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Killing Magnetic Curves in $\mathbb{H}^3$

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We consider magnetic curves corresponding to the Killing magnetic fields in hyperbolic 3-space.

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AMS Subject Classification (2020): *Primary: 53C30 ; Secondary: 53C22; 53Z05.*

1. Introduction

The purpose of this article is to study Killing magnetic curves in hyperbolic 3-space $\mathbb{H}^3$. More precisely, we deduce the differential equations of the magnetic trajectories derived from Killing vector fields on the hyperbolic 3-space. We will give some particular solutions to Killing magnetic trajectories.

Let us explain differential geometric background and motivation of the present study.

Electromagnetism and Killing vector fields over spacetimes have been motivating the development of Riemannian geometry of spacetimes.

Professor Krishan Lal Duggal (1929–2022) contributed geometric study of electromagnetic equation on spacetimes and relation to Killing vector fields [7, 8, 9, 10, 11].

Magnetic curves represent trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field.

A *magnetic field* F on a Riemannian manifold (M, g) of arbitrary dimension is a closed 2-form which is associated to a $(1,1)$-tensor field Φ on M, called the *Lorentz force*. Magnetic field and Lorentz force are related by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M.

A curve $\gamma(t)$ is called a *magnetic curve* (also called a magnetic trajectory) if it satisfies the *Lorentz equation*:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \Phi(\dot{\gamma}), \tag{1.1}$$

where ∇ is the Levi-Civita connection of g. One can see that magnetic curves have constant speed.

Obviously, Lorentz equation for a magnetic curve is a generalization of the geodesic equation. In fact if $\Phi = 0$, i.e., $F = 0$ the differential equation (1.1) reduces to the geodesic equation.

As is well known, geodesic equation is a Hamiltonian system on the tangent bundle of a Riemannian manifold whose Hamiltonian is the kinetic energy. Arnol'd [2, 3], Anosov and Sinaĭ [1] observed that Lorentz equation is a Hamiltonian system on the tangent bundle with perturbed symplectic form. The Hamiltonian is still the kinetic energy. Motivated by this observation, the study of trajectories of magnetic fields on Riemannian manifolds of arbitrary dimension has grown up an active and attractive area of mathematics as well as mathematical physics. See e.g., [17].

Now let us turn our attention to 3-dimensional Riemannian geometry.

Assume that (M^3, g) is oriented by a volume element dv_g. Then the *cross product* of two vector fields X, $Y \in \mathfrak{X}(M)$ is defined as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \mathfrak{X}(M). \tag{1.2}$$

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Moreover, two-forms are identified with vector fields. Indeed, for every vector field V, one can associate a two form $F = F_V$ defined $F_V = \iota_V dv_g$, where ι denotes the inner product.

The closing condition of F corresponds to the *Gauss's law* $\operatorname{div} V = 0$ for the corresponding vector field V of F. The Lorentz equation (1.1) is reformulated as

$$\nabla_{\dot{\gamma}}\dot{\gamma} = V \times \dot{\gamma}. \tag{1.3}$$

Recall that the vector field V on M is a Killing vector field if its local flows are isometries. Killing vector fields are characterized as vector fields satisfying the *Killing equation*:

$$g(\nabla_Y V, Z) + g(Y, \nabla_Z V) = 0, \quad \forall\, Y, Z \in \mathfrak{X}(M). \tag{1.4}$$

Since Killing vector fields are divergence free, they define an important class of magnetic fields which are called *Killing magnetic fields*. The trajectories corresponding to the Killing magnetic fields are called *Killing magnetic curves*.

Killing magnetic curves in 3-dimensional homogenous Riemannian spaces, especially model spaces of Thurston geometry [27, 28] have been studied extensively in this decade. We refer the reader to the following articles: Euclidean 3-space $\mathbb{E}^3$ [6], the 3-sphere [18, 19], Heisenberg group Nil_3 [4, 5], the special linear group $\mathrm{SL}_2\mathbb{R}$ [5, 12, 16, 20], the space Sol_3 [13], product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ [24, 25]. For more information we refer to a survey article [21].

As far as the authors know, Killing magnetic curves in the hyperbolic 3-space $\mathbb{H}^3$ is not well studied yet. In this article we study Killing magnetic curves in the hyperbolic 3-space $\mathbb{H}^3$.

This paper is organized as follows. In Section 2 we describe the Lie algebra of Killing vector fields of the hyperbolic 3-space $\mathbb{H}^3$ in detail. We start our investigation of magnetic curves in $\mathbb{H}^3$ in Section 3. First we classify homogeneous magnetic curves in $\mathbb{H}^3$. In Section 4 we study Killing magnetic curves in $\mathbb{H}^3$. We deduce the Lorentz equations for the basic Killing vector fields of $\mathbb{H}^3$. We exhibit some particular solutions to those Lorentz equations.

2. Killing vector fields on the hyperbolic 3-space

2.1. *The hyperbolic 3-space and its Lie group structure*

Let us consider the upper half-space model

$$\mathbb{H}^3 = (\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}, g), \quad g = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

of the hyperbolic 3-space of constant curvature -1.

To describe the Lie algebra $i(\mathbb{H}^3)$ of Killing vector fields, here we represent $\mathbb{H}^3$ as a homogeneous Riemannian space.

As is well known, hyperbolic 3-space $\mathbb{H}^3$ is represented by $\mathbb{H}^3 = \mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2)$ as a Riemannian symmetric space. The upper half-space model of the hyperbolic 3-space is identified with the solvable part

$$\left\{ \begin{pmatrix} \sqrt{z} & (x+yi)/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \middle|\ x, y, z \in \mathbb{R},\ z > 0 \right\}$$

in the Iwasawa decomposition $\mathrm{SL}_2\mathbb{C} = \mathbb{H}^3 \cdot \mathrm{SU}(2)$.

The multiplication law is given explicitly by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + z_1 x_2, y_1 + z_1 y_2, z_1 z_2).$$

The identity element (origin) is $(0, 0, 1)$.

Remark 2.1. Take an element $A \in \mathrm{SL}_2\mathbb{C}$. We decompose A along the Iwasawa decomposition as

$$A = A_S A_U, \quad A_S \in \mathbb{H}^3,\ A_U \in \mathrm{SU}(2).$$

By using the Iwasawa decomposition, $\mathrm{SL}_2\mathbb{C}$ acts isometrically and transitively on $\mathbb{H}^3$ by

$$A \cdot X = (AX)_S, \quad A \in \mathrm{SL}_2\mathbb{C},\ X \in \mathbb{H}^3.$$

When $A \in \mathbb{H}^3$, the action of A is just a left translation.

The Lie algebra $\mathfrak{h}^3$ of $\mathbb{H}^3$ is spanned by the basis

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The commutation relations are

$$[E_1, E_2] = 0, \quad [E_2, E_3] = -E_2, \quad [E_3, E_1] = E_1.$$

The exponential map $\exp : \mathfrak{h}^3 \to \mathbb{H}^3$ is given by

$$\exp(xE_1) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \exp(yE_2) = \begin{pmatrix} 1 & yi \\ 0 & 1 \end{pmatrix},$$
$$\exp((\ln z)E_3) = \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix}.$$

Remark 2.2. The Lie algebra

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} u_3 i & -u_2 + u_1 i \\ u_2 + u_1 i & -u_3 i \end{pmatrix} \middle| \, u_1, u_2, u_3 \in \mathbb{R} \right\}$$

of $\mathrm{SU}(2)$ is spanned by the basis

$$\vec{i} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \vec{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \vec{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is decomposed as $\mathfrak{sl}_2\mathbb{C} = \mathfrak{su}(2) \oplus \mathfrak{h}^3$. Along this decomposition, every $X = (x_{ij}) \in \mathfrak{sl}_2\mathbb{C}$ is decomposed as

$$X = \begin{pmatrix} i \,\mathrm{Im}\, x_{11} & -\overline{x_{21}} \\ x_{21} & -i \,\mathrm{Im}\, x_{11} \end{pmatrix} + \begin{pmatrix} \mathrm{Re}\, x_{11} & x_{12} + \overline{x_{21}} \\ 0 & -\mathrm{Re}\, x_{11} \end{pmatrix}.$$

2.2. *The Levi-Civita connection*

Translating E_1, E_2 and E_3, we obtain the following left invariant vector fields.

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

The Levi-Civita connection is given by

$$\begin{array}{lll} \nabla_{e_1} e_1 = e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array} \tag{2.2}$$

From this table one can confirm that the left invariant metric is of constant curvature -1. Here we remark that e_1 and e_2 are divergence free, but $\mathrm{div}\, e_3 = -2 \neq 0$.

Let us recall the symmetric bilinear map U introduced by Nomizu (see [22]):

$$2\langle \mathsf{U}(X, Y), Z\rangle = \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle, \quad X, Y, Z \in \mathfrak{h}^3.$$

This bilinear map is given by

$$\mathsf{U}(E_1, E_1) = \mathsf{U}(E_2, E_2) = E_3, \quad \mathsf{U}(E_1, E_2) = \mathsf{U}(E_3, E_3) = 0,$$

$$\mathsf{U}(E_1, E_3) = -\frac{1}{2}E_1, \quad \mathsf{U}(E_2, E_3) = -\frac{1}{2}E_2.$$

Thus for any vector $X = X_1E_1 + X_2E_2 + X_3E_3 \in \mathfrak{h}^3$, we have

$$\mathsf{U}(X, X) = -X_3(X_1E_1 + X_2E_2) + (X_1^2 + X_2^2)E_3. \tag{2.3}$$

2.3. The basic Killing vector fields

The isometry group $\mathrm{Iso}(\mathbb{H}^3)$ of $\mathbb{H}^3$ is 6-dimensional. Let us denote by $\mathfrak{iso}(\mathbb{H}^3)$ the Lie algebra of $\mathrm{Iso}(\mathbb{H}^3)$. Then the Lie algebra $i(\mathbb{H}^3)$ of all Killing vector fields is anti-isometric to the Lie algebra $\mathfrak{iso}(\mathbb{H}^3)$ (see [26, 33. Proposition]) and generated by the following basic Killing vector fields:

$$\begin{aligned}
\xi_1 =& \frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial y}, \quad \xi_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\\
\xi_4 =& -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\\
\xi_5 =& \frac{1}{2}(x^2 - y^2 - z^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z},\\
\xi_6 =& xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}.
\end{aligned}$$

The isometric flow $\mathrm{Exp}(t\xi_1)$ of ξ_1 is a left translation by $(t, 0, 1)$, i.e.,

$$\mathrm{Exp}(t\xi_1)(x, y, z) = (t, 0, 1) * (x, y, z) = (x + t, y, z).$$

Analogously we have

$$\mathrm{Exp}(t\xi_2)(x, y, z) = (0, t, 1) * (x, y, z) = (x, y + t, z).$$

The isometric flow $\mathrm{Exp}(t\xi_3)$ of ξ_3 is given by

$$\mathrm{Exp}(t\xi_3)(x, y, z) = (0, 0, e^t) * (x, y, z) = (e^t x, e^t y, e^t z).$$

The isometric flow $\mathrm{Exp}(t\xi_4)$ of ξ_4 is a rotation i.e.,

$$\mathrm{Exp}(t\xi_4)(x, y, z) = (\cos t\, x - \sin t\, y, \sin t\, x + \cos t\, y, z).$$

Let v be a left invariant vector field on $\mathbb{H}^3$ and set $V = v|_{(0,0,1)} \in \mathfrak{h}^3$. Then v is complete and has a global flow ([26, p. 256, 34. Lemma],[14, Proposition 2]):

$$\mathrm{Exp}(tv)(x, y, z) = (x, y, z) * \exp(tV).$$

Hence we have

$$\begin{aligned}
\mathrm{Exp}(te_1)(x, y, z) =& (x, y, z) * (t, 0, 1) = (x + tz, y, z),\\
\mathrm{Exp}(te_2)(x, y, z) =& (x, y, z) * (0, t, 1) = (x, y + tz, z),\\
\mathrm{Exp}(te_3)(x, y, z) =& (x, y, z) * (0, 0, e^t) = (x, y, e^t z).
\end{aligned}$$

On the other hand a tangent vector $X \in \mathfrak{h}^3$ induces a Killing vector field $X^\sharp$ on $\mathbb{H}^3$ defined by

$$X^\sharp_{(x,y,z)} = \left.\frac{d}{dt}\right|_{t=0} (\exp(tX)(x, y, z))_{(x,y,z)}.$$

One can see that

$$E_1^\sharp = \xi_1, \quad E_2^\sharp = \xi_2, \quad E_3^\sharp = \xi_3.$$

The remaining basic Killing vector fields ξ_4, ξ_5 and ξ_6 are derived from the Lie algebra $\mathfrak{su}(2)$. Namely there exist the vector fields E_4, E_5 and $E_6 \in \mathfrak{su}(2)$ so that

$$E_4^\sharp = \xi_4, \quad E_5^\sharp = \xi_5, \quad E_6^\sharp = \xi_6.$$

3. Magnetic curves and Frenet equation in $\mathbb{H}^3$

Let us consider an arc length parameterized curve $\gamma(t) = (x(t), y(t), z(t))$ in $\mathbb{H}^3$. Then the unit tangent vector field is given by

$$T(t) := \dot{\gamma}(t) = \dot{x}(t)\frac{\partial}{\partial x} + \dot{y}(t)\frac{\partial}{\partial y} + \dot{z}(t)\frac{\partial}{\partial z} = \frac{\dot{x}(t)}{z(t)}e_1 + \frac{\dot{y}(t)}{z(t)}e_2 + \frac{\dot{z}(t)}{z(t)}e_3.$$

The unit speed condition is

$$\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = z(t)^2. \tag{3.1}$$

Next, using (2.2), we compute the acceleration vector field as

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \left(\frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2}\right)\cdot e_1 + \left(\frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2}\right)\cdot e_2 + \left(\frac{\ddot{z}}{z} + \frac{(\dot{x})^2 + (\dot{y})^2 - (\dot{z})^2}{z^2}\right)\cdot e_3. \tag{3.2}$$

As we mentioned in Introduction, a curve $\gamma(t)$ in $\mathbb{H}^3$ is said to be a *magnetic curve* with respect to a magnetic field V if it satisfies the Lorentz equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = V \times \dot{\gamma}.$$

The Lorentz equation implies that a magnetic curve has constant speed. Indeed,

$$\frac{d}{dt}g(\dot{\gamma}, \dot{\gamma}) = 2g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = 2g(V \times \dot{\gamma}, \dot{\gamma}) = 0.$$

Obviously, a magnetic curve reduces to a geodesic if the magnetic field is absent. In hyperbolic 3-space, even if magnetic field V is non-trivial, magnetic curves may be geodesics. This phenomena happens when $V = 0$ along a geodesic γ (This phenomena does not occur for Kähler magnetic curves, or more generally for J-trajectories [14, 15]).

Example 3.1 (Vertical geodesic)**.** Let $\gamma(t) = \exp(tE_3) = (0, 0, e^t)$ be the flow of e_3 starting at the origin $(0, 0, 1)$. Then γ is a geodesic. On the other hand, the radial Killing vector field ξ_3 vanishes along γ. Thus γ satisfies the Lorentz equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \xi_3 \times \dot{\gamma}$.

3.1. The Frenet equation

Let $\gamma(t) = (x(t), y(t), z(t))$ be a unit speed curve in $\mathbb{H}^3$. Then the *unit normal vector field* $N(t)$ and the *curvature* $\kappa(t)$ are defined by

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa N.$$

Next we can define the *binormal vector field* B by $B = T \times N$. Then the orthonormal frame field $F(t) = (T(t), N(t), B(t))$ along $\gamma(t)$ is called the *Frenet frame field* and satisfies the following Frenet formula:

$$\begin{aligned} \nabla_{\dot{\gamma}}T &= \kappa N, \\ \nabla_{\dot{\gamma}}N &= -\kappa T + \tau B, \\ \nabla_{\dot{\gamma}}B &= -\tau N, \end{aligned} \tag{3.3}$$

where the function τ is called the *torsion*. One can see that if γ has vanishing torsion, then there exists a totally geodesic surface Σ which contains whole γ. Geodesic are regarded as curves with vanishing curvature.

Now let us assume that γ is an arc length parameterized magnetic curve satisfying the Lorentz equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = qW \times \dot{\gamma}.$$

Here $V = qW$ is a magnetic field given as a product of non-zero constant q (called the *charge*) and a divergence free vector field W.

Then its curvature κ is computed as

$$\kappa(t)^2 = q^2 g(W \times \dot{\gamma}, W \times \dot{\gamma}) = q^2(g(W, W) - g(W, \dot{\gamma})^2).$$

Let us denote by θ the angle function of V and $\dot{\gamma}$, then

$$g_{\gamma(t)}(W, \dot{\gamma}(t)) = \|W_{\gamma(t)}\| \cos\theta(t). \tag{3.4}$$

Hence

$$\kappa(t)^2 = q^2\, g_{\gamma(t)}(W, W)\sin^2\theta(t). \tag{3.5}$$

Now let us assume that V is a *Killing magnetic field*, then we have the following *conservation law*

Proposition 3.1. *Let γ be a unit speed Killing magnetic curve in $\mathbb{H}^3$ under the influence of Killing magnetic field V. Then $g(V,\dot\gamma)$ is constant along γ.*

Proof. Since V is Killing, we have

$$\begin{aligned}\frac{d}{dt}g(V,\dot\gamma) =&g(\nabla_{\dot\gamma}V,\dot\gamma)+g(V,\nabla_{\dot\gamma}\dot\gamma)=g(V,\nabla_{\dot\gamma}\dot\gamma)\\ =&g(V,V\times\dot\gamma)=0.\end{aligned}$$

□

Here we introduce the following notion.

Definition 3.1. A Killing magnetic curve γ is said to be *slant* if it makes a constant angle with the Killing magnetic field V.

3.2. Homogeneous magnetic curves

In this subsection we exhibited some simple examples of magnetic curves in $\mathbb{H}^3$. Motivated by homogeneous geodesics [23], we study homogeneous magnetic curves in $\mathbb{H}^3$.

Definition 3.2. A curve $\gamma(t)$ of the form $\gamma(t)=\exp(tX)$ for some non-zero vector X of $\mathbb{H}^3$ at the origin is called a *homogeneous curve* in $\mathbb{H}^3$.

Precisely speaking, the notion of homogeneous curve depends on the coset space representation of a homogeneous space. In this section we regard $\mathbb{H}^3$ as a coset space $\mathbb{H}^3/\{(0,0,1)\}$.

Let us fix a *left invariant* divergence free vector field W on the hyperbolic 3-space $\mathbb{H}^3$. Take a unit vector

$$X=X_1E_1+X_2E_2+X_3E_3\in\mathfrak{h}^3,\quad X_1^2+X_2^2+X_3^2=1.$$

Then the homogeneous curve $\gamma(t)=\exp(tX)$ is a magnetic curve under the influence of $V=qW$ if and only if

$$\mathsf{U}(X,X)=qW|_{(0,0,1)}\times X.$$

First we observe homogeneous geodesics, i.e., $q=0$. In this case $\gamma(t)$ is a homogenous geodesic when and only when $X=\pm E_3$. Hence the only homogeneous geodesics of the solvable Lie group model of $\mathbb{H}^3$ are the flows of the left invariant vector field e_3 parameterized as

$$\gamma(t)=(0,0,e^{\pm t}).$$

Example 3.2 ($W=e_1$)**.** Let us choose $W=e_1$ which is a left invariant magnetic field, but *not* Killing. Since $E_1\times X=-X_3E_2+X_2E_3$, the magnetic equation is the system

$$\begin{cases}X_3X_1=0,\\ X_3X_2=qX_3,\\ X_1^2+X_2^2=qX_2.\end{cases}$$

The possible solutions are

$$\begin{cases}X=qE_2\pm\sqrt{1-q^2}E_3, & 0<|q|\leq 1.\\ X=\pm\frac{\sqrt{q^2-1}}{q}E_1+\frac{1}{q}E_2, & |q|\geq 1.\end{cases}$$

The former curve is

$$x(t)=0,\quad y(t)=\frac{\pm 2qe^{\pm\frac{\sqrt{1-q^2}\,t}{2}}}{\sqrt{1-q^2}}\sinh\frac{\sqrt{1-q^2}\,t}{2},\quad z(t)=e^{\pm\sqrt{1-q^2}t}. \tag{3.6}$$

This magnetic curve lies in the totally geodesic hyperbolic plane $x=0$. Thus, the torsion vanishes. Direct computation shows that

$$\nabla_{\dot\gamma}\dot\gamma=-\delta q\sqrt{1-q^2}e_2+q^2e_3,\quad \delta=\pm 1.$$

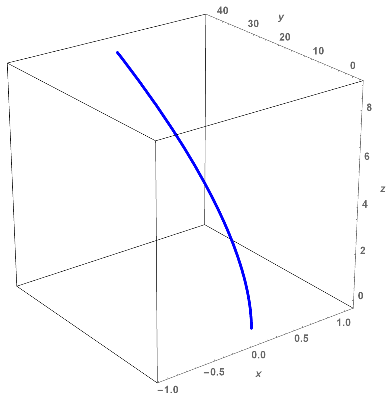

Figure 1. Homogeneous magnetic curve in plane $x = 0$

Hence, the curvature is $|q|$. The unit normal is given by

$$N = \frac{1}{|q|}\left(-\delta q\sqrt{1-q^2}e_2 + q^2 e_3\right).$$

The binormal vector field is e_1 and hence $\tau = 0$. This curve has constant angle $\theta = \pi/2$ with e_1. One can see that γ is a parabola $z = \{(1-q^2)/(4q^2)\}y^2$.
Fig.1 presents homogeneous magnetic curves (3.6) for $q = \frac{1}{2}$.

The latter curve is parameterized as

$$x(t) = \frac{\pm t\sqrt{q^2-1}}{q}, \quad y(t) = \frac{t}{q}, \quad z(t) = 1.$$

Thus the magnetic curve lies in the horosphere $z = 1$ and it is expressed as

$$x = \pm\sqrt{q^2-1}\, y.$$

The magnetic curve has constant curvature 1 and vanishing torsion.
When $q = 1$, the magnetic curve $\gamma(t) = \exp(tE_2)$ is an open Riemannian circle

$$x(t) = 0, \quad y(t) = t, \quad z(t) = 1 \tag{3.7}$$

of curvature 1 which lies in the totally geodesic hyperbolic plane $x = 0$ (Thus $\tau = 0$). The angle function is a constant $\pi/2$. This curve also lies in the horosphere $z = 1$.

Proposition 3.2. *Some of homogeneous magnetic curves in $\mathbb{H}^3$ are*

(a) curves in totally geodesic plane $x = 0$ given by

$$x(t) = 0, \quad y(t) = \frac{\pm 2qe^{\pm\frac{\sqrt{1-q^2}\,t}{2}}}{\sqrt{1-q^2}}\sinh\frac{\sqrt{1-q^2}\,t}{2}, \quad z(t) = e^{\pm\sqrt{1-q^2}t}, \qquad \forall q \in \langle -1, 1\rangle,$$

(b) curves in the horosphere $z = 1$ given by

$$x(t) = \frac{\pm t\sqrt{q^2-1}}{q}, \quad y(t) = \frac{t}{q}, \quad z(t) = 1, \qquad \forall q \in \mathbb{R}\setminus\langle -1, 1\rangle.$$

4. Killing magnetic curves in $\mathbb{H}^3$

Let us start our investigation on Killing magnetic curves. Since $\mathbb{H}^3$ is homogeneous, we may assume that Killing magnetic curves satisfy the initial conditions

$$x(0) = y(0) = 0, \quad z(0) = 1, \quad \dot{x}(0) = u, \quad \dot{y}(0) = v, \quad \dot{z}(0) = w \tag{4.1}$$

with $u^2 + v^2 + w^2 = 1$.

4.1. Killing magnetic curves corresponding to the Killing vector field ξ_1

First we investigate unit speed Killing magnetic curves with respect to ξ_1.
The Killing vector field $\xi_1 = \partial_x = \frac{1}{z}e_1$ has *non-constant* length $\|\xi_1\| = 1/z$.

Theorem 4.1. *The Killing magnetic curves in $\mathbb{H}^3$, corresponding to the Killing vector field $\xi_1 = \partial_x$ are solutions of the following system of differential equations*

$$\begin{aligned} \frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} &= 0, \\ \frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} &= -\frac{\dot{z}}{z^2}, \\ \frac{\ddot{z}}{z} + \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{z^2} &= \frac{\dot{y}}{z^2}. \end{aligned} \tag{4.2}$$

In particular, the only magnetic curve with constant angle function θ is the homogeneous curve $\exp(tE_2)$ with parametrization (3.7) for $\cos\theta = 0$.

Proof. Using formula (1.2) we have

$$\xi_1 \times \dot{\gamma} = -\frac{\dot{z}}{z^2}\, e_2 + \frac{\dot{y}}{z^2}\, e_3. \tag{4.3}$$

From the magnetic curve equation (1.3), using (3.2) and (4.3), we obtain the system of ordinary differential equations (4.2). Next, we solve the obtained system (4.2) under the initial condition (4.1).

Using separation of variables, the first equation gives

$$x(t) = u \int_0^t z(\tau)^2 d\tau.$$

Analogously, from the second equation of (4.2) we have

$$y(t) = \frac{t}{2} + \frac{(2v-1)}{2} \int_0^t z(\tau)^2 d\tau.$$

Substituting these in the third equation of (4.2), we have

$$z\ddot{z} - (\dot{z})^2 + \left(u^2 + \frac{(2v-1)^2}{4}\right) z^4 - \frac{1}{4} = 0.$$

Solutions of this equation are given by elliptic integrals. However, we will find later exact solution in some particular case.

On the other hand, the conserved quantity (3.4) given by

$$c := g(\xi_1, \dot{\gamma}) = \frac{\dot{x}(t)}{z(t)^2} = \frac{1}{z(t)} \cos\theta(t)$$

is constant along γ. From the initial condition (4.1), we have $c = u$. Hence,

$$\cos\theta(t) = uz(t), \quad \dot{x}(t) = u\, z(t)^2.$$

Thus, the curvature of γ is computed by the formula (3.5) as

$$\kappa(t)^2 = \frac{q^2 \sin^2\theta(t)}{z(t)^2} = \frac{q^2}{z(t)^2}(1 - u^2 z(t)^2) \geq 0.$$

Hence, if $u \neq 0$, we deduce the restriction

$$|z(t)| \leq \frac{1}{|u|}.$$

One can see that $u = 0$ if and only if $\cos\theta = 0$. In such a case, we have $\dot{x} = 0$ and hence $x(t) = 0$. Thus $\gamma(t)$ lies in the totally geodesic hyperbolic plane $\{(0, y, z) \in \mathbb{H}^3\}$.

In case $u \neq 0$, we obtain

$$T = \dot{\gamma} = \cos\theta(t)e_1 + \frac{\dot{y}(t)}{z(t)}e_2 - (\tan\theta)\dot{\theta}e_3.$$

This formula implies that γ is slant (with $\theta \neq \pm\pi/2$) if and only if z is constant.

Here we find slant particular solution of (4.2). If we assume that $u \neq 0$ and θ is a constant, then from the initial condition (4.1) we have $z = 1$. Hence, the system reduces to

$$\ddot{x}(t) = \ddot{y}(t) = 0, \quad \dot{x}(t)^2 + \dot{y}(t)^2 = \dot{y}(t).$$

Thus, both x and y coordinate functions are linear, i.e. $x(t) = ut$ and $y(t) = vt$. From the initial condition, we have $u = 0$ and $v = 0$. □

Remark 4.1. The other procedure to obtain the right hand side of the Lorentz equation, i.e. magnetic field, can be found in, e.g. [13].

4.2. *Killing magnetic curves corresponding to the Killing vector field ξ_2*

Theorem 4.2. *The Killing magnetic curves in $\mathbb{H}^3$, corresponding to the Killing vector field $\xi_2 = \partial_y$ are solutions of the following system of differential equations*

$$\begin{aligned} &\frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} = \frac{\dot{z}}{z^2}, \\ &\frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} = 0, \\ &\frac{\ddot{z}}{z} + \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{z^2} = -\frac{\dot{x}}{z^2}. \end{aligned} \tag{4.4}$$

In particular, the only slant Killing magnetic curve is a homogeneous magnetic curve $\exp(-tE_1)$ *parameterized as*

$$x(t) = -t, \quad y(t) = 0, \quad z(t) = 1,$$

Proof. Using formula (1.2) we have

$$\xi_2 \times \dot{\gamma} = \frac{\dot{z}}{z^2}\, e_1 - \frac{\dot{x}}{z^2}\, e_3. \tag{4.5}$$

Using (3.2) and (4.5), we obtain the system of ordinary differential equations (4.4). Analogously to the previous case, we find only a slant particular solution of (4.4). □

At the end of this subsection we show figures which present the Killing magnetic curves obtained as solutions of (4.2) and (4.4) using numerical integration in `Wolfram Mathematica` for $x(0) = y(0) = 0$, $z(0) = 1$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = \frac{\sqrt{3}}{3}$ and $t \in [-10, 10]$. Note that the shape of the obtained curves looks similar to a circular helix. Moreover, if initial velocities are $\dot{x}(0) = \dot{z}(0) = 0$, $\dot{y}(0) = 1$ and $\dot{y}(0) = \dot{z}(0) = 0$, $\dot{x}(0) = -1$, respectively, then "helices" degenerate to the homogeneous magnetic curves (Riemannian circles) mentioned before.

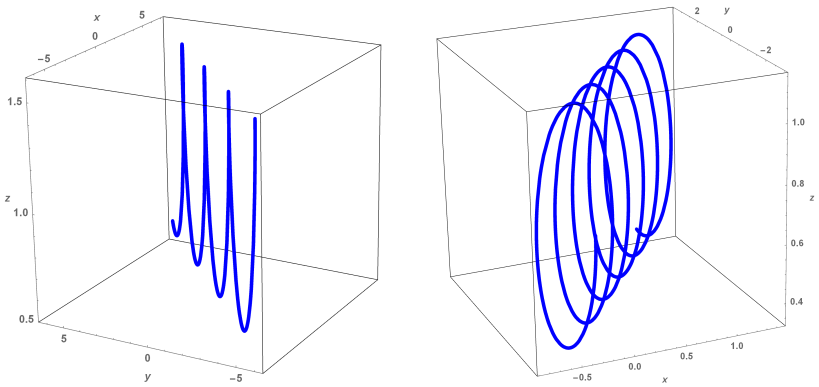

Figure 2. Killing magnetic curves corresponding to Killing vector fields ξ_1 and ξ_2 - numerical solution

4.3. Killing magnetic curves corresponding to the radial Killing vector field ξ_3

The radial Killing vector field $\xi_3 = x\partial_x + y\partial_y + z\partial_z = \frac{x}{z}e_1 + \frac{y}{z}e_2 + e_3$ has non-constant length

$$\|\xi_3\| = \sqrt{x^2 + y^2 + z^2}/z.$$

Theorem 4.3. *The Killing magnetic curves in $\mathbb{H}^3$ corresponding to the radial Killing vector field $\xi_3 = x\partial_x + y\partial_y + z\partial_z$ are solutions of the following system of differential equations*

$$\begin{aligned} \frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} &= \frac{y\dot{z}}{z^2} - \frac{\dot{y}}{z}, \\ \frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} &= -\frac{x\dot{z}}{z^2} + \frac{\dot{x}}{z}, \\ \frac{\ddot{z}}{z} + \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{z^2} &= \frac{x\dot{y} - y\dot{x}}{z^2}. \end{aligned} \tag{4.6}$$

In particular, some analytical solutions of the system (4.6) are

- *a vertical geodesic $\gamma(t) = (0, 0, e^t)$,*
- *a circle $\gamma(t) = (v\cos t + u\sin t, v\sin t - u\cos t, 1)$ with $u^2 + v^2 = 1$ which lies in the horosphere $z = 1$.*

Proof. By using formula (1.2) we have

$$\xi_3 \times \dot{\gamma} = \left(\frac{y\dot{z}}{z^2} - \frac{\dot{y}}{z}\right)e_1 + \left(-\frac{x\dot{z}}{z^2} + \frac{\dot{x}}{z}\right)e_2 + \left(\frac{x\dot{y} - y\dot{x}}{z^2}\right)e_3. \tag{4.7}$$

The conserved quantity is

$$c = g(\xi_3, \dot{\gamma}) = \frac{1}{z}\left\{\dot{x}\left(\frac{y\dot{z}}{z^2} - \frac{\dot{y}}{z}\right) + \dot{y}\left(-\frac{x\dot{z}}{z^2} + \frac{\dot{x}}{z}\right) + \dot{z}\left(\frac{x\dot{y} - y\dot{x}}{z^2}\right)\right\} = 0.$$

Taking into account (3.4), $\cos\theta = 0$ along the magnetic curve. Thus, the velocity vector field is always orthogonal to ξ_3. It should be remarked that along the vertical geodesic $\gamma(t) = \exp(tE_3)$, $\xi_3|_{\gamma(t)} = e_3|_{\gamma(t)}$ holds. Hence, the vertical geodesic is a solution of the Lorentz equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \xi_3 \times \dot{\gamma}$. Hereafter, we consider Killing magnetic curves other than vertical geodesic.

From Lorentz equation (1.3), using (3.2) and (4.7), we obtain the system of ordinary differential equations (4.6).

To find exact solutions of this system is a true challenge. Here we exhibit a particular solution. If we assume $z = \text{const} = 1$, then from the first and the second equation of (4.6) become $\ddot{x} = -\dot{y}$ and $\ddot{y} = \dot{x}$. The solution of this system is given by $x(t) = v\cos t + u\sin t$ and $y(t) = v\sin t - u\cos t$. $\square$

Figure 3 presents the Killing magnetic curves obtained as a solution of (4.6) using numerical integration in `Wolfram Mathematica` for $x(0) = y(0) = 0$, $z(0) = 1$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = \frac{\sqrt{3}}{3}$ and $t \in [-5, 5]$. Particularly, if initial velocities are $\dot{x}(0) = \dot{y}(0) = 0$, $\dot{z}(0) = 1$, then the curve degenerates to the vertical geodesic.

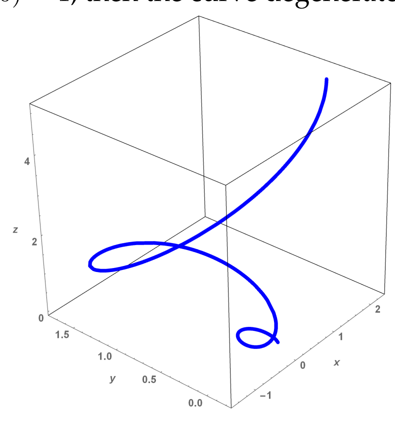

Figure 3. Killing magnetic curve corresponding to Killing vector field ξ_3 - numerical solution

4.4. Killing magnetic curves corresponding to the rotational Killing vector field ξ_4

The Killing vector field $\xi_4 = -y\partial_x + x\partial_y = -\frac{y}{z}e_1 + \frac{x}{z}e_2$ has the length $\|\xi_4\| = \sqrt{x^2+y^2}/z$.

Theorem 4.4. *The Killing magnetic curves in $\mathbb{H}^3$ away from z-axis corresponding to the rotational Killing vector field $\xi_4 = -y\partial_x + x\partial_y$ are solutions of the following system of differential equations*

$$\begin{aligned} &\frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} = \frac{x\dot{z}}{z^2}, \\ &\frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} = \frac{y\dot{z}}{z^2}, \\ &\frac{\ddot{z}}{z} + \frac{\dot{x}^2+\dot{y}^2-\dot{z}^2}{z^2} = -\frac{x\dot{x}+y\dot{y}}{z^2}. \end{aligned} \tag{4.8}$$

In particular, an analytical solution of the system (4.8) *is a Riemannian circle*

$$\gamma(t) = (e^{-t}\cos\phi_0, e^{-t}\sin\phi_0, e^{-t}).$$

Proof. It should be remarked that ξ_4 vanishes along the vertical geodesic (Example 3.1). Thus vertical geodesic $\gamma(t) = \exp(tE_3)$ satisfies the Lorentz equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \xi_4 \times \dot{\gamma}$. We are interested in Killing magnetic curves other than vertical geodesic. Hereafter we consider magnetic equation in the region $\mathbb{H}^3 \setminus \{(0,0,z) \in \mathbb{H}^3\}$.

Using formula (1.2) we have

$$\xi_4 \times \dot{\gamma} = \frac{x\dot{z}}{z^2}\,e_1 + \frac{y\dot{z}}{z^2}\,e_2 - \frac{x\dot{x}+y\dot{y}}{z^2}\,e_3. \tag{4.9}$$

On the other hand, the conserved quantity is computed as

$$c = g(\xi_4, \dot{\gamma}) = \frac{x\dot{y}-\dot{x}y}{z^2} = \frac{\sqrt{x^2+y^2}}{z}\cos\theta.$$

Thus $c = 0$ if and only if x-coordinate and y-coordinate have constant ratio.

From the Lorentz equation (1.3), using (3.2) and (4.9), we obtain the system of differential equations (4.8). The first equation can be written as homogeneous ODE

$$\ddot{x} - 2\frac{\dot{z}}{z}\dot{x} - \frac{\dot{z}}{z}x = 0. \tag{4.10}$$

A general solution of this equation is given by an elliptic integral.

In some particular cases we can find explicit solutions. We choose the initial condition

$$x(0) = y(0) = z(0) = 1, \quad \dot{x}(0) = u,\ \ \dot{y}(0) = v,\ \ \dot{z}(0) = w,\ \ u^2+v^2+w^2 = 1.$$

Firstly, we assume that $z = \mathrm{const}$, i.e., $z = 1$. From the first and the second equation of (4.8) we have $x(t) = ut+1$ and $y(t) = vt+1$. Note that in this case $c = 0$. Substituting these in the third equation we obtain $u = v = 0$, which leads to the contradiction.

Next, for $\frac{\dot{z}}{z} = w = \mathrm{const} \neq 0$, i.e. $z(t) = e^{wt}$, we can solve (4.10). In physics this equation is usually called *equation of damped vibrations*. The solution is given by

$$x(t) = e^{wt}\left(\frac{w-u}{\sqrt{w+w^2}}\sinh(\sqrt{w+w^2}\,t) + \cosh(\sqrt{w+w^2}\,t)\right).$$

Quite analogously, from the second equation of (4.8) we have

$$y(t) = e^{wt}\left(\frac{w-v}{\sqrt{w+w^2}}\sinh(\sqrt{w+w^2}\,t) + \cosh(\sqrt{w+w^2}\,t)\right).$$

Substituting these expressions in the third equation, we have a contradiction.

Let us introduce a cylindrical coordinate system (ρ, ϕ, z) in $\mathbb{H}^3$:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

Using this substitution, the system (4.8) can be rewritten as

$$\begin{aligned}
&\ddot{\rho} - 2\dot{\rho}\frac{\dot{z}}{z} - \rho\left(\dot{\phi}^2 + \frac{\dot{z}}{z}\right) = 0,\\
&\ddot{\phi} + 2\dot{\phi}\left(\frac{\dot{\rho}}{\rho} - \frac{\dot{z}}{z}\right) = 0, \\
&\frac{z\ddot{z} - \dot{z}^2}{z^2} + \frac{1}{z^2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \rho\dot{\rho}\right) = 0.
\end{aligned} \tag{4.11}$$

We obtain a particular solution in cylindrical coordinates $(\rho, \phi, z) = (e^{-t}, \phi_0, e^{-t})$ under the assumption $\phi = \text{const.}$, i.e. in (x, y, z)-coordinates, the curve $\gamma(t) = (e^{-t}\cos\phi_0, e^{-t}\sin\phi_0, e^{-t})$. The Killing magnetic curve $\gamma(t)$ has constant speed $\sqrt{2}$. One can check that $\gamma(t)$ has constant curvature $1/\sqrt{2}$ and vanishing torsion. Moreover $\gamma(t)$ lies in the surface

$$M(\phi_0) = \{(x, y, z) \in \mathbb{H}^3 \mid \sin\phi_0\, x - \cos\phi_0\, y = 0\}.$$

□

Remark 4.2. Killing magnetic curves in Euclidean 3-space with respect to the rotational Killing vector field $\xi = -y\partial_x + x\partial_y$ are investigated in [6].

Figure 4 presents the Killing magnetic curves obtained as a solution of (4.8) using numerical integration in `Wolfram Mathematica` for $x(0) = y(0) = z(0) = 1$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = \frac{\sqrt{3}}{3}$ and $t \in [-5, 5]$.

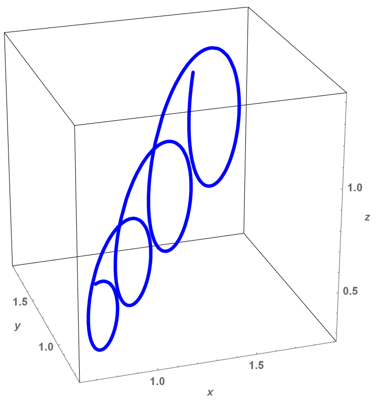

Figure 4. Killing magnetic curve corresponding to Killing vector field ξ_4 - numerical solution

4.5. Killing magnetic curves corresponding to Killing vector field ξ_5

The Killing vector field

$$\xi_5 = \frac{1}{2}(x^2 - y^2 - z^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} = \frac{1}{2z}(x^2 - y^2 - z^2)e_1 + \frac{xy}{z}e_2 + xe_3$$

has length

$$\|\xi_5\| = \sqrt{\frac{(x^2 - y^2 - z^2)^2}{4z^2} + \frac{x^2y^2}{z^2} + x^2} = \frac{x^2 + y^2 + z^2}{2z}.$$

Theorem 4.5. *The Killing magnetic curves in $\mathbb{H}^3$ corresponding to the Killing vector field $\xi_5 = \frac{1}{2}(x^2 - y^2 - z^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$ are solutions of the following system of differential equations*

$$\begin{aligned}
&\frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} = \frac{xy\dot{z}}{z^2} - \frac{x\dot{y}}{z},\\
&\frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} = \frac{x\dot{x}}{z} - \frac{\dot{z}(x^2 - y^2 - z^2)}{2z^2}, \\
&\frac{\ddot{z}}{z} + \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{z^2} = -\frac{xy\dot{x}}{z^2} + \frac{\dot{y}(x^2 - y^2 - z^2)}{2z^2}.
\end{aligned} \tag{4.12}$$

Proof. Using formula (1.2) we have

$$\xi_5 \times \dot{\gamma} = \left(\frac{xy\dot{z}}{z^2} - \frac{x\dot{y}}{z}\right) e_1 + \left(\frac{x\dot{x}}{z} - \frac{\dot{z}(x^2 - y^2 - z^2)}{2z^2}\right) e_2 + \left(-\frac{xy\dot{x}}{z^2} + \frac{\dot{y}(x^2 - y^2 - z^2)}{2z^2}\right) e_3. \tag{4.13}$$

From the magnetic curve equation (1.3), using (3.2) and (4.13), we obtain the system of ordinary differential equations (4.12).

If we assume $z = \text{const} = 1$, then from the first and the second equation of (4.12) we have $\ddot{x}\dot{x} + \ddot{y}\dot{y} = 0$. The solution of this system is given by $x(t) = v\cos t + u\sin t$ and $y(t) = v\sin t - u\cos t$, but substituting these expressions in the third equation, we have identity only for $u = v = 0$. □

4.6. Killing magnetic curves corresponding to Killing vector field ξ_6

The Killing vector field

$$\xi_6 = xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} = \frac{xy}{z}e_1 + \frac{1}{2z}(y^2 - x^2 - z^2)e_2 + ye_3$$

has length

$$\|\xi_6\| = \frac{x^2 + y^2 + z^2}{2z}.$$

Theorem 4.6. *The Killing magnetic curves in $\mathbb{H}^3$ corresponding to the Killing vector field $\xi_6 = xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$ are solutions of the following system of differential equations*

$$\begin{aligned}
&\frac{\ddot{x}}{z} - \frac{2\dot{x}\dot{z}}{z^2} = -\frac{y\dot{y}}{z} - \frac{\dot{z}(x^2 - y^2 + z^2)}{2z^2}, \\
&\frac{\ddot{y}}{z} - \frac{2\dot{y}\dot{z}}{z^2} = -\frac{xy\dot{z}}{z^2} + \frac{y\dot{x}}{z}, \\
&\frac{\ddot{z}}{z} + \frac{\dot{x}^2 + \dot{y}^2 - \dot{z}^2}{z^2} = \frac{xy\dot{y}}{z^2} + \frac{\dot{x}(x^2 - y^2 + z^2)}{2z^2}.
\end{aligned} \tag{4.14}$$

Proof. Using formula (1.2) we have

$$\xi_6 \times \dot{\gamma} = \left(-\frac{y\dot{y}}{z} - \frac{\dot{z}(x^2 - y^2 + z^2)}{2z^2}\right) e_1 + \left(\frac{y\dot{x}}{z} - \frac{xy\dot{z}}{z^2}\right) e_2 + \left(\frac{xy\dot{y}}{z^2} + \frac{\dot{x}(x^2 - y^2 + z^2)}{2z^2}\right) e_3. \tag{4.15}$$

From the magnetic curve equation (1.3), using (3.2) and (4.15), we obtain the system of ordinary differential equations (4.14). □

At the end of this subsection we show figures which present the Killing magnetic curves obtained as solutions of (4.12) and (4.14) using numerical integration in `Wolfram Mathematica` for $x(0) = y(0) = 0$, $z(0) = 1$, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = \frac{\sqrt{3}}{3}$ and $t \in [-5, 5]$.

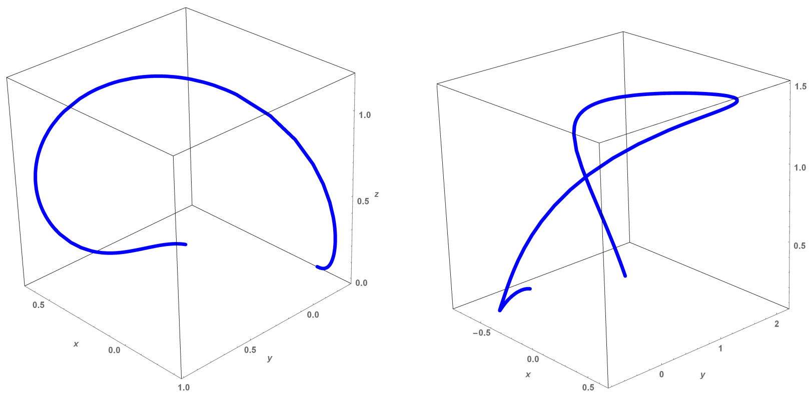

Figure 5. Killing magnetic curves corresponding to Killing vector fields ξ_5 and ξ_6 - numerical solution

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Author's contributions

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