12\gamma_1+4}}}}\times$$

$$\left(18\,JacobiSN\left(\begin{array}{c}\frac{1}{18(2+3\gamma_1)}\sqrt{\begin{array}{c}24-162\gamma_1^3+18\sqrt{\begin{array}{c}81\gamma_1^4+54\gamma_1^3\\-27\gamma_1^2-108\delta-12\gamma_1+4\end{array}}\gamma_1\\-162\gamma_1^2+12\sqrt{\begin{array}{c}81\gamma_1^4+54\gamma_1^3\\-27\gamma_1^2-108\delta-12\gamma_1+4\end{array}}\end{array}}(3t\gamma_1+2t-3x),\\ \frac{1}{18}\left(\frac{\sqrt{6}}{\delta}\sqrt{\begin{array}{c}81\gamma_1^4+9\sqrt{81\gamma_1^4+54\gamma_1^3-27\gamma_1^2-108\delta-12\gamma_1+4}\gamma_1^2\\+54\gamma_1^3+3\sqrt{81\gamma_1^4+54\gamma_1^3-27\gamma_1^2-108\delta-12\gamma_1+4}\gamma_1\\-27\gamma_1^2-2\sqrt{\begin{array}{c}81\gamma_1^4+54\gamma_1^3-27\gamma_1^2-108\delta\\-12\gamma_1+4\end{array}}-54\delta-12\gamma_1+4\end{array}}\right)\end{array}\right)\right)$$

4. Lagrangian of the NDMBBME in Potential Form Using the Variational Problem Method

It is crucial that if an equation has odd order, it does not accept a variational problem, but thanks to the potential form Δ_v, this equation accepts a variational problem [18–20].
The NDMBBME

$$\Delta_w : w_t + w_x - \delta w^2 w_x + w_{xxx} = 0$$

is in an odd order. Frechet derivative of Δ_w is

$$\wp_{\Delta_w} : \wp_t + \wp_x - \delta w_x^2 \wp - 2\delta w w_x + \wp_x^3.$$

Note that $\wp_{\Delta_w} \neq \wp^*_{\Delta_w}$. We say that the NDMBBME does not accept a variational problem. The NDMBBME in Δ_v is got by the lustrous differential substitution $w = v_x$,

$$\Delta_v = v_{xt} + v_{xx} - \delta v_x^2 v_{xx} + v_{xxxx} = 0. \tag{4.1}$$

Eq. (4.1) is named "the NDMBBME in the potential form" and its Frechet derivative is

$$\wp_{\Delta_v} = \wp_x \wp_t + \wp_x^2 - \delta v_x^2 \wp_x^2 - 2\delta v_x v_{xx} \wp_x + \wp_x^4. \tag{4.2}$$

Note that Eq. (4.2) is self-adjoint. Thanks to the Theorem 2.2, the NDMBBME in Δ_v has a Lagrangian of the form

$$\begin{aligned} L[v] &= \int_0^1 v\Delta_v[\lambda v] d\lambda \\ &= -\frac{1}{2} v_x v_t - \frac{1}{2} v_x^2 + \frac{\delta}{12} v_x^4 + \frac{1}{2} v_{xx}^2 + DivP. \end{aligned}$$

Thus, we have

$$\mathscr{L}_{\Delta_v}[v] = -\frac{1}{2}(v_x v_t + v_x^2 - \frac{\delta}{6} v_x^4 - v_{xx}^2). \tag{4.3}$$

5. Application of the μ-Conservation Laws of the NDMBBME

In this part, first of all, we will compute the conservation laws of μ for the NDMBBME as Δ_v. Consider the second-order Lagrangian (4.3) for the NDMBBME as Δ_v

$$\begin{aligned} \Delta_v &= v_{xt} + v_{xx} - \delta v_x^2 v_{xx} + v_{xxxx} \\ &= \grave{E}(\mathscr{L}_{\Delta_v}). \end{aligned} \tag{5.1}$$

Surmise that for $\mathscr{L}_{\Delta_v}[v]$, $\Upsilon = \varphi \partial_v$ be a vector field. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a semi-basic one-form such that $\wp_x \lambda_2 = \wp_t \lambda_1$ when $\Delta_v = 0$. Thanks to the (2.4), Ω and its coefficients are

$$\Omega = \varphi \frac{\partial}{\partial v} + \psi^x \frac{\partial}{\partial v_x} + \psi^t \frac{\partial}{\partial v_t} + \psi^{xx} \frac{\partial}{\partial v_{xx}},$$

$$\psi^x = (\wp_x + \lambda_1)\varphi, \quad \psi^t = (\wp_t + \lambda_2)\varphi, \quad \psi^{xx} = (\wp_x + \lambda_1)\psi^x.$$

By applying the $\mu-$prolongation Ω to Eq. (5.1) and substituting $\frac{1}{v_x}(-v_x^2 + \frac{\delta}{6} v_x^4 + v_{xx}^2)$ for v_t, we get

$$\begin{gathered} \lambda_1 \varphi + \varphi_x = 0, \quad -2\varphi_{vv} = 0, \\ -\frac{\delta}{3}\varphi_v = 0, \quad -\frac{\delta}{2}(\lambda_1 \varphi + \varphi_x) = 0, \\ \varphi_x + \varphi_t + \lambda_2 \varphi + \lambda_1 \varphi = 0, \\ -2\lambda_{1v}\varphi - 4\lambda_1 \varphi_v - 4\varphi_{vx} = 0, \\ -2\varphi_{xx} - 2\lambda_1^2 \varphi - 4\lambda_1 \varphi_x - 2\lambda_{1x}\varphi = 0. \end{gathered} \tag{5.2}$$

Consider $\varphi = \Xi$, and $\mathscr{L}_{\Delta_v}[v] = 0$. A particular solution of the system (5.2) is given by

$$\lambda_1 = -\frac{\Xi_x}{\Xi}, \quad \lambda_2 = -\frac{\Xi_t}{\Xi}.$$

Therefore, for $\mathscr{L}_{\Delta_v}[v]$, $\Upsilon = \Xi \frac{\partial}{\partial v}$ is a $\mu-$symmetry. Then, by using Theorem 2.2, there exists an $\aleph$-vector P^i which is conservation law of μ, that is, $(\wp_i + \lambda_i)P^i = 0$. Then, by of (2.5), the $\aleph$-vector P^i for $\mathscr{L}_{\Delta_v}[v]$ is got

$$P^1 = -\Xi(\frac{1}{2}v_t + v_x - \frac{\delta}{3}v_x^3 + v_{xxx}),$$

$$P^2 = -\frac{v_x}{2}\Xi. \tag{5.3}$$

So, for $\mathscr{L}_{\Delta_v}[v]$, conservation law of μ is the form $\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$.

Corollary 5.1. *Conservation law of μ for the NDMBBME in $\Delta_v = \grave{E}(\mathscr{L}_{\Delta_v})$ is as*

$$\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0,$$

where P^1 and P^2 are the $\aleph$-vector P^i of (5.3).

Remark 5.2. *Conservation law of μ for the NDMBBME in Δ_v, satisfying to the Noether's Theorem for $\mu-$symmetry, that is to say*

$$\begin{aligned}(\wp_i + \lambda_i)P^i &= -\Xi(v_{xt} + v_{xx} - \delta v_x^2 v_{xx} + v_{xxxx}) \\ &= Q\grave{E}(\mathscr{L}_{\Delta_v}).\end{aligned}$$

Secondly, let us consider the NDMBBME as Δ_v

$$\Delta_v = v_{xt} + v_{xx} - \delta v_x^2 v_{xx} + v_{xxxx} = 0. \tag{5.4}$$

Eq. (5.4) corresponds to

$$\wp_x(v_t + v_x - \frac{\delta}{3}v_x^3 + v_{xxx}) = 0,$$

or equivalently

$$v_t + v_x - \frac{\delta}{3}v_x^3 + v_{xxx} = \Theta_1(t),$$

where $\Theta_1(t) = \Theta_1$ is an arbitrary function. If we put

$$\Theta_1 - v_x + \frac{\delta}{3}v_x^3 - v_{xxx}$$

for v_t and substitute w for v_x in the $\aleph$-vector P^i of (5.3), then, we get the $\aleph$-vectors P^1 and P^2 as:

$$P^1 = -\Xi(\frac{1}{2}\Theta_1 + \frac{1}{2}w - \frac{\delta}{6}w^3 + \frac{1}{2}w_{xx}),$$

$$P^2 = -\frac{w}{2}\Xi. \tag{5.5}$$

Corollary 5.3. *Conservation law of μ for the NDMBBME Δ_w is*

$$\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0,$$

where P^1 and P^2 are the $\aleph$-vector P^i of (5.5).

Remark 5.4. *The NDMBBME Δ_w satisfies the characteristic form, that is to say*

$$\begin{aligned}(\wp_i + \lambda_i)P^i &= -\Xi(w_x + w_t - \delta w^2 w_x + w_{xxx}) \\ &= Q\Delta_w.\end{aligned}$$

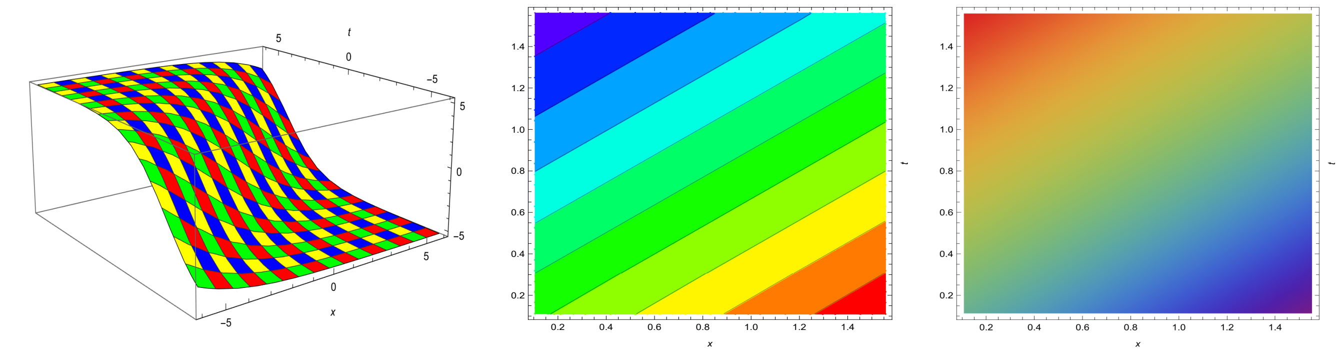

Figure 5.1: The 3−dimensional, contour and density figures of $w(x,t)$ in (3.12)

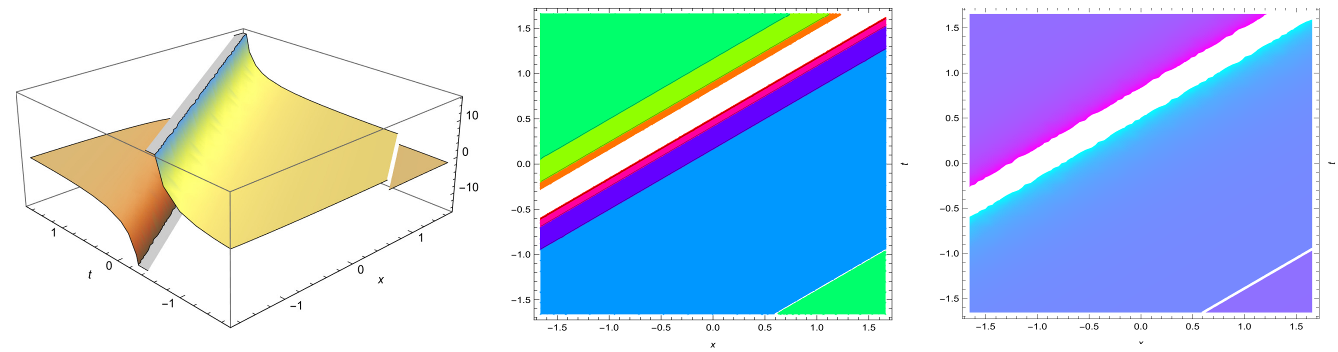

Figure 5.2: The 3−dimensional, contour and density figures of $w(x,t)$ in (3.13)

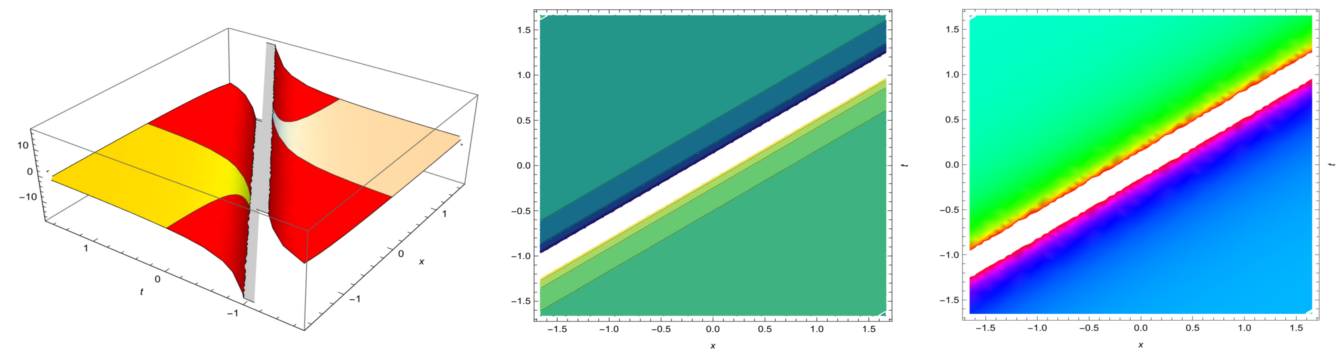

Figure 5.3: The 3−dimensional, contour and density figures of $w(x,t)$ in (3.14)

6. Conclusions

In this study, we considered the NDMBBME to scrutinize the $\mu-$symmetries, symmetry reductions, invariant solutions, and conservation laws. To begin with, some essential properties of the $\mu-$symmetries and conservation law were given. The vital situation in this approach is a semi-basic one-form $\mu = \lambda_i dx_i$, which must satisfy compatibility conditions. Then we demonstrated that the approach of the $\mu-$symmetry reduction can also be analyzed in terms of the formulation of the Noether theorem when $\mu-$symmetries were regarded to discover the invariant solutions of PDEs, which are named the $\mu-$invariant solutions. Moreover, we obtained Lagrangian in potential by using the variational problem method and the Frechet derivative. In this context, the equation must have Lagrangian necessary and sufficient condition its Frechet derivative is self-adjoint. Finally, the conservation law of μ was investigated. The main novelty of this paper is NDMBBM equation is first studied using the $\mu-$symmetry method and conservation law of μ. The 3d, contour, and density figures of the reached solutions were drawn with the aid of Mathematica. The accuracy of the solutions acquired was tested and proved in Maple.

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