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Research Article

On Optimal Control of the Heat Flux at the Left-Hand Side in a Heat Conductivity System

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Abstract: We deal with an optimal boundary control problem in a 1-d heat equation with Neumann boundary conditions. We search for a boundary function which is the minimum element of a quadratic cost functional involving the H^1-norm of boundary controls. We prove that the cost functional has a unique minimum element and is Fréchet differentiable. We give a necessary condition for the optimal solution and construct a minimizing sequence using the gradient of the cost functional.

Keywords: Optimal control problems, heat equation, Fréchet differentiability, adjoint problem.

1. Introduction

Control problems are used to improve efficiency in many fields such as economics, biology, agriculture, robotics industry, chemical reactions, and gas dynamics. Mathematical modeling of many physical phenomena is known to lead to differential equations [1, 6–8, 21, 22, 24–26]. Therefore, it is important to study the control problems related to PDEs. Optimal control problems for parabolic equations arise in various areas of science including chemical reactions, heat transfer, and population dynamics and they have been widely studied due to their importance in the natural sciences and their applications. The boundary control problem for heat transfer systems is one of the most addressed control problems for PDEs. Some detailed works of problems in these areas can be found in [2, 3, 5, 9, 10, 14, 15, 17, 19, 20].

Lions [17] studied the optimal control problem in the parabolic system with the aim of finding a boundary condition that ensures the approach of the solution of the parabolic problem at the terminal time to the given desired function. He chose the Lebesque space L_2 as the space of bound-

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ary controls. Hasanoğlu [12] considered the problem of finding unknown pair $\{h(t,x), f(t)\}$ in the equation $y_t - (a(x)y_x)_x = h(t,x)$ with conditions $y_x(t,0) = 0, -a(L)y_x(t,L) = v[y(t,L) - f(t)]$ from the final overdetermination. Sadek and Bokhari [23] examined the controlling of Neumann boundary conditions for the heat conduction equation by minimizing the energy-based performance measure involving boundary controls.

Şener and Subaşi [27] analyzed the optimal control problem of the boundary function $s(t)$ in the system

$$\begin{cases} y_t = ay_{xx} + b(t,x), & (t,x) \in (0,T) \times (0,L), \\ y(0,x) = v(x), & 0 < x < L, \\ y_x(t,0) = 0, \ y_x(t,L) = s(t), & 0 < t < T. \end{cases}$$

They obtained the optimal solution as a minimum element of the cost functional

$$J_\alpha(s) = \int_0^L [y(T,x;s) - f(x)]^2 dx + \alpha \|s\|^2_{H^1(0,T)}$$

for the given target function $f(x) \in L_2(0,L)$ and $\alpha > 0$.

In this study, we consider the following mathematical model

$$\begin{cases} w_t = aw_{xx} + b(t,x), & (t,x) \in \Omega := (0,T) \times (0,L), \\ w(0,x) = w_0(x), & 0 < x < L, \\ w_x(t,0) = \mu(t), \ w_x(t,L) = 0, & 0 < t < T, \end{cases} \tag{1}$$

where T is a given final time, a is a positive constant, $b(t,x)$, $w_0(x)$ are given functions and $\mu(t)$ is an unknown function. Physically speaking, a is the heat conductivity, $b(t,x)$ is the heat source, $w_0(x)$ is the initial temperature, and $\mu(t)$ is the heat flux.

The aim of this study is to find a boundary function $\mu \in H^1(0,T)$ such that the corresponding solution to the system (1) approaches to the given desired $\nu(t,x) \in L_2(\Omega)$. More precisely, we want to minimize the cost functional

$$J_\alpha(\mu) = \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dx dt + \alpha \|\mu - \mu^+\|^2_{H^1(0,T)} \tag{2}$$

in the admissible controls set $M_{ad} \subset H^1(0,T)$. Here the function $\mu^+(t) \in H^1(0,T)$ is an initial guess for the optimal solution and $\alpha > 0$ is a regularization parameter. $w(t,x;\mu)$ stands for the dependence of the solution $w(t,x)$ of the system (1) on the boundary control $\mu(t)$.

This paper differs from existing works in the literature in view of the functional space of the controls and the choice of the cost functional. Previous studies propose the usage of the space L_2 as the control set [5, 12, 17, 23]. Moreover, this study investigates a different target than the study in [27]. With the choice of the functional in (2), we use $w(t,x;\mu)$ for the boundary control $\mu(t)$.

This paper is organized as follows: Firstly, we show that the conditions of the Goebel Theorem are valid for the optimal control problem considered. So, we prove that the optimal solution exists and is unique by this theorem. Then, we introduce an adjoint problem by the Lagrange multiplier method and calculate the Fréchet derivate of the cost functional via the adjoint approach. Finally, we state a necessary optimality condition and establish a minimizing sequence.

2. Existence and Uniqueness of a Minimizer for the Cost Functional

This section is dedicated to proving the conditions for the existence of the unique optimal solution to the optimal control problem (1)-(2). We denote the set of admissible boundary control functions with M_{ad}. Let M_{ad} be a non-empty subset of the space $H^1(0,T)$. Furthermore, we assume that M_{ad} is closed, convex, and bounded.

We know that for every $w_0(x) \in H^1(0,L)$, $b(t,x) \in L_2(\Omega)$ and $\mu(t) \in H^1(0,T)$, the parabolic system (1) has a unique solution $w \in H^{2,1}(\Omega)$ satisfies the following estimate:

$$\|w\|^2_{H^{2,1}(\Omega)} \le c_1(\|b\|^2_{L_2(\Omega)} + \|w_0\|^2_{H^1(0,L)} + \|\mu\|^2_{H^1(0,T)}), \tag{3}$$

where c_1 is a constant independent from b, w_0 and μ [18]. We refer to [16] for definitions of the spaces $H^{2,1}(\Omega)$, $H^1(0,L)$ and $L_2(\Omega)$.

Let $\delta\mu \in M_{ad}$ be an increment of the control at $\mu \in M_{ad}$ such that $\mu + \delta\mu \in M_{ad}$. Let us denote by $w_\delta = w(t,x;\mu+\delta\mu)$ the solution of the system (1) corresponding to the boundary condition $\mu + \delta\mu \in M_{ad}$. Then, the function $\delta w(t,x;\mu) = w(t,x;\mu+\delta\mu) - w(t,x;\mu) = w_\delta - w$ is the solution to the following difference problem

$$\begin{cases} \delta w_t = a\delta w_{xx}, & (t,x) \in \Omega, \\ \delta w(0,x) = 0, & 0 < x < L, \\ \delta w_x(t,0) = \delta\mu(t),\ \ \delta w_x(t,L) = 0, & 0 < t < T. \end{cases} \tag{4}$$

Furthermore, the difference problem is of the same type as the problem (1). So, it can be proven that the solution $\delta w(t,x;\mu)$ of the problem (4) satisfies the following inequality:

$$\|\delta w(t,x;\mu)\|^2_{L_2(\Omega)} \le c_2 \|\delta\mu\|^2_{H^1(0,T)}, \quad t \in [0,T]. \tag{5}$$

Here c_2 is independent from $\delta\mu$.

We can use the Goebel Theorem [11] widely referred to for the existence of a minimum element in optimal control problems. The following theorem states the existence and uniqueness of the solution to the optimal control problem under consideration.

Theorem 2.1 *Let $\mu^+ \in H^1(0,T)$ be a given element. There is a dense subset $G \in H^1(0,T)$ such that the cost functional $J_\alpha(\mu)$ has a unique minimum in the set M_{ad} for all $\mu^+ \in G$ and $\alpha > 0$.*

Proof We know that $H^1(0,T)$ is a uniformly convex Banach space [4] and the admissible set M_{ad} is a bounded, closed and convex subset of $H^1(0,T)$. Let's rewrite the cost functional as

$$J_\alpha(\mu) = J(\mu) + \alpha\|\mu\|^2_{H^1(0,T)},$$

where

$$J(\mu) = \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dxdt.$$

The functional $J(\mu)$ is bounded from below in the set M_{ad} since $J(\mu) \ge 0$ for any $\mu \in M_{ad}$. It is sufficient to show that the functional $J(\mu)$ is lower semi-continuous in the set M_{ad}. Let us evaluate the increment $\delta J(\mu) = J(\mu + \delta\mu) - J(\mu)$ for any $\mu \in M_{ad}$. We obtain

$$\begin{aligned}
\delta J(\mu) &= \int_0^T \int_0^L [w(t,x;\mu+\delta\mu) - \nu(t,x)]^2 dxdt - \int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]^2 dxdt \\
&= 2\int_0^T \int_0^L [w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu) dxdt \\
&+ \int_0^T \int_0^L [\delta w(t,x;\mu)]^2 dxdt.
\end{aligned} \tag{6}$$

Taking into account the inequalities (3) and (5), we can write that

$$|\delta J(\mu)| \le c_3(\|\delta\mu\|_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}). \tag{7}$$

Here c_3 is independent from $\delta\mu$.

(7) implies that the functional $J(\mu)$ is lower semi-continuous in the set M_{ad}. According to Goebel Theorem, there is a dense subset G of $H^1(0,T)$ such that the functional $J_\alpha(\mu)$ takes its minimum value at a unique point for every $\mu^+ \in G$. □

3. Fréchet Differentiability of the Cost Functional

In this section, we first apply the Lagrange multipliers method to obtain the adjoint problem and then find the Fréchet derivative of the functional $J_\alpha(\mu)$. In order to construct a minimizing sequence, it is important to prove that the cost functional is continuously differentiable.

Lagrange functional is defined by

$$L(w,\mu,\varphi) = J_\alpha(\mu) + \langle \varphi, w_t - a w_{xx} - b \rangle_{L_2(\Omega)}$$

, where the functional $J_\alpha(\mu)$ is the cost functional given in (2) and φ is called the Lagrange function.

It can be easily seen that the first variation for the Lagrange functional is:

$$\begin{aligned} \delta L &= \int_0^T \int_0^L 2[w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu)dxdt \\ &- \int_0^T \int_0^L [\varphi_t + a\varphi_{xx}]\delta w(t,x;\mu)dxdt + \int_0^L \varphi(T,x)\delta w(T,x;\mu)dx \\ &+ \int_0^T \varphi_x(t,L)\delta w(t,L)dt - \int_0^T \varphi_x(t,0)\delta w(t,0)dt, \end{aligned} \tag{8}$$

where $\delta w(t,x;\mu)$ is the solution to the problem (4).

Using the $\delta L = 0$ stationarity condition, we get the following adjoint problem:

$$\begin{cases} \varphi_t + a\varphi_{xx} = 2[w(t,x;\mu) - \nu(t,x)], & (t,x) \in \Omega, \\ \varphi(T,x) = 0, & 0 < x < L, \\ \varphi_x(t,0) = 0, \ \ \varphi_x(t,L) = 0, & 0 < t < T. \end{cases} \tag{9}$$

If we replace t in (9) by new variable $\tau = T - t$, then we obtain a boundary value problem in the same type as the problem (1). The adjoint problem has a weak solution φ in $H^{2,1}(\Omega)$ since $w - \nu \in L_2(\Omega)$ [18].

Lemma 3.1 *Let $\mu, \mu + \delta\mu \in M_{ad}$ be given elements. If $w = w(t,x;\mu)$ is the solution to the problem (1) and $\varphi(t,x;\mu)$ is the solution to the adjoint problem (9), then the following identity holds:*

$$\int_0^T \int_0^L 2[w(t,x;\mu) - \nu(t,x)]\delta w(t,x;\mu)dxdt = a\int_0^T \delta\mu(t)\varphi(t,0)dt \tag{10}$$

for all $\mu \in M_{ad}$.

Proof Using the equation (9) and applying integration by parts, we write the left side of (10) as follows:

$$
\begin{aligned}
&2\int_0^T\int_0^L [w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&=\int_0^T\int_0^L [\varphi_t(t,x)+a\varphi_{xx}(t,x)]\delta w(t,x;\mu)dxdt\\
&=\int_0^L \Big\{[a\varphi(t,x)\delta w(t,x;\mu)]_{t=0}^{t=T}-\int_0^T \varphi(t,x)\delta w_t(t,x;\mu)dt\Big\}dx\\
&\quad+\int_0^T \Big\{[a\varphi_x(t,x)\delta w(t,x;\mu)]_{x=0}^{x=L}-\int_0^L a\varphi_x(t,x)\delta w_x(t,x;\mu)dx\Big\}dt.
\end{aligned}
$$

From (4) and (9), we get

$$
\begin{aligned}
&2\int_0^T\int_0^L [w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&=-\int_0^T\int_0^L \varphi(t,x)\delta w_t(t,x;\mu)dxdt\\
&\quad-\int_0^T \Big\{[a\varphi(t,x)\delta w_x(t,x;\mu)]_{x=0}^{x=L}-\int_0^L a\varphi(t,x)\delta w_{xx}(t,x;\mu)dx\Big\}dt\\
&=-\int_0^T\int_0^L [\delta w_t(t,x;\mu)-a\delta w_{xx}(t,x;\mu)]\varphi(t,x)dxdt\\
&\quad+\int_0^T a\varphi(t,0)\delta\mu(t)dt.
\end{aligned}
$$

Considering the equation (4), the integral identity (10) is obtained.

□

Let's evaluate the first variation of $J_\alpha(\mu)$. We write

$$
\begin{aligned}
\delta J_\alpha(\mu) &= J_\alpha(\mu+\delta\mu)-J_\alpha(\mu)\\
&=2\int_0^T\int_0^L [w(t,x;\mu)-\nu(t,x)]\delta w(t,x;\mu)dxdt\\
&\quad+\int_0^T\int_0^L [\delta w(t,x;\mu)]^2dxdt\\
&\quad+2\alpha\langle\mu-\mu^+,\delta\mu\rangle_{H^1(0,T)}+\|\delta\mu\|^2_{H^1(0,T)},
\end{aligned}
\tag{11}
$$

where $\mu+\delta\mu\in M_{ad}$ and $\delta w(t,x;\mu)$ is the solution to the problem (4).

Using the integral identity (10) on the formula (11) we can write the first variation of the cost functional $J_\alpha(\mu)$ as follows:

$$\begin{aligned}\delta J_\alpha(\mu) = \int_0^T a\varphi(t,0)\delta\mu(t)dt + \int_0^T\int_0^L [\delta w(t,x;\mu)]^2 dxdt \\ + 2\alpha\langle \mu - \mu^+, \delta\mu\rangle_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}.\end{aligned} \tag{12}$$

In order to get the Fréchet derivative of the cost functional, the first term on the right-hand side of (12) must be written as the inner product in the space $H^1(0,T)$. To do this we define the following problem

$$\begin{cases} \theta''(t) - \theta(t) = -a\varphi(t,0), \quad t \in (0,T), \\ \theta'(0) = 0, \ \theta'(T) = 0. \end{cases} \tag{13}$$

Using (13), the formula (12) can be written as

$$\begin{aligned}\delta J_\alpha(\mu) = \int_0^T (\theta(t)\delta\mu(t) + \theta'(t)\delta\mu'(t))dt + \int_0^T\int_0^L [\delta w(t,x;\mu)]^2 dxdt \\ + 2\alpha\langle \mu - \mu^+, \delta\mu\rangle_{H^1(0,T)} + \|\delta\mu\|^2_{H^1(0,T)}.\end{aligned} \tag{14}$$

The estimate (5) yields that the second term on the right-hand side of (14) is of the order $o(\|\delta\mu\|^2_{H^1(0,T)})$. The formula (14) becomes

$$\delta J_\alpha(\mu) = \langle \theta + 2\alpha(\mu - \mu^+), \delta\mu\rangle_{H^1(0,T)} + o(\|\delta\mu\|^2_{H^1(0,T)}).$$

So, the cost functional is Fréchet differentiable, that is $J_\alpha(\mu) \in C^1(M_{ad})$. The operator

$$J'_\alpha(\mu) = \theta + 2\alpha(\mu - \mu^+) \tag{15}$$

is the Fréchet derivative of the cost functional. Here $\theta(t)$ is the solution of (13).

4. Necessary Condition for the Optimal Solution and a Minimizing Sequence

We construct a minimizing sequence based on the gradient methods. According to the gradient method, a minimizer for the cost functional is chosen by the formula

$$\mu^{(j+1)} = \mu^{(j)} - \beta_j J'_\alpha\big(\mu^{(j)}\big), \quad j = 0,1,2,..., \tag{16}$$

where $\mu^{(0)} \in M_{ad}$ is a given initial element and $J'_\alpha\big(\mu^{(j)}\big)$ is the Fréchet derivative corresponding to $\mu^{(j)}$. The β_j is called the relaxation parameter. From the definition of Fréchet differentiability, we can obtain that

$$J_\alpha\big(\mu^{(j+1)}\big) - J_\alpha\big(\mu^{(j)}\big) = \beta_j\Big[-\|J'_\alpha\big(\mu^{(j)}\big)\|^2 + \frac{o(\beta_j)}{\beta_j}\Big] < 0$$

for sufficiently small $\beta_j > 0$ [13]. The choice of the relaxation parameter defines various gradient methods and this choice is very important.

To stop the iteration process, one of the following stopping criterion can be selected:

$$\|\mu^{(j+1)} - \mu^{(j)}\| < \epsilon_1, \quad \|J_\alpha\big(\mu^{(j+1)}\big) - J_\alpha\big(\mu^{(j)}\big)\| < \epsilon_2, \quad \|J'_\alpha\big(\mu^{(j)}\big)\| < \epsilon_3. \tag{17}$$

Now, we can state the optimality condition in view of [28]. Let $\mu^* \in M_{ad}$ be the optimal solution to the problem (1)-(2) and let us denote the solution of the adjoint problem corresponding to the optimal solution μ^* with $\varphi^*(t,x)$. We know that the cost functional $J_\alpha(\mu)$ is a continuously differentiable in the control set M_{ad}. In this case, the following inequality is provided for all $\mu \in M_{ad}$ [28]:

$$\langle J'_\alpha(\mu^*), \mu - \mu^*\rangle_{H^1(0,T)} \geq 0. \tag{18}$$

The following variational inequality states the necessary condition for the optimal solution:

$$\langle \theta^* + 2\alpha(\mu^* - \mu^+), \mu - \mu^*\rangle_{H^1(0,T)} \geq 0 \tag{19}$$

for all $\mu \in M_{ad}$, where $\theta^*(t)$ is the solution of the problem (13) corresponding to $\varphi^*(t,0)$.

5. Conclusions

In this study, we focus on investigating the optimality conditions in the optimal control problem governed by the parabolic system and obtaining a minimizer for the chosen cost functional. We prove that the boundary condition $w_x(t,0) = \mu(t)$ in the parabolic problem can be controlled from target $w(x,t) = \nu(x,t)$ using H^1-norm. The admissible control set is chosen as a bounded, convex, and closed subset of the space $H^1(0,T)$. Using Goebel Theorem, we prove that the optimal boundary control problem considered has a unique solution. Obtaining the explicit formula for the gradient of the cost functional allows the usage of the gradient method to construct a minimization sequence. Fréchet differentiable of the cost functional in the admissible controls set is proved and the explicit formula of this derivative is obtained by adjoint approach. The obtained results permit one to acquire the necessary optimality condition. This study provides some results for numerical research on obtaining the optimal solution.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Taha Koç]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%45).

Author [Yeşim Akbulut]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%35).

Author [Seher Aslancı]: Contributed to completing the research and solving the problem (%20).

Conflicts of Interest

The authors declare no conflict of interest.

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