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On Some Spectral Properties of Discrete Sturm-Liouville Problem

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Abstract

Time scale theory helps us to combine differential equations with difference equations. Especially in models such as biology, medicine, and economics, since the independent variable is handled discrete, it requires us to analyze in discrete clusters. In these cases, the difference equations defined in $\mathbb{Z}$ are considered. Boundary value problems (BVP's) are used to solve and model problems in many physical areas. In this study, we examined spectral features of the discrete Sturm-Liouville problem. We have given some examples to make the subject understandable. The discrete Sturm-Liouville problem is solved by using the discrete Laplace transform. In the classical case, the discrete Laplace transform is preferred because it is a very useful method in differential equations and it is thought that the discrete Laplace transform will show similar properties. The other method obtained for the solution of this problem is the solutions obtained according to the states of the characteristic equation and λ parameter. In this solution, discrete Wronskian and Cramer methods are used.

1. Introduction

A time scale $\mathbb{T}$ is a non-empty, arbitrary, closed subset of $\mathbb{R}$. This theory was first studied by Hilger in his doctoral thesis [1]. Later, Bohner and Peterson expressed Δ-derivative, Δ-integral and some properties in [2]. Bohner and Georgiev studied the concepts for multivariate functions on time scale [3] . There are many studies in different years on this theory [4, 5]. For instance, time scale population model is used in many important areas such as wound healing, maximization and minimization problems in economy, epidemic problems.

The special case of $\mathbb{T}=\mathbb{Z}$ has many applications in literature. Due to the difficulties posed by derivative and integral in general case, the special cases of time scale are frequently used in many applications. Difference equation is a type of equation that have applications in many fields such as biology, medicine and population. Examples of these applications can be given such as population growth model, logistics surplus model, competition model and infectious disease model. First studies on BVP's for linear Δ-difference equations on time scale are in the relevant references [6, 7]. In addition, various studies have been carried out on the general theory of difference equations [8], non-regular cases of linear ordinary difference equations, asymptotic behavior of difference equation systems and difference equations [9], finite difference calculus. Other important studies that deal with the properties of difference equations are [10] and [11]. In addition, there are many studies that examine discrete versions of Sturm-Liouville, Bessel, Dirac on time scales (see [12–29]).

To give basic results, we should recall substantial concepts of time scale theory. For $t\in\mathbb{T}$, forward and backward jump operators [2] are expressed by

$$\sigma(t)=inf\{s\in\mathbb{T};s>t\}$$

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and

$$\rho(t) = sup\{s \in \mathbb{T}; s < t\}.$$

Let's state a set $\mathbb{T}^{\kappa}$ which is derived from $\mathbb{T}$, and necessary for the definition of delta derivatives. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - m$. In other cases, $\mathbb{T}^{\kappa} = \mathbb{T}$. Moreover, let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Then, one can define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon < 0$, there is a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that

$$\left|[f(\sigma(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. $f^{\Delta}(t)$ is known as Δ-derivative of f at $t \in \mathbb{T}^{\kappa}$. Now, let's express another important concept that is necessary when defining an integral on $\mathbb{T}$. $f : \mathbb{T} \to \mathbb{R}$ is regulated if its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. f is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. $C_{rd}(\mathbb{T})$ indicates the set of all rd-continuous functions. By the special selections of $\mathbb{T}$, we can have below derivatives [7].

1. $f^{\Delta}(t) = f^{'}(t)$ for all $t \in \mathbb{R}$ if $\mathbb{T} = \mathbb{R}$.
2. $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$ if $\mathbb{T} = \mathbb{Z}$.

Let's start to express the concept of integral, which is very important for the subject we are working on, gradually. There exists a function F which is pre-differentiable with region of differentiation D where $F^{\Delta}(t) = f(t)$ holds for all $t \in D$ where f is regulated. For functions f and F satisfying these conditions and an arbitrary constant C, we express indefinite delta integral of f by

$$\int f(t)\Delta t = F(t) + C.$$

In same logic, Cauchy integral of f on $[r,s]$ is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r),$$

for all $r, s \in \mathbb{T}$. The definitions of delta derivative and delta integral, which are generally given in arbitrary time scales, have different representations in different time scales.
We will now express some spectral results for a special case of $\mathbb{T}$. In this study, we take into account below discrete Sturm-Liouville problem

$$L_{\Delta}y(t) = -\Delta^2 y(t) + q(t)y(t) = \lambda y(t), \quad 0 < t < N, \tag{1.1}$$

with separated discrete boundary conditions

$$\Delta y(0) - hy(0) = 0, \tag{1.2}$$

$$\Delta y(N) + Hy(N) = 0, \tag{1.3}$$

where λ is spectral parameter, $q \in L_2^{\mathbb{Z}}[0,N]$, $N \in \mathbb{Z}^+$ and $H, h \in \mathbb{R}$. Considering this problem, which is very important in terms of mathematical physics in classical analysis, in a discrete situation will give very important results. By setting $\mathbb{T} = \mathbb{R}$ in (1.1), it reduces to Sturm-Liouville equation on $\mathbb{R}$ as

$$Ly(t) = -y^{''}(t) + q(t)y(t) = \lambda y(t).$$

Discrete version of $L_2[0,N]$ will play a key role in the study, while the main results are obtained in spectral point of view. So let's define this space.The discrete $L_2^{\mathbb{Z}}[0,N]$ space is defined by [according to Theorem 1.79 (iv) in [2]]

$$L_2^{\mathbb{Z}}[0,N] = \left\{ x(t) : \sum_{t=0}^{N-1} |x(t)|^2 < \infty \right\}.$$

Inner product on $L_2^{\mathbb{Z}}[0,N]$ is defined by

$$\langle x, y \rangle = \int_0^N x(t)y(t)\Delta t = \sum_{t=0}^{N-1} x(t)y(t),$$

where $x, y \in L_2^{\mathbb{Z}}[0,N]$. Another concept that we will use in the study is discrete Laplace transform (or L-transform) [2, 11]. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is regulated where $N_a = \{a, a+1, a+2, \ldots\}$ and $a \in \mathbb{R}$. Then, discrete L-transform of f based at a is defined by

$$L_a\{f\}(s) = F_a(k) = \int_0^{\infty} \frac{f(a+k)}{(s+1)^{k+1}} \Delta k = \sum_{k=0}^{\infty} \frac{f(a+k)}{(s+1)^{k+1}},$$

for all complex numbers $s \neq -1$ when the improper integral converges. Here, $L_a\{f\}(s) = F_a(k)$, $L_a^{-1}\{F_a(k)\} = f(s)$ where L_a^{-1} is inverse discrete L-transform [11].
Let's continue with another concept that is extremely important for L-transform. f is of exponential order $r > 0$ if there exists a constant $A > 0$ such that $|f(t)| \leq Ar^t$ for $t \in \mathbb{N}_a$. Let f be exponential order $r > 0$. Then, for any $N \in \mathbb{Z}^+$ [11],

$$L_a\left\{\Delta^N f\right\}(s) = s^N F_a(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a),$$

for $|s+1| > r$. Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and discrete L-transforms of f and g converge for $|s+1| > r$ where $r > 0$. Then, discrete L-transform of $c_1 f + c_2 g$ converges for $|s+1| > r$ and

$$L_a\{c_1 f + c_2 g\}(s) = c_1 L_a\{f\}(s) + c_2 L_a\{g\}(s),$$

for $|s+1| > r$, $c_1, c_2 \in \mathbb{R}$. Addition, assume that $p \neq \pm i$. Then,

$$L_a\{\cos_p(t,a)\}(s) = \frac{s}{s^2 + p^2},$$

and

$$L_a\{\sin_p(t,a)\}(s) = \frac{p}{s^2 + p^2}.$$

This study is planned as follows: In Section 2, we give proofs of some basic theorems for spectral properties of discrete Strum-Liouville equation. Using some methods, we get eigenfunctions of (1.1)-(1.3) discrete Sturm-Liouville problem in Section 3.

2. Some spectral properties of discrete Sturm-Liouville equation

The eigenvalues and eigenfunctions of differential operators need to be found in solving problems encountered in many fields such as analysis, applied mathematics and mathematical physics. For this reason, spectral properties of Sturm-Liouville problem, which has applications in many fields, have been an important subject of study.
Orthogonality of eigenfunctions, simplicity and reality of eigenvalues, formally self-adjointness property of operator are well-known properties in usual spectral analysis. The following results are generalized to discrete case. All the features that will be given below will allow to better understand and explain the physical phenomenon expressed by the problem expressed in the discrete situation.

Theorem 2.1. *The eigenfunctions corresponding to the distinct eigenvalues of problem (1.1)-(1.3) are orthogonal.*

Proof. We have to show that there are $\lambda_1 \neq \lambda_2$ for $y_1(t,\lambda_1)$ and $y_2(t,\lambda_2)$ such that $\langle y_1, y_2 \rangle = 0$.

$$-\Delta^2 y_1 + q(t) y_1 = \lambda_1 y_1,$$

$$-\Delta^2 y_2 + q(t) y_2 = \lambda_2 y_2.$$

If necessary adjustments are made here, we get

$$-y_2(\Delta^2 y_1) + y_1(\Delta^2 y_2) = (\lambda_1 - \lambda_2) y_1 y_2.$$

Let's take discrete integral of both sides on $[0,N]$ to get

$$-\int_0^N y_2(\Delta^2 y_1)\Delta t + \int_0^N y_1(\Delta^2 y_2)\Delta t = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t,$$

$$\int_0^N \Delta[(\Delta y_2) y_1 - (\Delta y_1) y_2]\Delta t = (\lambda_1 - \lambda_2)\int_0^N y_1 y_2 \Delta t,$$

$$[(\Delta y_2)y_1-(\Delta y_1)y_2]_0^N=(\lambda_1-\lambda_2)\int_0^N y_1y_2\Delta t,$$

$$\Delta y_2(N)y_1(N)-\Delta y_1(N)y_2(N)-\Delta y_2(0)y_1(0)+\Delta y_1(0)y_2(0)=(\lambda_1-\lambda_2)\int_0^N y_1y_2\Delta t.$$

If we substitute the boundary conditions of the (1.1)-(1.3) problem, we get

$$-Hy_2(N)y_1(N)+Hy_1(N)y_2(N)-hy_2(0)y_1(0)+hy_1(0)y_2(0)=(\lambda_1-\lambda_2)\int_0^N y_1y_2\Delta t$$

$$0=(\lambda_1-\lambda_2)\int_0^N y_1y_2\Delta t.$$

Since $\lambda_1\neq\lambda_2$, we get

$$\int_0^N y_1y_2\Delta t=0$$

or

$$\sum_{t=0}^{N-1} y_1(t)y_2(t)=0.$$

Thus,

$$\langle y_1,y_2\rangle=0.$$

□

Theorem 2.2. *Eigenvalues corresponding to the discrete Sturm-Liouville problem (1.1)-(1.3) are all real.*

Proof. Let λ be an eigenvalue and u be eigenfunction corresponding to λ. Since L_Δ is symmetric where $L_\Delta u=\lambda u$, we get

$$\langle L_\Delta u,u\rangle=\langle\lambda u,u\rangle,$$

$$\langle L_\Delta u,u\rangle=\langle u,\bar{L_\Delta}u\rangle=\left\langle u,\bar{\lambda}u\right\rangle.$$

Then,

$$\langle\lambda u,u\rangle=\left\langle u,\bar{\lambda u}\right\rangle,$$

$$\langle\lambda u,u\rangle-\left\langle u,\bar{\lambda u}\right\rangle=0,$$

$$(\lambda-\bar{\lambda})\langle u,u\rangle=0.$$

Since u is the eigenvalue, $\langle u,u\rangle\neq 0$, we get

$$(\lambda-\bar{\lambda})=0$$

and

$$\lambda=\bar{\lambda}.$$

It yields that, the eigenvalues are all real. This completes the proof. □

Theorem 2.3. *Eigenvalues corresponding to discrete Sturm-Liouville problem (1.1)-(1.3) are all simple.*

Proof. To prove this, we will consider two eigenfunctions corresponding to the same eigenvalue. We will show that these eigenfunctions are linearly dependent.

$$L_\Delta u = -\Delta^2 u + q(t)u = \lambda u,$$

$$L_\Delta v = -\Delta^2 v + q(t)v = \lambda v.$$

If necessary adjustments are made here, we get

$$v(-\Delta^2 u + q(t)u) - u(-\Delta^2 v + q(t)v) = 0,$$

$$\Delta[u(\Delta v) - v(\Delta u)] = 0,$$

$$u(\Delta v) - v(\Delta u) = c, \quad c \in \mathbb{R}. \tag{2.1}$$

Now let us set $t = 0$. Then, we get

$$u(0)\Delta v(0) - v(0)\Delta u(0) = c,$$

$$u(0)hv(0) - v(0)hu(0) = c,$$

and

$$c = 0.$$

If this expression is substituted in (2.1), we get

$$u\Delta v - v\Delta u = 0.$$

If the fact $\Delta\left(\frac{u}{v}\right) = \frac{(\Delta u)v - (\Delta v)u}{v^2}$ is used in the above equation, we get

$$\Delta\left(\frac{u}{v}\right) = 0,$$

$$\frac{u}{v} = c_1,$$

$$u = c_1 v, \quad c_1 \in \mathbb{R}.$$

This means u and v are linearly dependent. Proof is completed. □

Theorem 2.4. *Discrete Sturm-Liouville operator L_Δ is formally self-adjoint on $L_2^{\mathbb{Z}}[0,N]$.*

Proof. Let u and v be two eigenfunctions. We have to show that $\langle v, L_\Delta u\rangle = \langle L_\Delta v, u\rangle$. Let's consider the following equations.

$$L_\Delta u = -\Delta^2 u + q(t)u$$

$$L_\Delta v = -\Delta^2 v + q(t)v$$

If necessary arrangements are made, it yields

$$vL_\Delta u - uL_\Delta v = \Delta[-(\Delta u)v + (\Delta v)u].$$

Let's take discrete integral for both sides on $[0,N]$ to obtain

$$\begin{aligned}
\int_0^N vL_\Delta u\Delta t - \int_0^N uL_\Delta v\Delta t &= \int_0^N \Delta[-(\Delta u)v + (\Delta v)u]\Delta t \\
&= [(\Delta u)v - (\Delta v)u]_0^N \\
&= \Delta u(N)v(N) + \Delta v(N)u(N) + \Delta u(0)v(0) - \Delta v(0)u(0) \\
&= Hu(N)v(N) - Hu(N)v(N) + hu(0)v(0) - hv(0)u(0) \\
&= 0.
\end{aligned}$$

From here, we get $\langle v, L_\Delta u\rangle - \langle L_\Delta v, u\rangle = 0$ and $\langle v, L_\Delta u\rangle = \langle L_\Delta v, u\rangle$. □

The feature of being formally self-adjointness is important in terms of making sense of the problem that is handled physically.

3. Some examples on discrete Sturm-Liouville equation

In this section, the discrete eigenfunctions of the Sturm-Liouville problem (1.1)-(1.3) with various types of conditions will be obtained. In these examples, it is seen that obtaining the eigenfunctions in the discrete case is more troublesome and difficult than in the classical case.

Example 3.1. *Let us consider the below discrete BVP*

$$-\Delta^2 y = \lambda y,$$

$$y(0) = 0, \Delta y(2) = 0.$$

In this example, the discrete boundary value problem will be solved in this particular case. By taking the necessary derivatives in $-\Delta^2 y = \lambda y$, the following equation is obtained.

$$y(t+2) - 2y(t+1) + (\lambda+1)y(t) = 0.$$

Characteristic equation of this equation is $k^2 - 2k + (1+\lambda) = 0$ and its characteristic roots are $k_{1,2} = 1 \pm i\sqrt{-\lambda}$. There are three situations for these roots as $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$.

1. Let $\lambda = 0$. Since k is double-decker root as $k_{1,2} = 1$, we get

$$y(t) = c_1 + c_2 t.$$

   Since $c_1 = 0$, $c_2 = 0$ by the given conditions, λ is not an eigenvalue.
2. Let $\lambda < 0$. We get $k_1 = 1 - \sqrt{-\lambda}$, $k_2 = 1 + \sqrt{-\lambda}$ and

$$y(t) = c_1(1 - \sqrt{-\lambda})^t + c_2(1 + \sqrt{-\lambda})^t.$$

   By the conditions, we get $c_1 = 0$ and $c_2 = 0$. So, λ is not an eigenvalue.
3. Let $\lambda > 0$. We get $k_1 = 1 - i\sqrt{\lambda}$, $k_2 = 1 + i\sqrt{\lambda}$. Since $r = \sqrt{1+\lambda}$, and $\theta = \tan^{-1}(\sqrt{\lambda})$, y has the following form.

$$y(t) = (c_1 \cos\theta t + c_2 \sin\theta t)(1+\lambda)^{\frac{t}{2}}.$$

   Since $y(0) = 0$ and $c_1 = 0$, it should be $\sin 3\theta = 0$ for $c_2 \neq 0$. Then, it yields $\theta = \frac{\pi z}{3}, z = 1, 2, 3, \ldots$ Therefore,

$$y(t) = c_2(1+\lambda)^{\frac{t}{2}} \sin\frac{\pi z}{3} t.$$

Example 3.2. *Consider discrete L-transform to solve below discrete IVP*

$$-\Delta^2 y(t) = \lambda y(t),$$

$$y(0) = 2, \quad \Delta y(0) = 4.$$

Let's apply discrete L-transform to both sides of equation as

$$L_\alpha\left(-\Delta^2 y(t)\right) = L_\alpha\left(\lambda y(t)\right),$$

$$-s^2 Y(s) + sy(0) + \Delta y(0) = \lambda Y(s),$$

$$Y(s) = \frac{2s+4}{\lambda + s^2}.$$

By using inverse discrete L-transform, we get

$$y(t) = 2\cos_{\sqrt{\lambda}}(t,0) + \frac{4}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0).$$

Now let's get the eigenfunctions of discrete Sturm-Liouville problem in general case, which includes $q(t) \in L_2^{\mathbb{Z}}[0,N]$.

Example 3.3. *Consider following discrete equation*

$$-\Delta^2 y(t) + q(t)y(t) = \lambda y(t).$$

First, let's find the homogeneous solution of this equation. Since $y(t+2)-2y(t+1)+(\lambda+1)y(t)=0$, characteristic equation is $k^2-2k+(1+\lambda)=0$. There are three situations for the roots as $\lambda>0$, $\lambda=0$ and $\lambda<0$.

1. Let $\lambda>0$. Then, homogeneous solution is

$$y_h(t)=c_1(1+\lambda)^{\frac{t}{2}}\cos\theta t+c_2(1+\lambda)^{\frac{t}{2}}\sin\theta t.$$

Here, $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ can be written as a particular solution. Since $y_1=(1+\lambda)^{\frac{t}{2}}\cos\theta t$ and $y_2=(1+\lambda)^{\frac{t}{2}}\sin\theta t$, we get

$$y_p(t)=u_1(t)(1+\lambda)^{\frac{t}{2}}\cos\theta t+u_2(t)(1+\lambda)^{\frac{t}{2}}\sin\theta t.$$

The following operations can be performed.

$$\Delta u_1(t)(1+\lambda)^{\frac{t+1}{2}}\cos\theta(t+1)+\Delta u_2(t)(t+1)(1+\lambda)^{\frac{t+1}{2}}\sin\theta(t+1)=0,$$

$$\Delta u_1(t)(1+\lambda)^{\frac{t+2}{2}}\cos\theta(t+2)+\Delta u_2(t)(t+2)(1+\lambda)^{\frac{t+2}{2}}\sin\theta(t+2)=-q(t)y(t).$$

When necessary solutions are made, we get

$$u_1(t)=\sum_{i=0}^{t-1}\frac{-q(i)y_2(i+1)}{W_{\mathbb{Z}}(i+1)}$$

and

$$u_2(t)=\sum_{i=0}^{t-1}\frac{q(i)y_1(i+1)}{W_{\mathbb{Z}}(i+1)}$$

where

$$W_{\mathbb{Z}}(y_1(i+1),y_2(i+2))=\begin{vmatrix}y_1(i+1) & y_2(i+1)\\ y_1(i+2) & y_2(i+2)\end{vmatrix}.$$

$$\begin{aligned}W_{\mathbb{Z}}(y_1(i+1),y_2(i+2)) &= \begin{vmatrix}(1+\lambda)^{\frac{i+1}{2}}\cos\theta(i+1) & (1+\lambda)^{\frac{i+1}{2}}\sin\theta(i+1)\\ (1+\lambda)^{\frac{i+2}{2}}\cos\theta(i+2) & (1+\lambda)^{\frac{i+2}{2}}\sin\theta(i+2)\end{vmatrix}\\ &= (1+\lambda)^{\frac{2i+3}{2}}\sin\theta\end{aligned}$$

Finally, general solution of given discrete equation is

$$\begin{aligned}y(t,\lambda) &= c_1(1+\lambda)^{\frac{t}{2}}\cos\theta t+c_2(1+\lambda)^{\frac{t}{2}}\sin\theta t+(1+\lambda)^{\frac{t}{2}}\cos\theta t\sum_{i=0}^{t-1}\frac{-q(i)y(i)(1+\lambda)^{\frac{i+1}{2}}\sin\theta(i+1)}{(1+\lambda)^{\frac{2i+5}{2}}\sin\theta}\\ &\quad +(1+\lambda)^{\frac{t}{2}}\sin\theta t\sum_{i=0}^{t-1}\frac{q(i)y(i)(1+\lambda)^{\frac{i+1}{2}}\cos\theta(i+1)}{(1+\lambda)^{\frac{2i+5}{2}}\sin\theta}.\end{aligned}$$

2. Let $\lambda=0$. Homogeneous solution is

$$y_h(t)=c_1+c_2t.$$

Similarly, $y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ can be written as a particular solution.

$$\Delta u_1(t)+\Delta u_2(t)(t+1)=0$$

$$\Delta u_1(t)+\Delta u_2(t)(t+2)=-q(t)y(t).$$

Wronskian of y_1 and y_2 is as follows.

$$W_{\mathbb{Z}}(y_1(i+1),y_2(i+2))=\begin{vmatrix}1 & i+1\\ 1 & i+2\end{vmatrix}=1.$$

As a result, general solution is obtained as

$$y(t)=c_1+c_2t-\sum_{i=0}^{t-1}q(i)y(i)+t\sum_{i=0}^{t-1}q(i)y(i)(i+1).$$

3. Let $\lambda < 0$. In this case, homogeneous solution is

$$y_h(t) = c_1(1-\sqrt{-\lambda})^t + c_2(1+\sqrt{-\lambda})^t.$$

When the necessary calculations are made, we get

$$W_{\mathbb{Z}}(y_1(i+1), y_2(i+2)) = (1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda}).$$

Considering the definitions of u_1 and u_2, general solution is obtained as follows.

$$\begin{aligned} y(t) &= c_1(1-\sqrt{-\lambda})^t + c_2(1+\sqrt{-\lambda})^t - (1+\sqrt{-\lambda})^t \sum_{i=0}^{t-1} \frac{q(i)y(i)(1-\sqrt{-\lambda})^t}{(1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda})} \\ &\quad + (1-\sqrt{-\lambda})^t \sum_{i=0}^{t-1} \frac{q(i)y(i)(1+\sqrt{-\lambda})^t}{(1+i)^i(-2\sqrt{-\lambda} - 2\lambda\sqrt{-\lambda})} \end{aligned}$$

Example 3.4. *Consider discrete L-transform to solve below discrete IVP*

$$-\Delta^2 y(t) + q(t)y(t) = \lambda y(t)$$

$$y(0) = c_1, \Delta y(0) = c_2.$$

Let $q(t)y(t) = f(t)$. Applying discrete L-transform to both sides of equation, it yields

$$L\left\{-\Delta^2 y\right\}(s) + L\{f\}(s) = \lambda L\{y\}(s)$$

$$-s^2 Y(s) + sy(0) + \Delta y(0) + L\{f\}(s) = \lambda Y(s)$$

$$Y(s) = c_1 \frac{s}{s^2 + (\sqrt{\lambda})^2} + \frac{c_2}{\sqrt{\lambda}} \frac{\sqrt{\lambda}}{s^2 + (\sqrt{\lambda})^2} + \frac{1}{\lambda + s^2} L\{f\}(s).$$

By applying discrete inverse L-transform in last equation, we get

$$y(t) = c_1 \cos_{\sqrt{\lambda}}(t,0) + \frac{c_2}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0) + L^{-1}\left\{\frac{1}{\lambda + s^2} L\{f\}(s)\right\}.$$

Let's apply discrete convolution to last expression on right-hand side of equation.

$$\begin{aligned} L^{-1}\left\{\frac{1}{\lambda + s^2} L\{f\}(s)\right\} &= \frac{\sin_{\sqrt{\lambda}}(t,0)}{\sqrt{\lambda}} * q(t)y(t) \\ &= \sum_{r=0}^{t-1} \frac{\sin_{\sqrt{\lambda}}(r,0)}{\sqrt{\lambda}} q(t-\sigma(r))y(t-\sigma(r)) \end{aligned}$$

If we consider expression that we found in $y(t)$ solution, we get

$$y(t) = c_1 \cos_{\sqrt{\lambda}}(t,0) + \frac{c_2}{\sqrt{\lambda}} \sin_{\sqrt{\lambda}}(t,0) + \sum_{r=0}^{t-1} \frac{\sin_{\sqrt{\lambda}}(r,0)}{\sqrt{\lambda}} q(t-\sigma(r))y(t-\sigma(r)).$$

Finally, let us express two important concepts in the solution of the examples given in this section as a reminder.

Remark 3.5 ([2]). *Let y_1 and y_2 be delta differentiable functions. Discrete Wronskian of these functions is defined by*

$$W_{\mathbb{Z}} = \begin{pmatrix} y_1(t) & y_2(t) \\ \Delta y_1(t) & \Delta y_2(t) \end{pmatrix}.$$

Remark 3.6 ([11]). *Let $f, g : \mathbb{N}_a \to \mathbb{R}$. Delta convolution product of f and g is defined by*

$$(f * g)(t) = \sum_{r=a}^{t-1} f(r)g(t-\sigma(r)+a)$$

for $t \in \mathbb{N}_a$.

4. Conclusion

Difference equations are used in mathematical models and numerical solutions of differential equations in various fields. Sturm-Liouville problems are used to solve problems in many physical fields. In this study, we examined discrete Sturm-Liouville operator and its spectral properties. We have obtained a solution for the discrete Sturm-Liouville problem we are considering using some methods. We did one of these solutions by considering the existing discrete L-transform. We proved the basic spectral properties of the operator for the discrete Sturm-Liouville difference equation, such as self-adjointness, orthogonality of eigenfunctions, and realness of eigenvalues. We hope that the study will guide researchers for discrete case of Sturm-Liouville problem.

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