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Bright and Dark Solitary Love Waves in a Heterogeneous Semi-Space Coated with a Heterogeneous Elastic Layer

Elastik Heterojen Bir Tabakayla Kaplı Heterojen Yarım Uzayda Parlak ve Karanlık Soliter Love Dalgaları

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ABSTRACT

The propagation of nonlinear Love waves in an elastic, vertically heterogeneous crust laid upon an elastic, heterogeneous semi-space is considered. By employing the multiple scales method, the amplitude function of Love waves is represented by a nonlinear Schrödinger equation which includes the nonlinear material and heterogeneity parameters of the layered semi-space in its coefficients. This study numerically investigates the influence of heterogeneity as well as the nonlinear properties of the media on the presence of bright and dark solitary Love waves. Moreover, the remarkable effects of nonlinear and heterogeneous material properties of both layer and semi-space on the wave evolution of bright and dark solions are graphically shown.

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ÖZET

Bu çalışmada, dalga yayılımına dik yönde heterojen, elastik bir tabakayla kaplı heterojen yarım uzayda doğrusal olmayan Love dalgalarının yayılması problemi göz önüne alınmıştır. Çoklu ölçekler yöntemi kullanılarak, doğrusal olmayan Love dalgalarının öz etkileşimini karakterize eden, katsayıları ortamın malzeme özelliklerine, dolayısıyla tabaka ve yarım uzayı oluşturan malzemelerin heterojenlik parametrelerine bağlı, doğrusal olmayan bir Schrödinger denklemi türetilmiştir. Ortamın doğrusal olmayan özelliklerinin yanı sıra heterojenliğinin de parlak ve karanlık soliter Love dalgalarının varlığı üzerindeki etkileri nümerik olarak incelenmiştir. Ayrıca hem tabakaya hem de yarım uzaya ait doğrusal olmayan ve heterojen malzeme özelliklerinin parlak ve karanlık dalga evrimi üzerindeki kaydadeğer etkileri grafiksel olarak gösterilmiştir.

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1. INTRODUCTION

Wave evolution in an elastic media is generally determined by three important factors which are dispersion, nonlinearity and inhomogeneity. Love [1] theoretically demonstrated the presence of dispersive waves with displacements perpendicular to the propagation plane in a homogeneous, linear, semi-space underlying a uniform layer with different materials, called Love waves. Love waves, which occur near the surface where the two elastic layers are in contact, are the fastest transverse surface waves which we directly feel throughout an earthquake [2]. Moreover, owing to its extensive application areas such as seismology, continuum mechanics, petroleum engineering, biomechanics and metallurgy, the propagation of horizontally polarized shear (SH) waves in a linear, elastic, homogeneous semi-space coated with a regular layer has been the subject of many studies [3-9]. Especially, considering that the earth is composed of heterogeneous stratified elastic media whose rigidity and density change depending on depth, for more realistic research on the Love wave propagation, the heterogeneous constitutional characteristics of the medium must be taken into account. First, investigation of Love waves has been extended to a heterogeneous semi-space covered with a homogeneous layer by Meissner, who considered the linear variation in density and quadratic variation in rigidity of the semi-space [10]. Then, the surface SH waves in a homogeneous, linear, elastic layer over a heterogeneous semi-space whose shear wave velocity, density and rigidity are functions of depth has been investigated by several authors [11-15]. Since the analysis of wave propagation in a stratified medium for any type of inhomogeneity is very complicated, the researchers have considered the problem for particular types of variations in the constituent materials such as exponential, harmonic, quadratic or linear change with depth, which are summarized in [16]. Since Love waves occur near the surface where the layer and semi-space are in contact, it is desirable to examine a model where not only the semi-space but also the layer are taken as inhomogeneous. Hence, the effects of both the layer and the semi-space's heterogeneity on Love wave propagation have been the subject of many studies [17-21].

The effects of nonlinear constituent materials as well as heterogeneity of the medium on solitary SH wave propagation have been extensively studied in recent years. Teymur has examined SH wave propagation in a homogeneous, nonlinear, elastic semi-infinite medium coated with a nonlinear crust [22]. Deliktas et. al. have studied the nonlinear modulation of SH waves in a vertically heterogeneous two-layered plate and investigate the influence of heterogeneity on the presence of envelope solitary waves [23]. Then, Deliktas has extended the study of Love waves to the heterogeneous nonlinear layer between two different semi-spaces and reveal the remarkable influences of material characteristics of intermediate layer on both existence and nonlinear evolutions of solitary Love-type waves [24].

In the present study, the influence of both nonlinear and nonhomogeneous constituent materials of the media consisting of a layer laid upon a semi-space having different elastic material properties on the bright and dark solitary Love waves is investigated. It is assumed that the densities and strain energy functions of the media have an exponential change in the thickness direction. When nonlinearity is ignored, considered problem reduces to the propagation of Love waves studied by Sidhu [17]. By using the derivative expansion method, an NLS equation whose coefficients depend on the medium's material properties, wave number and also heterogeneity parameters is derived for nonlinear modulation of waves. When heterogeneity parameters of the media tend to zero, derived NLS equation reduce to that for the layer laid upon semi-space consisting of homogeneous elastic materials. The effects of heterogeneity as well as the nonlinearity of the media on both existence and nonlinear evolutions of bright and dark solitary Love wave solutions have been examined and the results are presented graphically.

2. FORMULATION

Let the ordered triples (x_1, x_2, x_3) and (X_1, X_2, X_3) represent, respectively, the spatial and material coordinates of a point referred to the same rectangular Cartesian system. We take into consideration a layer with uniform thickness over a semi-space consisting of different, vertically nonhomogeneous, nonlinear, elastic materials whose linear shear modulus, nonlinear material functions and densities change exponentially in the depth direction. In the reference frame (X_1, X_2, X_3), the regions occupying $0 < X_2 < h$ and $-\infty < X_2 < 0$ are the layer (P_1) and the semi-space (P_2), respectively; here h is a positive constant (see Figure 1).

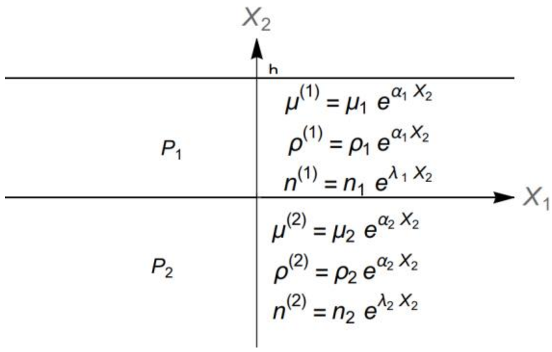


Figure 1. Geometry of the media.

The SH waves are defined as follows

$$x_1 = X_1,\quad x_2 = X_2,\quad x_3 = X_3 + u^{(s)}(X_1, X_2, t) \qquad s = 1,2. \tag{1}$$

Here, t denotes the time, superscript s indicates to the region P_s, $u^{(s)}$ represents the particle's displacement in the X_3 direction. It is assumed that $X_2 = h$ is traction free, displacements and stresses are continuous at $X_2 = 0$ and radiation condition satisfies in semi-space.

The materials of the media are considered as nonlinear, isotropic, incompressible hyper-elastic and made of different generalized neo-Hookean materials [25]. The densities and strain energy functions of both layer and semi-space are assumed to be functions of thickness variable. For such a material

$$\rho^{(1)} = \rho_1(X_2),\ \rho^{(2)} = \rho_2(X_2),\ \Sigma^{(1)} = \Sigma^{(1)}\left(I^{(1)}, X_2\right),\quad \Sigma^{(2)} = \Sigma^{(2)}\left(I^{(2)}, X_2\right) \tag{2}$$

where $I^{(s)}$=tr $\boldsymbol{c}^{(-1)}$ such that $\boldsymbol{c}^{(-1)} = \left[x_{k,K}\ x_{l,K}\right]$, $\Sigma^{(s)}$ is the strain energy function. Let $X = X_1,\ Y = X_2,\ Z = X_3$. The following equations of motion and boundary conditions can be obtained (for detailed analysis see e.g., [26])

$$\frac{\partial^2 u^{(1)}}{\partial t^2} - c_1^2\left(\frac{\partial^2 u^{(1)}}{\partial X^2} + \frac{\partial^2 u^{(1)}}{\partial Y^2}\right) - \frac{1}{\rho^{(1)}}\frac{d\mu^{(1)}}{dY}\frac{\partial u^{(1)}}{\partial Y} = n^{(1)}\left\{\frac{\partial}{\partial X}\left(\frac{\partial u^{(1)}}{\partial X}\mathcal{N}\left(u^{(1)}\right)\right) + \frac{\partial}{\partial Y}\left(\frac{\partial u^{(1)}}{\partial Y}\mathcal{N}\left(u^{(1)}\right)\right)\right\} + \frac{\mathcal{N}(u^{(1)})}{\rho^{(1)}}\frac{d(\rho^{(1)}n^{(1)})}{dY}\frac{\partial u^{(1)}}{\partial Y} \ \text{ in } P_1 \tag{3}$$

$$\frac{\partial^2 u^{(2)}}{\partial t^2} - c_2^2\left(\frac{\partial^2 u^{(2)}}{\partial X^2} + \frac{\partial^2 u^{(2)}}{\partial Y^2}\right) - \frac{1}{\rho^{(2)}}\frac{d\mu^{(2)}}{dY}\frac{\partial u^{(2)}}{\partial Y} = n^{(2)}\left\{\frac{\partial}{\partial X}\left(\frac{\partial u^{(2)}}{\partial X}\mathcal{N}\left(u^{(2)}\right)\right) + \frac{\partial}{\partial Y}\left(\frac{\partial u^{(2)}}{\partial Y}\mathcal{N}\left(u^{(2)}\right)\right)\right\} + \frac{\mathcal{N}(u^{(2)})}{\rho^{(2)}}\frac{d(\rho^{(2)}n^{(2)})}{dY}\frac{\partial u^{(2)}}{\partial Y} \quad \text{in } P_2 \tag{4}$$

$$\frac{\partial u^{(1)}}{\partial Y} = 0 \text{ on } \ Y = \text{h}, \tag{5}$$

$$u^{(1)} = u^{(2)} \text{ and } \frac{d\Sigma^{(1)}}{dI^{(1)}}\frac{\partial u^{(1)}}{\partial Y} = \frac{d\Sigma^{(2)}}{dI^{(2)}}\frac{\partial u^{(2)}}{\partial Y} \ \text{ on } Y = 0, \tag{6}$$

$$u^{(2)} \to 0 \text{ as } Y \to\ -\infty. \tag{7}$$

where

$$\mathcal{N}\left(u^{(s)}\right) = \left(\frac{\partial u^{(s)}}{\partial X}\right)^2 + \left(\frac{\partial u^{(s)}}{\partial Y}\right)^2, \qquad s = 1,2.$$

Here, linear shear velocities c_s, $s = 1,2$, are $c_s^2 = \mu^{(s)}/\rho^{(s)}$ where $\mu^{(s)} = 2\frac{d\Sigma^{(s)}}{dI^{(s)}}(3, Y)$ are the linear shear modulus. Nonlinear material functions of the media are $n^{(s)} = 2\frac{\frac{d^2\Sigma^{(s)}}{dI^2}(3,Y)}{\rho^{(s)}(Y)}$.

3. ASYMPTOTIC ANALYSIS

The self-modulation of small and finite amplitude SH waves by employing the multiple scales method (see e.g. [27]) with the following scales

$$x_i = \varepsilon^i X,\qquad t_i = \varepsilon^i t,\qquad y = Y,\qquad i = 0,1,2 \tag{8}$$

where $\varepsilon > 0$, a small parameter, represents the strength of non-linearity, $\{x_1, x_2, t_1, t_2\}$ are the slow variables which describe slow variations, whereas $\{x_0, t_0, y\}$ are the fast variables representing fast variations. Next, we expand $u^{(s)}$ in the following asymptotic series:

$$u^{(s)} = \sum_{n=1}^{\infty}\varepsilon^n u_n^{(s)}(x_0, x_1, x_2, t_0, t_1, t_2, y) \quad s = 1,2. \tag{9}$$

Rewriting (3)-(7) in terms of (8) and applying (9) give a hierarchy of problems that enable $u_n^{(s)}$ to be obtained, successively. First three of them can be written as follows:

$$O(\varepsilon):\quad \mathcal{L}_s\left(u_1^{(s)}\right) \triangleq \frac{\partial^2 u_1^{(s)}}{\partial t_0^2} - c_s^2\left(\frac{\partial^2 u_1^{(s)}}{\partial x_0^2} + \frac{\partial^2 u_1^{(s)}}{\partial y^2}\right) - \frac{1}{\rho^{(s)}}\frac{d\mu^{(s)}}{dy}\frac{\partial u_1^{(s)}}{\partial y} \quad \text{in } \ P_s,\quad s = 1,2. \tag{10}$$

$$\frac{\partial u_1^{(1)}}{\partial y} = 0 \text{ on } \ y = \text{h}, \tag{11}$$

$$u_1^{(1)} = u_1^{(2)} \quad \text{and} \quad \frac{\partial u_1^{(1)}}{\partial y} - \gamma\frac{\partial u_1^{(2)}}{\partial y} = 0 \text{ on } \ y = 0, \tag{12}$$

$$u_1^{(2)} \to 0 \text{ as } \ y \to\ -\infty. \tag{13}$$

$$O(\varepsilon^2): \qquad \mathcal{L}_s(u_2^{(s)}) = 2\left(c_s^2 \frac{\partial^2 u_1^{(s)}}{\partial x_0 \partial x_1} - \frac{\partial^2 u_1^{(s)}}{\partial t_0 \partial t_1}\right), \ s = 1,2. \tag{14}$$

$$\frac{\partial u_2^{(1)}}{\partial y} = 0 \ \text{ on } \ y = \mathrm{h}, \tag{15}$$

$$u_2^{(1)} = u_2^{(2)} \quad \text{and} \ \frac{\partial u_2^{(1)}}{\partial y} - \gamma \frac{\partial u_2^{(2)}}{\partial y} = 0 \ \text{on} \ y = 0, \tag{16}$$

$$u_2^{(2)} \to 0 \ \text{as} \ y \to -\infty. \tag{17}$$

$$O(\varepsilon^3): \qquad \mathcal{L}_s(u_3^{(s)}) = 2\left(c_s^2 \frac{\partial^2 u_2^{(s)}}{\partial x_0 \partial x_1} - \frac{\partial^2 u_2^{(s)}}{\partial t_0 \partial t_1}\right) + c_s^2\left(\frac{\partial^2 u_1^{(s)}}{\partial x_1^2} + 2\frac{\partial^2 u_1^{(s)}}{\partial x_0 x_2}\right) - \frac{\partial^2 u_1^{(s)}}{\partial t_1^2} - 2\frac{\partial^2 u_1^{(s)}}{\partial t_0 t_2}$$
$$+ n^{(s)}\left(\frac{\partial}{\partial x_0}\left(\frac{\partial u_1^{(s)}}{\partial x_0}\mathcal{N}_0(u_1^{(s)})\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_1^{(s)}}{\partial y}\mathcal{N}_0(u_1^{(s)})\right)\right) + \frac{\mathcal{N}_0\left(u_1^{(s)}\right)}{\rho^{(s)}} \frac{d(\rho^{(s)} n^{(s)})}{dy} \frac{\partial u_1^{(s)}}{\partial y} \quad \text{in } P_s \tag{18}$$

$$\text{On } \ y = \mathrm{h} \qquad \frac{\partial u_3^{(1)}}{\partial y} = 0, \tag{19}$$

$$\text{On } y = 0 \qquad \frac{\partial u_3^{(1)}}{\partial y} - \gamma \frac{\partial u_3^{(2)}}{\partial y} = \gamma \beta_2 \frac{\partial u_1^{(2)}}{\partial y} \mathcal{N}_0(u_1^{(2)}) - \beta_1 \frac{\partial u_1^{(1)}}{\partial y} \mathcal{N}_0(u_1^{(1)}) \ \text{and} \ u_3^{(1)} = u_3^{(2)}, \tag{20}$$

$$u_3^{(2)} \to 0 \ \text{as} \ y \to -\infty \tag{21}$$

where

$$\mathcal{N}_0(\psi) = \left(\frac{\partial \psi}{\partial x_0}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2. \ \gamma = \mu^{(2)}/\mu^{(1)}, \ \beta_s = n^{(s)}/c_s^2$$

Note that the first order problem in ε is the propagation of linear Love waves in the vertically nonhomogeneous layer overlying the nonhomogeneous semi-space investigated in [17]. For harmonic wave solutions of this problem, different types of variation in $\rho^{(s)}$ and $\mu^{(s)}$, $s = 1,2$, such as linear, quadratic, harmonic or exponential variation, are examined by several authors [16, 28, 29]. In this analysis, we consider depth dependent exponential variations in the nonhomogeneous constituent materials. Hence, the densities, linear shear modulus and nonlinear material functions are taken to be, respectively,

$$\rho^{(s)} = \rho_s e^{\alpha_s y}, \quad \mu^{(s)} = \mu_s e^{\alpha_s y}, \quad n^{(s)} = n_s e^{\lambda_s y}, \ s = 1,2 \tag{22}$$

where μ_s and ρ_s, $s = 1,2$, are rigidities and densities at the interface, respectively, and α_s are the linear heterogeneity parameters, λ_s are the nonlinear heterogeneity parameters. n_s are the nonlinear material constants. When $n_s > 0$, the material has the property of shear hardening otherwise softening.

By employing the separation of variables method and using (13), and making the substitution $u_1^{(s)} = U_1^{(s)}/\sqrt{\mu^{(s)}}$ the solutions of the equations (10) are expressed as follows

$$u_1^{(1)} = \frac{1}{\sqrt{\mu_1 e^{\alpha_1 y}}} \sum_{l=1}^{\infty} \left\{ A_1^{(l)}(x_1, x_2, t_1, t_2) e^{ilkp_l y} + B_1^{(l)}(x_1, x_2, t_1, t_2) e^{-ilkp_l y} \right\} e^{il\theta} + c.c. \tag{23}$$

$$u_1^{(2)} = \frac{1}{\sqrt{\mu_2 e^{\alpha_2 y}}} \sum_{l=1}^{\infty} C_1^{(l)}(x_1, x_2, t_1, t_2) e^{lkv_l y} e^{il\theta} + c.c., \tag{24}$$

where

$$\theta = kx_0 - \omega t_0, \ p_l = (c^2/c_1^2 - 1 - {\alpha_1}^2/(4k^2 l^2))^{\frac{1}{2}}, \ v_l = (1 + {\alpha_2}^2/(4k^2 l^2) - c^2/c_2^2)^{1/2}.$$

$A_1^{(l)}$, $B_1^{(l)}$, $C_1^{(l)}$ are the first order amplitude functions dependent on $\{x_1, x_2, t_1, t_2\}$, k and ω are the wave number and angular frequency, respectively, $c = \omega/k$ is the phase velocity and "c.c." represents the complex conjugate of the former terms.

Clearly, a surface SH wave propagates when the following inequality holds for the phase velocity c

$$c_1 \left(1 + \frac{{\alpha_1}^2}{4k^2 l^2}\right)^{\frac{1}{2}} < c < c_2 \left(1 + \frac{{\alpha_2}^2}{4k^2 l^2}\right)^{\frac{1}{2}}. \tag{25}$$

Substituting of (23)-(24) in (11)-(12) gives

$$\boldsymbol{W}_l \boldsymbol{U}_1^{(l)} = \boldsymbol{0}, \ l = 1,2,.. \tag{26}$$

where $\boldsymbol{U}_1^{(l)} = (A_1^{(l)}, B_1^{(l)}, C_1^{(l)})^T$ and $\boldsymbol{W}_l$ is dispersion matrix given in the Appendix. For the nontrivial solutions of (26), the condition $\det \boldsymbol{W}_1 = 0$ gives the dispersion relation first derived in [17]

$$2kp(2kv - \alpha_2)\gamma_0 - \left(4k^2p^2 + 2kv\alpha_1\gamma_0 + \alpha_1(\alpha_1 - \alpha_2\gamma_0)\right)tan(kph) = 0 \tag{27}$$

where $v = v_1, p = p_1$ and $\gamma_0 = \mu_2/\mu_1$. For $\alpha_1 = \alpha_2 = 0$, (27) reduces to

$$\nu\gamma_0 - p\tan(kph) = 0 \tag{28}$$

which coincides with the frequency equation of Love waves in a homogeneous semi-space covered with a homogeneous layer [22]. Since the focus of this work is on the nonlinear self-modulation, the analysis does not include the harmonic-resonance case. Therefore, we assume

$$\det \boldsymbol{W}_l \neq 0, \text{for } \ l \neq 1. \tag{29}$$

Thus, the solutions of (26) can be obtained as

$$\boldsymbol{U}_1^{(1)} = \mathcal{A}_1\boldsymbol{R}, \quad \boldsymbol{U}_1^{(l)} = \boldsymbol{0} \text{ for } \ l \neq 1\ . \tag{30}$$

Here, $\mathcal{A}_1$, a complex function of $\{x_1, x_2, t_1, t_2\}$, represents the first order amplitude of the nonlinear modulation. The components of **R** satisfying $\boldsymbol{W}_1\boldsymbol{R} = 0$ can be found in the Appendix. Thus, (23)-(24) are given as follows

$$u_1^{(1)} = \frac{\mathcal{A}_1}{\sqrt{\mu_1 e^{\alpha_1 y}}}(R_1 e^{ikpy} + R_2 e^{-ikpy})e^{i\theta} + c.c., \tag{31}$$

$$u_1^{(2)} = \frac{\mathcal{A}_1}{\sqrt{\mu_2 e^{\alpha_2 y}}}R_3 e^{k\nu y}e^{i\theta} + c.c. \tag{32}$$

Note that the condition

$$k\nu > \frac{\alpha_2}{2} \tag{33}$$

which is necessary for the $u_1^{(2)}$ to vanish as $y \to -\infty$, is satisfied only if

$$k^2(1 - \frac{c^2}{c_2^2}) > 0, \text{ consequently } c < c_2\,. \tag{34}$$

It follows from (25) and (34) that

$$c_1\left(1 + \frac{{\alpha_1}^2}{4k^2}\right)^{\frac{1}{2}} < c \ < \ c_2 < \ c_2\left(1 + \frac{{\alpha_2}^2}{4k^2}\right)^{\frac{1}{2}}. \tag{35}$$

With nondimensional linear heterogeneous parameters $A_1 = \alpha_1/k, A_2 = \alpha_2/k$ and nondimensional phase velocity $C = c/c_1$, (35) can be written as follows

$$\left(1 + \frac{{A_1}^2}{4}\right)^{\frac{1}{2}} < C < \frac{c_2}{c_1} \ < \frac{c_2}{c_1}\left(1 + \frac{{A_2}^2}{4}\right)^{\frac{1}{2}}. \tag{36}$$

Hence a surface SH wave satisfying radiation condition (13) propagates in an exponentially heterogeneous layered semi-space when (36) holds for the phase velocity C.
To finalize the first order solution, we must determine $\mathcal{A}_1$. This can be accomplished by higher order problems. Substituting (31)-(32) in the problem of $O(\varepsilon^2)$, one can easily solve (14) via method of undetermined coefficients for $u_2^{(s)}$. Then the nonhomogeneous boundary conditions (15)-(16) gives the following compatibility condition

$$\boldsymbol{L}.\boldsymbol{b}_2^{(l)} = 0 \tag{37}$$

where $\boldsymbol{b}_2^{(l)}$ are given as

$$\boldsymbol{b}_2^{(1)} = -i\left(\frac{\partial\mathcal{A}_1}{\partial t_1}\frac{\partial\boldsymbol{W}_1}{\partial\omega} - \frac{\partial\mathcal{A}_1}{\partial x_1}\frac{\partial\boldsymbol{W}_1}{\partial k}\right)\boldsymbol{R} \ \text{ and } \boldsymbol{b}_2^{(l)} = \boldsymbol{0} \ \text{ for all } l\ > 1. \tag{38}$$

L is a left row vector such that $\boldsymbol{L}\boldsymbol{W}_1 = 0$. The components of $\boldsymbol{L} = (L_1, L_2, L_3\ )$ are given in the Appendix.
From the compatibility condition (37), following equation for $\mathcal{A}_1$ is obtained

$$\frac{\partial\mathcal{A}_1}{\partial t_1} + V_g\frac{\partial\mathcal{A}_1}{\partial x_1} = 0, \tag{39}$$

where the group velocity, V_g, is expressed by

$$V_g = -\left(\boldsymbol{L}\frac{\partial\boldsymbol{W}_1}{\partial k}\boldsymbol{R}\right)/\boldsymbol{L}\frac{\partial\boldsymbol{W}_1}{\partial\omega}\boldsymbol{R} \tag{40}$$

(39) implies that

$$\mathcal{A}_1 = \mathcal{A}_1\left(x_1 - \ V_g x_1, t_1, t_2\right) \tag{41}$$

Substituting $u_1^{(s)}$ and $u_2^{(s)}$ into the third order perturbation problems (18) yields

$$\mathcal{L}_1\left(u_3^{(1)}\right) = \textstyle\sum_{j=1}^{12} D_j f_j(y)e^{i\theta} + \text{terms in } \left(e^{\pm 3i\theta}\right) + c.c. \tag{42}$$

$$\mathcal{L}_2\left(u_3^{(2)}\right) = \textstyle\sum_{j=13}^{16} D_j f_j(y)e^{i\theta} + \text{terms in } \left(e^{\pm 3i\theta}\right) + c.c. \tag{43}$$

We give the explicit forms of $f_j(\mathrm{y})$ and $\ D_j,\ j = 1,2,..,16$ in the Appendix. $u_3^{(s)}$ are written as

$$u_3^{(s)} = \bar{u}_3^{(s)} + \hat{u}_3^{(s)}, \quad s = 1,2 \tag{44}$$

where $\hat{u}_3^{(s)}$ are the homogeneous solutions. They are constructed as in the (23)-(24) by writing third order amplitude functions $\boldsymbol{U}_3^{(l)} = \left(A_3^{(l)},\ B_3^{(l)}, C_3^{(l)}\right)$ instead of $\boldsymbol{U}_1^{(l)}$. The particular solutions $\bar{u}_3^{(s)}$ can be solved by means of the method of undetermined coefficients or the method of variation of parameters. For detailed analysis, we refer to [23] and [24] where the methods are applied in detail, respectively. Then, substituting $u_3^{(s)}$ and $u_1^{(s)}$ in (19)-(20) one obtains the following system

$$\boldsymbol{W}_l \boldsymbol{U}_3^{(l)} = \boldsymbol{b}_3^{(l)}, \tag{45}$$

Where $\boldsymbol{b}_3^{(1)} \neq \boldsymbol{0}, \boldsymbol{b}_3^{(3)} \neq \boldsymbol{0}$ and $\boldsymbol{b}_3^{(l)} = \boldsymbol{0}$ for $l \neq 1,3$. $\boldsymbol{b}_3^{(1)}$ is symbolized by

$$\boldsymbol{b}_3^{(1)} = \left[-i\left(\frac{\partial \boldsymbol{W}_1}{\partial \omega}\frac{\partial \mathcal{A}_1}{\partial t_2} - \frac{\partial \boldsymbol{W}_1}{\partial k}\frac{\partial \mathcal{A}_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial^2 \boldsymbol{W}_1}{\partial \omega^2}\frac{\partial^2 \mathcal{A}_1}{\partial t_1^2} - 2\frac{\partial^2 \boldsymbol{W}_1}{\partial k \partial \omega}\frac{\partial^2 \mathcal{A}_1}{\partial x_1 \partial t_1} + \frac{\partial^2 \boldsymbol{W}_1}{\partial k^2}\frac{\partial^2 \mathcal{A}_1}{\partial x_1^2}\right)\right]\boldsymbol{R} + \left(\frac{\partial \boldsymbol{W}_1}{\partial k}\frac{\partial^2 \mathcal{A}_1}{\partial x_1^2} - \frac{\partial \boldsymbol{W}_1}{\partial \omega}\frac{\partial^2 \mathcal{A}_1}{\partial x_1 \partial t_1}\right)\left(\frac{\partial \boldsymbol{R}}{\partial k} + V_g\frac{\partial \boldsymbol{R}}{\partial \omega}\right) + \mathbf{F}|\mathcal{A}_1|^2\mathcal{A}_1. \tag{46}$$

Here **F** is the constant vector. Since its components are so long, they are not written explicitly. The solvability condition $\boldsymbol{L}.\boldsymbol{b}_3^{(1)} = 0$ must be satisfied for (45). Consequently, the following NLS equation is obtained

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \Gamma\frac{\partial^2 \mathcal{A}}{\partial \xi^2} + \Delta|\mathcal{A}|^2 = 0 \tag{47}$$

where nondimensional variables are described by

$$\tau = \omega t_2,\ \ \xi = k\left(x_1 - V_g t_1\right),\ \ \mathcal{A} = k\mathcal{A}_1\,. \tag{48}$$

The simple representation of Γ and Δ are

$$\Gamma = \frac{k^2}{2\omega}\frac{d^2\omega}{dk^2}\ ,\ \Delta = -\frac{1}{\omega k^2}(\mathbf{L}.\mathbf{F})\Big/\left(\mathbf{L}\frac{\partial \boldsymbol{W}_1}{\partial \omega}\mathbf{R}\right). \tag{49}$$

Now, we search for the soliton solutions of the NLS equation (47) through the following ansatz

$$\mathcal{A}(\xi,\tau) = g(\xi)e^{ir\tau},\ \ r: \text{constant}. \tag{50}$$

For bright soliton solutions, we choose $g(\xi) = g_0 sech(\xi)$ such that $g_0 \neq 0$. Hence putting (50) in the NLS equation (47) yields

$$g_0 = \sqrt{\frac{2\Gamma}{\Delta}} \ \ \text{and}\ r = \Gamma\,. \tag{51}$$

Thus, the following bright soliton solution is constructed by means of (50)

$$\mathcal{A}(\xi,\tau) = \sqrt{\frac{2\Gamma}{\Delta}}\ sech(\xi)e^{i\Gamma\tau},\ \ \text{for}\ \Gamma\Delta > 0. \tag{52}$$

When $\Gamma\Delta < 0$, to get the dark soliton solutions, we choose $g(\xi)$ in (50) as $g(\xi) = g_0 tanh(\xi)$. Putting (50) with $g(\xi)$ in the NLS equation gives

$$g_0 = \sqrt{-\frac{2\Gamma}{\Delta}} \ \ \text{and}\ r = -2\Gamma\,. \tag{53}$$

Hence, the dark soliton solution is identified as

$$\mathcal{A}(\xi,\tau) = \sqrt{-\frac{2\Gamma}{\Delta}}\ tanh(\xi)e^{-2i\Gamma\tau},\ \ \text{for}\ \Gamma\Delta < 0. \tag{54}$$

Since it is decisive in the solutions of (47), the $\Gamma\Delta$ sign should be investigated.

4. NUMERICAL EVALUATION

The effects of nonlinear constituent materials as well as heterogeneity of the layered semi-space on the $\Gamma\Delta$ sign and hence solitary Love wave propagation are studied. In the calculations, the following parameters are chosen as in the geophysical model given in [7]

γ_0=2.159, $\quad c_2/c_1 = 1.297$.

Firstly, we investigate the influence of dimensionless linear heterogeneity parameters of the layer and semi-space $A_1 = \alpha_1/k,\ A_2 = \alpha_2/k,$ respectively, on the change of nondimensional phase velocity $C = c/c_1$ versus nondimensional wave number $K = kh$ for the dispersion relation's first branch. Figure 2 illustrates the results for different values of (A_1, A_2) which have been chosen as $\{(0.1,0.3),(0.3,0.1),(0.1,0.1),(0.3,0.3)\}$. It is seen that the curves having same A_2 approach to each other for small wave numbers whereas the curves with same A_1 approach to each other for $K \gg 1$. Consequently, linear heterogeneity of the layer dominates C for short waves while that of half space is effective on C for long waves. It is also observed that the upper limit of C is c_2/c_1 chosen as 1.297, and the lower limit which is greater than 1 changes depending on A_1, which is consistent with the inequality (36). The change of Γ versus K is also shown in Figure 3 for different values of (A_1, A_2). The sign of Γ, initially negative, changes at a certain K values with variation in A_1 and A_2.

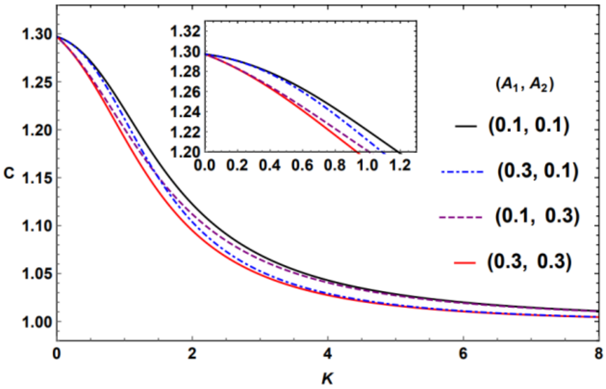


Figure 2. C vs. K for various (A_1, A_2).

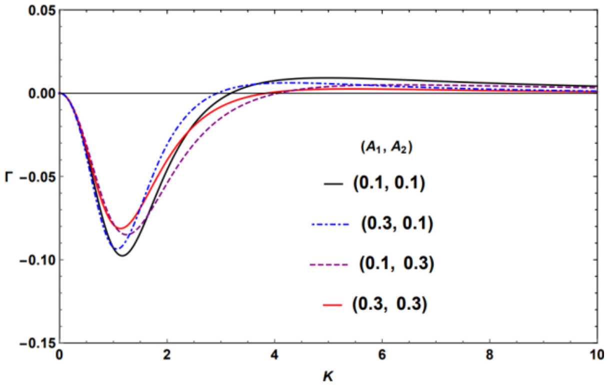
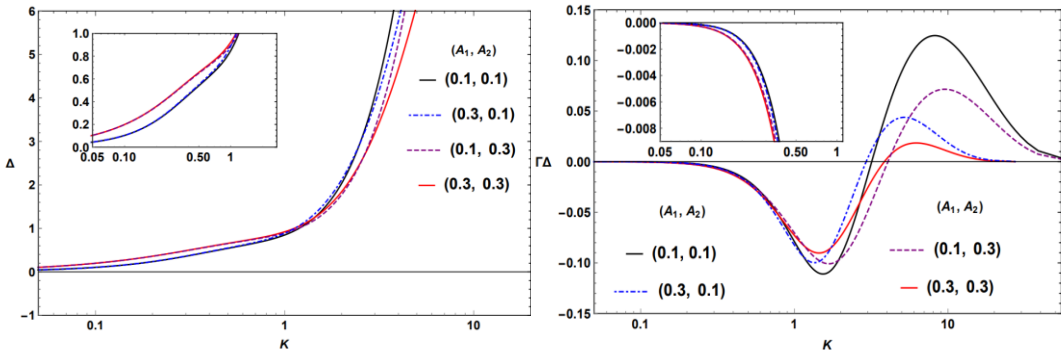


Figure 3. Γ vs. K for various (A_1, A_2).

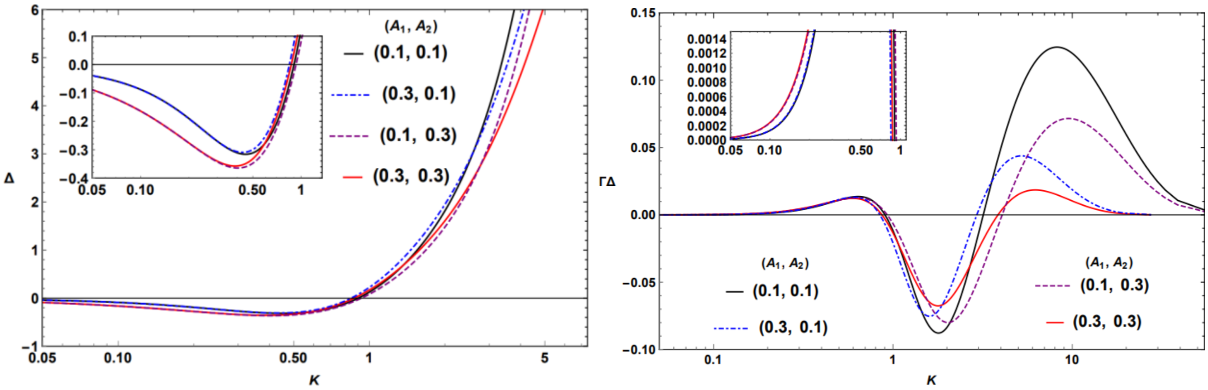
To examine the effect of heterogeneity associated with the linear constitution of both layer and semi-space on nonlinear wave propagation, in the numerical evaluations of $\Gamma\Delta$, nonlinear material constants $\beta_{01} = n_1/c_1^2$, $\beta_{02} = n_2/c_2^2$ and the nondimensional nonlinear heterogeneity parameters $\Lambda_1 = \lambda_1 h$ and $\Lambda_2 = \lambda_2 h$ are fixed while (A_1, A_2) is being changed. Note that when $\beta_{0s} > 0$, the material has shear hardening (H) otherwise softening (S) properties. Firstly, we examine the effect of (A_1, A_2) choosing as $\{(0.1,0.3), (0.3,0.1), (0.1,0.1), (0.3,0.3)\}$ on the sign of Δ and $\Gamma\Delta$ for a softening layer and a hardening semi-space, (S, H) material model, with fixed $\beta_{01} = -1$, $\beta_{02} = 1$ and the nonlinear heterogeneity parameters (Λ_1, Λ_2)=(0.1, 0.1). The results are illustrated in Figures 4a-4b. As shown in Figure 4a, Δ >0 for all K >0. Hence $\Gamma\Delta$ is negative in the interval in which Γ <0. Therefore, the presence of (54) is possible in this interval that is changing with (A_1, A_2).



(a) (b)

Figure 4. For various (A_1, A_2) and for (S, H) material model with $\beta_{01} = -1$ and $\beta_{02} = 1$, $(\Lambda_1, \Lambda_2) = (0.1, 0.1)$ a) Δ vs. K, b) $\Gamma\Delta$ vs. K.

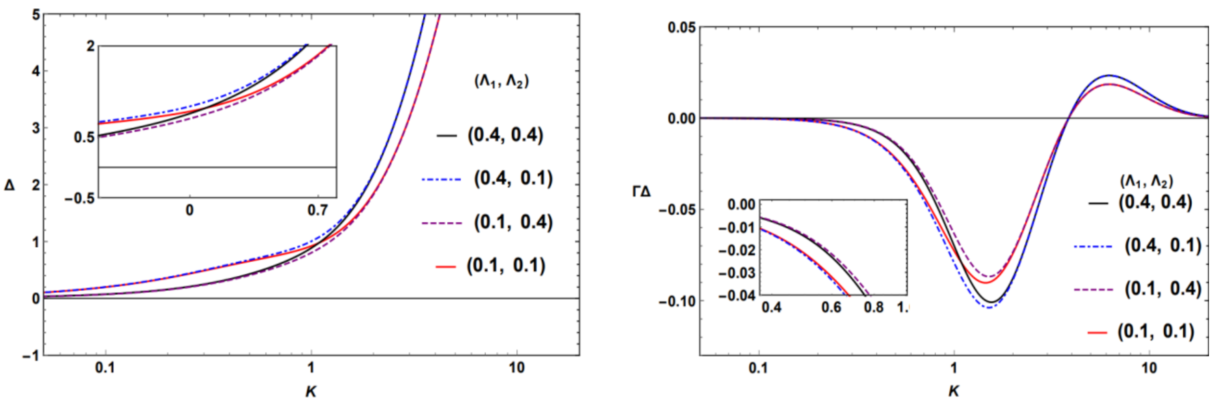
When the media consists of the softening layered half-space, for (S,S) material model having nonlinear material constants β_{01}= $\beta_{02} = -1$, changes of $\varDelta$ and $\Gamma\varDelta$ versus K are shown in Figures 5a-5b, respectively, for various values of (A_1, A_2) and for fixed (Λ_1, Λ_2)=(0.1, 0.1). As can be seen, $\Gamma\varDelta$ is positive initially, thus existence of bright solitary waves is possible. The each $\Gamma\varDelta$ curve has two zeros that belong to $\varDelta$ and Γ, respectively. The positive intervals in which bright solitons (52) exist are varying with the change in A_1 and A_2. It is seen that the $\Gamma\varDelta$ curves having same A_2 approach to each other for small wave numbers whereas the curves with same A_1 approach to each other for $K \gg 1$. Consequently, linear heterogeneity of the layer dominates nonlinear modulation for short waves while that of half space is effective on nonlinear waves for long waves. Note that different choice of β_{02} makes the curves in Figure 4 and Figure 5 different from each other. Thus, the effect of not only (A_1, A_2) but also β_{02} on the presence of solitary SH waves is demonstrated.



(a) (b)

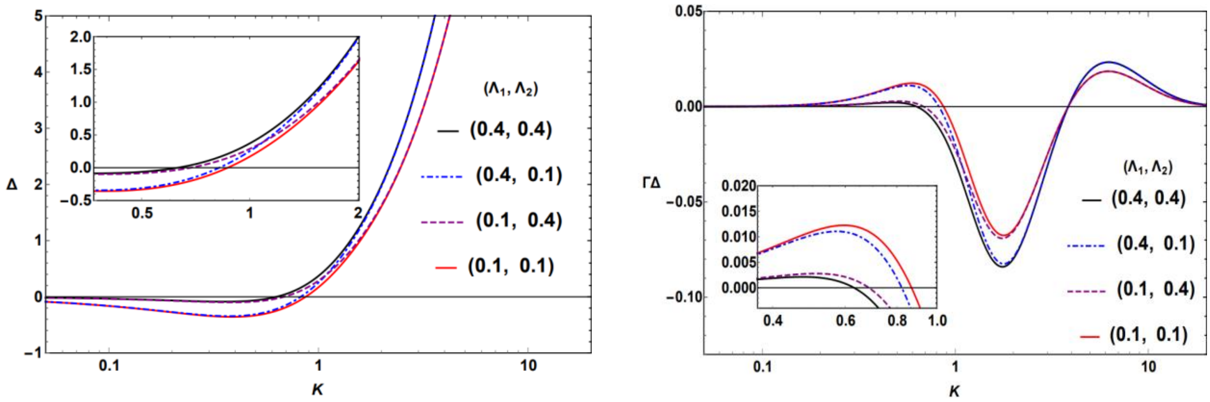
Figure 5. For various (A_1, A_2) and for (S, S) material model with β_{01}=$\beta_{02} = -1$, $(\Lambda_1, \Lambda_2) = (0.1, 0.1)$ a) $\varDelta$ vs. K, b) $\Gamma\varDelta$ vs. K.

Now, to investigate the effect of nonlinear heterogeneity of both layer and semi-space, $\varDelta$ and $\Gamma\varDelta$ curves are depicted with fixed $(A_1, A_2) = (0.3, 0.3)$ and for various values of (Λ_1, Λ_2) which have been chosen as $\{(0.1,0.1), (0.4,0.1), (0.1,0.4), (0.4,0.4)\}$. $\varDelta$ and $\Gamma\varDelta$ versus K for (S, H) material model with $(\beta_{01}, \beta_{02}) = (-1, 1)$ are presented in Figures 6a-6b, respectively. As can be observed in Figure 6a, $\varDelta > 0$ for all K. In Figure 6b, the sign of each $\Gamma\varDelta$ curve changes at $K = 3.85$ in which $\Gamma = 0$. Dark solitary SH waves exist for 0< K <3.85 in which $\Gamma\varDelta$ <0. Note that, for (S, H) material model, the wave numbers where dark solitary waves exist do not affected by the change in (Λ_1, Λ_2) . A similar examination is carried out for the media consisting of the softening layered half-space for $\beta_{01} = \beta_{02} = -1$, with Figures 7a-7b. It is seen that $\Gamma\varDelta$ curves have two zeros such that the first belongs to $\varDelta$ and the second belongs to Γ. Since Γ does not dependent on nonlinear material parameters, the second zeros do not change, whereas the first zeros vary with the variation of (Λ_1, Λ_2) . It is also seen that $\Gamma\varDelta$ curves having same Λ_2 approach to each other when $K \ll 1$, $\Gamma\varDelta$ curves with same Λ_1 approach to each other for large wave numbers. This observation is consistent with the conclusion highlighted in [22] that the layer's nonlinearity for short waves and the semi-space's nonlinearity for long waves dominate the wave modulation. Consequently, intervals where bright and dark solitons exist change depending on the nonlinear heterogeneous structures of layer and semi-space. The reason why the curves in Figure 6 and Figure 7 are different from each other is the different β_{02} selection. Thus, the effect of not only (Λ_1, Λ_2) but also β_{02} of the half-space on the presences of solitons is observed.



(a) (b)

Figure 6. For various (Λ_1, Λ_2) and for (S, H) material model with $\beta_{01} = -1, \beta_{02} = 1$, $(A_1, A_2) = (0.3, 0.3)$ a) $\varDelta$ vs. K, b) $\Gamma\varDelta$ vs. K.



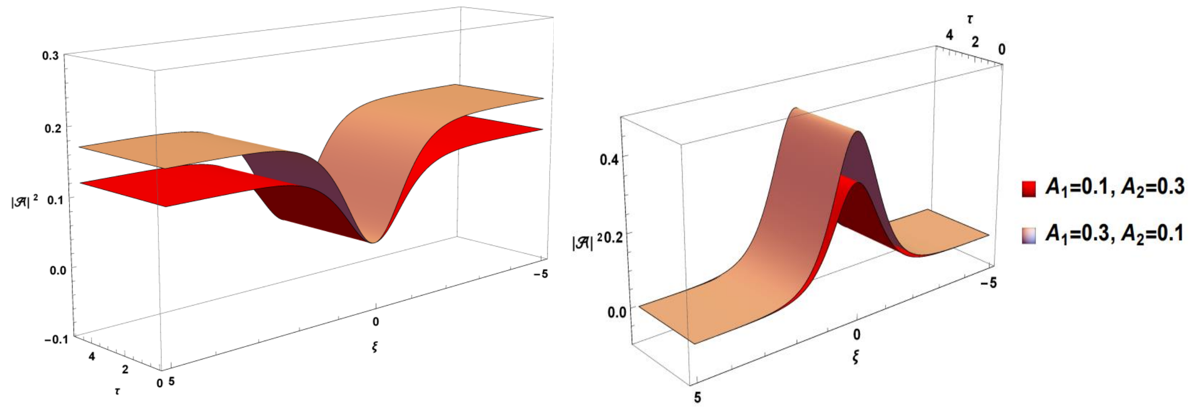
(a) (b)

Figure 7. For various (Λ_1,Λ_2) and for（S, S）material model with $\beta_{01}=-1, \beta_{02}=-1,\ (A_1,A_2)=(0.3{,}0.3)$ **a)** Δ vs. K, **b)** $\Gamma\Delta$ vs. K.

For all models having hardening half-space, $\Gamma\Delta$ is negative for $K \ll 1$, thus dark solitary waves propagate. However, for all models having softening half-space, $\Gamma\Delta$ is positive for $K \ll 1$, hence bright solitons propagate. It is concluded that the semi-space's nonlinearity dominates the wave motion for long waves.

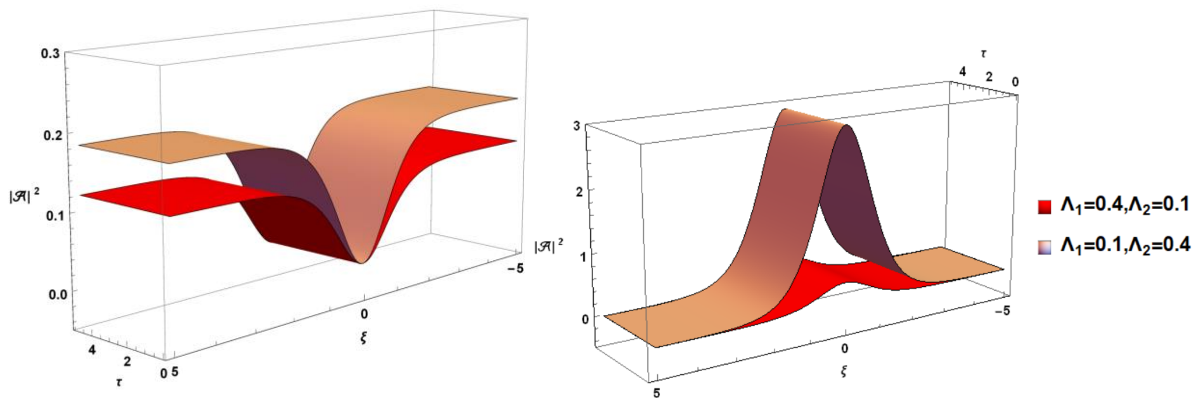
Δ and $\Gamma\Delta$ curves for hardening layer overlying the softening half-space and for hardening layer overlying the hardening half-space are not given due to limited space. These curves are symmetrical about the K axes of the opposite sign (β_{01}, β_{02}) curves in Figures 4-7.

We also examine the influence of nonlinearity and nonhomogeneity on the evolution of solitary Love waves. As shown in Figures 4b-5b when K =0.6, for (S, S) models $\Gamma\Delta$ >0 and bright solitons propagate, for (S, H) models $\Gamma\Delta$ <0 and dark soliton propagation exists. Hence nonlinear evolutions of dark and bright solitons are presented in Figures 8a-8b for (S, H) and (S, S) material models, respectively, for different (A_1,A_2) selected as $\{(0.1{,}0.3),(0.3{,}0.1)\}$, with fixed K =0.6 and (Λ_1,Λ_2)=(0.1, 0.1). Consequently, the considerable effects of both (A_1,A_2) and β_2 on the nonlinear evolutions of waves are demonstrated. Similar observation is made for different (Λ_1,Λ_2) values selected as $(\Lambda_1,\Lambda_2)=\{(0.4{,}0.1),(0.1{,}0.4)\}$ with K=0.6 and fixed $(A_1,A_2)=(0.3{,}0.3)$ in Figures 9a-9b, respectively, for (S, H) and (S, S) models. Thus, the influence of not only (Λ_1,Λ_2) but also β_2 on the nonlinear evolutions of waves is observed. Notice that for the (S, H) material model, though the change in (Λ_1,Λ_2) does not affect the interval of existence of dark solitons, it has a significant effect on the nonlinear evolution of waves.



(a) (b)

Figure 8. For various (A_1,A_2) with $(\Lambda_1,\Lambda_2)=\ (0.1, 0.1)$ and K=0.6 **a)** Nonlinear evolution of the dark solitons in the (S, H) model with $\beta_{01}=-1$, $\beta_{02}=1$ **b)** Nonlinear evolution of the bright solitons in the (S, S) model with $\beta_{01}=-1,\ \beta_{02}=-1$.



(a) (b)

Figure 9. For various (Λ_1, Λ_2) with $(A_1, A_2) = (0.3, 0.3)$ and K=0.6 **a**) Nonlinear evolution of the dark solitons in the (S, H) model with $\beta_{01} = -1$, $\beta_{02} = 1$ **b)** Nonlinear evolution of the bright solitons in the (S, S) model with $\beta_{01} = -1$, $\beta_{02} = -1$.

5. CONCLUDING REMARKS

Existence and nonlinear evolution of solitary Love waves in a layered semi-space consisting of different nonlinear, elastic, heterogeneous constituent materials varying exponentially with depth are examined. Firstly, dispersion relation is derived, and it is shown that linear heterogeneity of the layer dominates C for short waves while that of half space is dominant on C for long waves. Then an NLS equation is obtained for nonlinear modulation of waves via multiple scales method. For two different material models, (S, H) and (S, S), the variation of $\Gamma\Delta$ sign with heterogeneity of both layer and half-space is examined due to its distinctive effect on the presence of solitary wave solutions. As it is seen in the Figures 4-7, linear and nonlinear heterogeneity parameters of the semi-space affects strongly the existence of envelope solitary waves for long waves while those of layer dominate the existence of solitary waves for short waves. Furthermore, the considerable influence of heterogeneity properties of both layer and semi-space on the nonlinear evolutions of bright and dark solitons are shown graphically. It is observed that for the (S, H) material model, the change in (Λ_1, Λ_2) does not affect the interval of existence of dark solitons whereas it has a considerable effect on nonlinear evolution of dark solitons.

Author's Contribution

Ekin DELİKTAŞ-ÖZDEMİR contributed to the design and implementation of the research, to the analysis of the results, and to the writing, reviewing, and editing of this manuscript.

Conflict of Interest

All authors declare that they have no conflicts of interest.

APPENDIX

$$\boldsymbol{W}_l = \begin{pmatrix} (iklp_l - \alpha_1/2)e^{ilkp_l h} & (-iklp_l - \alpha_1/2)e^{-ilkp_l h} & 0 \\ iklp_l - \alpha_1/2 & -iklp_l - \alpha_1/2 & (-klv_l + \alpha_2/2)\sqrt{\gamma_0} \\ 1 & 1 & -1/\sqrt{\gamma_0} \end{pmatrix}$$

$$\boldsymbol{R} = (R_1, R_2, R_3)^T;\ \ R_1 = \frac{(2kp - i\alpha_1)(1 - i\tan(hkp))}{2\sqrt{\gamma_0}(2kp - \alpha_1 \tan(hkp))}, R_2 = \frac{(2kp + i\alpha_1)(1 + i\tan(hkp))}{2\sqrt{\gamma_0}(2kp - \alpha_1 \tan(hkp))}, R_3 = 1$$

$$\boldsymbol{L} = (L_1, L_2, L_3);\quad L_1 = \frac{2kp\sec(hkp)}{-2kp + \alpha_1 \tan(hkp)},\quad L_2 = 1,\quad L_3 = -\frac{(4k^2p^2 + \alpha_1{}^2)\tan(hkp)}{4kp - 2\alpha_1 \tan(hkp)}.$$

$$f_1 = n^{(1)}(y)\frac{e^{ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ \ f_2 = \frac{dn^{(1)}(y)}{dy}\frac{e^{ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ f_3 = \frac{e^{ikpy}}{\sqrt{\mu_1 e^{\alpha_1 y}}},\ \ f_4 = y\frac{e^{ikpy}}{\sqrt{\mu_1 e^{\alpha_1 y}}},$$

$$f_5 = n^{(1)}(y)\frac{e^{-ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ \ f_6 = \frac{dn^{(1)}(y)}{dy}\frac{e^{-ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ f_7 = \frac{e^{-ikpy}}{\sqrt{\mu_1 e^{\alpha_1 y}}},\ \ f_8 = y\frac{e^{-ikpy}}{\sqrt{\mu_1 e^{\alpha_1 y}}},$$

$$f_9 = n^{(1)}(y)\frac{e^{3ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ \ f_{10} = \frac{dn^{(1)}(y)}{dy}\frac{e^{3ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ f_{11} = n^{(1)}(y)\frac{e^{-3ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},\ \ f_{12} = \frac{dn^{(1)}(y)}{dy}\frac{e^{-3ikpy}}{(\mu_1 e^{\alpha_1 y})^{3/2}},$$

$$f_{13} = \frac{e^{kvy}}{\sqrt{\mu_2 e^{\alpha_2 y}}},\ \ f_{14} = y\frac{e^{kvy}}{\sqrt{\mu_2 e^{\alpha_2 y}}},\ f_{15} = n^{(2)}(y)\frac{e^{3kvy}}{(\mu_2 e^{\alpha_2 y})^{3/2}},\ \ f_{16} = \frac{dn^{(2)}(y)}{dy}\frac{e^{3kvy}}{(\mu_2 e^{\alpha_2 y})^{3/2}}.$$

$$D_1 = |A_1|^2 A_1 R_1^2 R_2\left(\frac{9}{16}\alpha_1{}^4 - \frac{9}{4}ikp\,\alpha_1{}^3 - ik^3\alpha_1(p + 9p^3) - k^4(9 + 2p^2 + 9p^4)\right),$$

$$D_2 = |A_1|^2 A_1 R_1^2 R_2 \frac{i}{8}(9i\,\alpha_1{}^3 + 18kp\,\alpha_1{}^2 + 12ik^2\alpha_1(1 + 3p^2) + 8k^3p(1 + 9p^2)),$$

$D_3 = 2iR_1\left(\mathcal{M}_{12}^{(1)} + \mathcal{M}_{21}^{(1)}\right) + R_1\mathcal{N}^{(1)} + 2\Lambda_1 \frac{\partial}{\partial x_1}\mathcal{M}_{11}^{(1)},$
$D_4 = -\frac{2iR_1}{pkc_1^2}\left(\omega\frac{\partial}{\partial t_1} + kc_1^2\frac{\partial}{\partial x_1}\right)\mathcal{M}_{11}^{(1)},$
$D_5 = |A_1|^2A_1R_2^2R_1\left(\frac{9}{16}\alpha_1{}^4 + \frac{9}{4}ikp\,\alpha_1{}^3 + ik^3\alpha_1(p+9p^3) - k^4(9+2p^2+9p^4)\right),$
$D_6 = |A_1|^2A_1R_2^2R_1\frac{1}{8}(-9\,\alpha_1{}^3 - 18ikp\,\alpha_1{}^2 - 12k^2\alpha_1(1+3p^2) - 8ik^3p(1+9p^2)),$
$D_7 = 2iR_2\left(\mathcal{M}_{12}^{(1)} + \mathcal{M}_{21}^{(1)}\right) + R_2\mathcal{N}^{(1)} + 2\Lambda_2 \frac{\partial}{\partial x_1}\mathcal{M}_{11}^{(1)},$
$D_8 = \frac{2iR_2}{pkc_1^2}\left(\omega\frac{\partial}{\partial t_1} + kc_1^2\frac{\partial}{\partial x_1}\right)\mathcal{M}_{11}^{(1)},$
$D_9 = |A_1|^2A_1R_1^3\left(\frac{3}{16}\alpha_1{}^4 - \frac{9}{4}ikp\,\alpha_1{}^3 - 9k^2p^2\alpha_1{}^2 + ik^3p\alpha_1(-1+15p^2) + k^4(-3-2p^2+9p^4)\right),$
$D_{10} = |A_1|^2A_1R_1^3\frac{1}{8}(-2ikp+\alpha_1)(4k^2(-1+3p^2) + 12\,ikp\,\alpha_1 - 3\,\alpha_1{}^2),$
$D_{11} = |A_1|^2A_1R_2^3\left(\frac{3}{16}\alpha_1{}^4 + \frac{9}{4}ikp\,\alpha_1{}^3 - 9k^2p^2\alpha_1{}^2 + ik^3p\alpha_1(1-15p^2) + k^4(-3-2p^2+9p^4)\right),$
$D_{12} = |A_1|^2A_1R_2^3\frac{1}{8}(2ikp+\alpha_1)(4k^2(-1+3p^2) - 12\,ikp\,\alpha_1 - 3\,\alpha_1{}^2),$
$D_{13} = 2iR_3\left(\mathcal{M}_{12}^{(2)} + \mathcal{M}_{21}^{(2)}\right) + R_3\mathcal{N}^{(2)} + 2\Lambda_3 \frac{\partial}{\partial x_1}\mathcal{M}_{11}^{(2)},$
$D_{14} = \frac{2R_3}{vkc_2^2}\left(\omega\frac{\partial}{\partial t_1} + kc_2^2\frac{\partial}{\partial x_1}\right)\mathcal{M}_{11}^{(2)},$
$D_{15} = \frac{1}{16}(3\alpha_2{}^4 + 16k^4(9v^4+2v^3-3) - 16\alpha_2k^3(15v^3+v) + 144\alpha_2{}^2k^2v^2 - 36\alpha_2{}^3kv)R_3^3|A_1|^2A_1\,,$
$D_{16} = \frac{1}{8}(2kv-\alpha_2)(4k^2(3v^2+1) - 12\alpha_2kv + 3\alpha_2{}^2)R_3^3|A_1|^2A_1$
$\mathcal{M}_{\beta\gamma}^{(\alpha)} = \omega\frac{\partial A_\beta}{\partial t_\gamma} + kc_\alpha^2\frac{\partial A_\beta}{\partial x_\gamma}, \quad \mathcal{N}^{(\alpha)} = c_\alpha^2\frac{\partial^2 A_1}{\partial x_1^2} - \frac{\partial^2 A_1}{\partial t_1^2}, \quad \Lambda_\alpha = \left(\frac{\partial R_\alpha}{\partial k} + V_g\frac{\partial R_\alpha}{\partial \omega}\right)$

REFERENCES

[1] A.E.H. Love “Some Problems of Geodynamics”, Cambridge University Press, 1911.

[2] S. Kundu, S. Gupta, and S. Manna “Propagation of Love wave in fiber-reinforced medium lying over an initially stressed orthotropic half-space”, International Journal for Numerical and Analytical Methods in Geomechanics, vol. 38, no. 11, pp. 1172–1182, 2014.

[3] J.D. Achenbach “Wave Propagation in Elastic Solids”, Amsterdam: North Holland Publ. Co., 1973.

[4] K. Bataille, and F. Lund “Nonlinear waves in elastic media”, Physica D: Nonlinear Phenomena, vol. 6, no. 1, pp. 95-104, 1982.

[5] T. Boxberger, M. Picozzi, and S. Parolai “Shallow geology characterization using Rayleigh and Love wave dispersion curves derived from seismic noise array measurements”, Journal of Applied Geophysics, vol. 75, no. 2, pp. 345-354, 2011.

[6] A.C. Eringen, and E.S. Suhubi “Elastodynamics”, New York: Academic Press, vol. 1, 1974.

[7] W.M. Ewing, W.S. Jardetzky, F. Press, and A. Beiser “Elastic waves in layered media”, Physics Today, vol. 10, no. 12, pp. 27, 1957.

[8] Y. Fukao, and K. Abe “Multi-mode Love waves excited by shallow and deep earthquakes”, Bull Earthq. Res. Inst, vol. 49, pp. 1-12, 1971.

[9] S.V. Kuznetsov “Love waves in nondestructive diagnostics of layered composites”, Survey Acoustical Physics, vol. 56, no. 6, pp. 877-892, 2010.

[10] E. Meissner “Elastische Oberflachenwellen mit Dispersion in einem inhomogenen Medium”, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich, vol. 66, pp. 181-195, 1921.

[11] S.C.D. Gupta “Note on Love waves in a homogeneous crust laid upon heterogeneous medium”, Journal of Applied Physics, vol. 23, no. 11, pp. 1276-1277, 1952.

[12] S.N. Bhattacharya “Love wave dispersion in a homogeneous crust overlying a continuous mantle with vertical in-homogeneity”, Geophysical Journal International, vol. 19, no. 4, pp. 335-349, 1970.

[13] K. Watanabe “Plane SH-waves in harmonically in-homogeneous elastic media”, Wave motion, vol. 6, no. 5, pp. 477-488, 1984.

[14] S. Majhi, P.C. Pal, and S. Kumar “Propagation of SH waves in a visco-elastic layer overlying an in-homogeneous isotropic half-space”, Ain Shams Engineering Journal, vol. 9, no. 4, pp. 675-680, 2018.

[15] S. Manna, S. Kundu, and S. Gupta “Effect of reinforcement and in-homogeneity on the propagation of Love waves”, International Journal of Geomechanics, vol. 16, no. 2, 2016.

[16] C.D. Wang, C.S. Tzeng, E. Pan, and J.J. Liao “Displacements and stresses due to a vertical point load in an inhomogeneous transversely isotropic half-space”, Int. J. Rock Mech. Min. Sci. vol. 40, no. 5, pp. 667–685, 2003.

[17] R.S. Sidhu “Propagation of love waves in a two-layered heterogeneous half space”, Pure and Applied Geophysics, vol. 91, pp. 76-94, 1971.

[18] S.N. Chatterjee “Propagation of Love waves in a medium containing a heterogeneous layer lying over a heterogeneous half space”, Pure and Applied Geophysics, vol. 90, pp. 53-60, 1971.

[19] M. Chatterjee, S. Dhua, A. Chattopadhyay and S. A. Sahu “Seismic waves in heterogeneous crust-mantle

layers under initial stresses", Journal of Earthquake Engineering, vol. 20, no. 1, pp. 39-61, 2016.

[20] S. Gupta et al. "Propagation of Love waves in non-homogeneous substratum over initially stressed heterogeneous half-space", Applied Mathematics and Mechanics, vol. 34, no. 2, pp. 249–258, 2013.

[21] A.M. Abd-Alla, and S.M. Ahmed "Propagation of Love waves in a non-homogeneous orthotropic elastic layer under initial stress overlying semi-infinite medium", Applied Mathematics and Computation, vol. 106, no. 2, pp. 265-275, 1999.

[22] M. Teymur "Nonlinear modulation of Love waves in a compressible hyperelastic layered half-space", Int. J. Eng. Sci., vol.26, pp. 907-927, 1988.

[23] E. Deliktas-Ozdemir, S. Ahmetolan, and D. Tuna "Existence of solitary SH waves in a heterogeneous elastic two-layered plate", Z. Angew. Math. Phys., vol. 73, pp. 220, 2022.

[24] E. Deliktas-Ozdemir "Nonlinear Love-type waves in a heterogeneous layer sandwiched between two semi-infinite media", Z. Angew. Math. Phys., vol. 74, pp. 1-7, 2023.

[25] J.K. Knowles "The finite anti-plane shear field near the tip of a crack for a class of in-compressible elastic solids", International Journal of Fracture, vol. 13, no. 5, pp. 611-639, 1977.

[26] E. Deliktas-Özdemir, and M. Teymür "Nonlinear surface SH waves in a half-space covered by an irregular layer", Z. Angew. Math. Phys., vol. 73, no. 4, pp. 1-18, 2022.

[27] A. Jeffrey, and T. Kawahara "Asymptotic Methods in Nonlinear Wave Theory", Boston Pitman, 1982.

[28] A. Chattopadhyay, S. Gupta, V.K. Sharma, and P. Kumari "Effect of point source and heterogeneity on the propagation of SH-waves", Int. J. of Appl. Math and Mech., vol. 6, no. 9, pp. 76-89, 2010.

[29] P. Avtar "Love waves in a two-layered crust overlying a vertically in-homogeneous halfspace", I. pure and applied geophysics, vol. 66, no. 1, pp. 48-68, 1967.