Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 15
DOI : 10.15672/hujms.xx

Research Article

High order monotonicity of a ratio of the modified Bessel function with applications

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Abstract

Let K_ν be the modified Bessel functions of the second kind of order ν and $Q_\nu(x) = xK_{\nu-1}(x)/K_\nu(x)$. In this paper, we proved that $Q_\nu'''(x) < (>)\, 0$ for $x > 0$ if $|\nu| > (<)\, 1/2$, which gives an affirmative answer to a guess. As applications, some monotonicity and concavity or convexity results as well functional inequalities involving $Q_\nu(x)$ are established. Moreover, several high order monotonicity of $x^k Q_\nu^{(n)}(x)$ on $(0,\infty)$ for certain integers k and n are given.

Mathematics Subject Classification (2020). 33C10, 26A51, 26A48, 39B62

Keywords. modified Bessel function of the second kind, high order monotonicity, complete monotonicity, functional inequality

1. Introduction

Let K_ν be the modified Bessel functions of the second kind of order ν(see [11]). The ratio

$$T_\nu(x) = \frac{K_\nu(x)}{xK_{\nu+1}(x)}$$

appeared in solving Schrödinger's equation with a rectangular potential well and related problems (see [16]). Kelker [6] and Ismail and Kelker [9] found that the Student t-distribution is infinitely divisible if and only if the ratio $T_{n-1/2}(\sqrt{x})$ for $n \in \mathbb{N}$ is completely monotonic on $(0,\infty)$. Whereafter, Ismail and Kelker [9] conjectured that $T_\nu(\sqrt{x})$ is completely monotonic on $(0,\infty)$ for all $\nu \geq -1$, which was solved by Grosswald [5]. As a corollary, Grosswald [5] presented an integral representation of $T_\nu(x)$:

$$T_{\nu-1}(x) = \frac{K_{\nu-1}(x)}{xK_\nu(x)} = \frac{4}{\pi^2}\int_0^\infty \frac{dt}{t(x^2+t^2)(J_\nu^2(t)+Y_\nu^2(t))} \tag{1.1}$$

for $\nu \geq 0$ and $x > 0$. One year later, using the representation theorem and inversion formula for Stieltjes transforms, Ismail [7] gave a simple proof of (1.1) and showed that (1.1) holds for complex x. More information including integral representations and complete

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Received: 30.01.2023; Accepted: 31.12.2023

monotonicity involving the ratio $T_\nu(x)$ can be found in [8, Theorems 5.1 and 5.3], [10, Lemma 2.6].

Another ratio $Q_\nu(x)$ defined by

$$Q_\nu(x) = \frac{xK_{\nu-1}(x)}{K_\nu(x)} = x^2 T_{\nu-1}(x) \tag{1.2}$$

appeared in physics [17, (4.25)] and probability [2] since the relation

$$\frac{1}{x} Q'_\nu(x) = \frac{K_{\nu-1}(x) K_{\nu+1}(x)}{K_\nu(x)^2} - 1.$$

By (1.1), the ratio $Q_\nu(x)$ has the following integral representation:

$$Q_\nu(x) = \frac{xK_{\nu-1}(x)}{K_\nu(x)} = \frac{4}{\pi^2}\int_0^\infty \frac{x^2}{x^2+t^2} \frac{dt}{t\left(J_\nu^2(t) + Y_\nu^2(t)\right)}, \tag{1.3}$$

and obeys the Riccati equation

$$xQ'_\nu(x) = Q_\nu(x)^2 + 2\nu Q_\nu(x) - x^2. \tag{1.4}$$

Some good properties of $Q_\nu(x)$ can be seen in [17, (4.25)], [20], [27].

Recently, Yang and Tian established the notion of incompletely monotonic functions as follows.

Definition 1.1 ([24, Definition 2])**.** Let the function F have derivatives of all orders on an interval I and assume that there exits an integer $n \geq 0$ such that

$$(-1)^n F^{(n)}(x) \geq 0 \text{ for } x \in I.$$

Then the function F is said to be incompletely monotonic on I, and the maximum such n is called the order of the incompletely monotonic function $F(x)$, denoted by

$$O_{\text{incm}}[F(I)] = \sup_{n\geq 0}\left\{n : (-1)^n F^{(n)}(x) \geq 0 \text{ for } x \in I\right\}.$$

In particular, if $O_{\text{incm}}[F(I)] = \infty$, then the function f is completely monotonic on I.

It was shown in [22, Property 7] that$x \mapsto Q_\nu(\sqrt{x})$ is a Bernstein function of x on $(0,\infty)$ for $\nu \geq 0$ by using (1.3). A problem naturally arises from this: *is $Q_\nu(x)$ still a Bernstein function of x? or equivalently, is $Q'_\nu(x)$ a completely monotonic function of x?* The authors [22] pointed out that the answer is negative since the fourth derivative of $Q_{5/2}(x)$ changes sign on $(0,\infty)$ (the details can be seen in Section 5 in this paper). From this we see that $Q'_\nu(x)$ for $|\nu| \geq 0$ is an incompletely monotonic function of x on $(0,\infty)$.

The authors [22, Conjecture 1] further guessed that

$$(-1)^n Q_\nu^{(n)}(x) > (<)\, 0 \text{ for all } x > 0 \text{ and } n = 2,3 \text{ if } |\nu| > (<)\, 1/2, \tag{1.5}$$

and proved this guess is valid for $n = 2$ in a complicated way. Then

$$Q_\nu(x) > 0, Q'_\nu(x) > 0, Q''_\nu(x) > 0 \text{ for all } x \in (0,\infty) \text{ if } |\nu| > 1/2,$$
$$Q_\nu(x) > 0, Q'_\nu(x) > 0, Q''_\nu(x) < 0 \text{ for all } x \in (0,\infty) \text{ if } |\nu| < 1/2,$$

which mean that $Q''_\nu(x)$ for $|\nu| > 1/2$ and $Q'_\nu(x)$ for $|\nu| < 1/2$ are 0-th and 1-th order incompletely monotonic functions, respectively. If the guess (1.5) is valid for $n = 3$, then since $Q_{5/2}^{(4)}(x)$ changes sign on $(0,\infty)$, we have

$$O_{\text{incm}}\left[Q''_\nu(0,\infty)\right] = 1 \text{ for } |\nu| > 1/2, \tag{1.6}$$

that is, the order of the incompletely monotonic function $Q''_\nu(x)$ for $|\nu| > 1/2$ is 1 (the strict proof is given in Section 5). If the guess (1.5) is valid for $n = 3$, then we have

$$Q''_\nu(x) > (<) \lim_{x\to\infty} Q''_\nu(x) = 0$$

for $x>0$ if $|\nu|>(<)\,1/2$, which will give a simple proof of Theorem 1 in [22]. Moreover, the validity of the guess (1.5) for $n=3$ can yield more monotonicity and convexity or concavity results related to the ratio $Q_\nu(x)$.

On that account, the aim of this paper is to verify the guess (1.5) for $n=3$. More precisely, we will prove the following theorem.

Theorem 1.2. *For $\nu\in\mathbb{R}$, let $Q_\nu(x)$ be defined by (1.2). Then function $Q_\nu'''(x)<(>)\,0$ for all $x>0$ if $|\nu|>(<)\,1/2$.*

From the recurrence formulas (see [18, p. 79])

$$xK_\nu'(x)-\nu K_\nu(x)=-xK_{\nu+1}(x),$$

$$xK_\nu'(x)+\nu K_\nu(x)=-xK_{\nu-1}(x),$$

we have

$$q_\nu(x)=\frac{xK_\nu'(x)}{K_\nu(x)}=-\frac{xK_{\nu-1}(x)}{K_\nu(x)}-\nu=-\frac{xK_{\nu+1}(x)}{K_\nu(x)}+\nu. \tag{1.7}$$

Then Theorem 1.2 is equivalent to the following

Theorem 1.3. *Let $\nu\in\mathbb{R}$. Then $[xK_\nu'(x)/K_\nu(x)]'''>(<)\,0$ for $x>0$ if $|\nu|>(<)\,1/2$.*

Since $K_{-\nu}(x)=K_\nu(x)$, by the relation (1.7) we have

$$q_{-\nu}(x)=q_\nu(x)\quad\text{and}\quad Q_{-\nu}(x)=Q_\nu(x)+2\nu$$

for $\nu\geq 0$, and therefore, $Q_\nu^{(n)}(x)=Q_{|\nu|}^{(n)}(x)$ for $n\in\mathbb{N}$. For this reason, we always assume that $\nu\geq 0$ in what follows.

The rest of this paper is organized as follows. Theorem 1.2 is proved in Section 2. As applications of Theorem 1.2, some monotonicity and concavity or convexity results involving $Q_\nu(x)$ are listed in Section 3. In Section 4, several functional inequalities involving $Q_\nu(x)$ are established. In Section 5, the high order monotonicity of the functions $Q_\nu^{(n)}(x)/x^k$ on $(0,\infty)$ for certain suitable integers k and n are found. In the last section, we give an answer to a guess and propose a problem on the sign of $[xI_\nu(x)/I_{\nu+1}(x)]'''$ for $x>0$ and $\nu>-1/2$, where I_ν is the modified Bessel functions of the first kind of order ν.

2. Proof of Theorem 1.2

Our tools used in this paper contain the integral representation (1.3) and the following two lemmas.

Lemma 2.1 ([18, p. 446])**.** *The function*

$$t\mapsto\Phi_\nu(t)=\frac{1}{t\left[J_\nu^2(t)+Y_\nu^2(t)\right]}$$

is increasing on $(0,\infty)$ if $\nu\geq 1/2$ and is decreasing on $(0,\infty)$ if $0\leq\nu<1/2$.

Lemma 2.2. *Suppose that*

(i) the functions f,g are continuous on $[a,b]$ $(a<b)$;

(ii) there is a $t_0\in(a,b)$ such that $f(t)<0$ for $t\in(a,t_0)$ and $f(t)>0$ for $t\in(a,t_0)$;

(iii) $g(t)$ is non-negative and increasing (decreasing) on $[a,b]$;

(iv) $\int_a^b f(t)\,dt\geq(\leq)\,0$.

Then

$$\int_a^b f(t)\,g(t)\,dt\geq(\leq)\,g(t_0)\int_a^b f(t)\,dt\geq(\leq)\,0.$$

Proof. Note that

$$\int_a^b f(t)[g(t)-g(t_0)]\,dt = \int_a^{t_0} f(t)[g(t)-g(t_0)]\,dt + \int_{t_0}^b f(t)[g(t)-g(t_0)]\,dt.$$

By the conditions (ii) and (iii), we see that

$$\begin{aligned} f(t)[g(t)-g(t_0)] &\geq (\leq)\,0 \text{ for } t\in[a,t_0], \\ f(t)[g(t)-g(t_0)] &\geq (\leq)\,0 \text{ for } t\in[t_0,b], \end{aligned}$$

which in combination with (iv) gives the desired result. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By (1.3), $Q_\nu(x)$ can be represented as

$$Q_\nu(x) = \frac{4}{\pi^2}\int_0^\infty \frac{x^2}{x^2+t^2}\Phi_\nu(t)\,dt.$$

Differentiation yields

$$Q_\nu'(x) = \frac{8}{\pi^2}\int_0^\infty \frac{xt^2}{(t^2+x^2)^2}\Phi_\nu(t)\,dt, \tag{2.1}$$

$$Q_\nu''(x) = \frac{8}{\pi^2}\int_0^\infty \frac{t^2(t^2-3x^2)}{(t^2+x^2)^3}\Phi_\nu(t)\,dt, \tag{2.2}$$

$$Q_\nu'''(x) = -\frac{96}{\pi^2}x\int_0^\infty \frac{t^2(t^2-x^2)}{(x^2+t^2)^4}\Phi_\nu(t)\,dt := -\frac{96}{\pi^2}x\int_0^\infty f_0(t)\Phi_\nu(t)\,dt,$$

where

$$f_0(t) = \frac{t^2(t^2-x^2)}{(x^2+t^2)^4}.$$

Clearly, $f_0(t)<0$ for $t\in(0,x)$ and $f_0(t)>0$ for $t\in(x,\infty)$; by Lemma 2.1, $\Phi_\nu(t)$ is increasing (decreasing) on $(0,\infty)$ if $\nu>(<1/2)$; and by a direct computation,

$$\int_0^\infty f_1(t)\,dt = \int_0^\infty \frac{t^2(t^2-x^2)}{(x^2+t^2)^4}dt = \left[-\frac{1}{3}\frac{t^3}{(t^2+x^2)^3}\right]_{t\to0}^{t\to\infty} = 0.$$

It thus follows from Lemma 2.2 that

$$\int_0^\infty f_0(t)\Phi_\nu(t)\,dt \geq (\leq)\,\Phi_\nu(x)\int_0^\infty f_0(t)\,dt = 0$$

if $\nu>(<1/2)$, which implies that $Q_\nu'''(x)\leq(\geq)\,0$ for $x>0$ if $\nu>(<1/2)$, and the proof is done. □

3. Monotonicity and concavity (convexity) results

Before we show the monotonicity and convexity results involving $Q_\nu(x)$, we introduce the asymptotic behavior of $Q_\nu^{(n)}(x)$ for $n=0,1,2$. Using the asymptotic formulas [1, p. 375 and p. 378]

$$\begin{aligned} K_\nu(x) &\sim 2^{\nu-1}\Gamma(\nu)\,x^{-\nu} \text{ for } \nu>0 \quad\text{and}\quad K_0(x)\sim -\ln x \quad \text{as } x\to0, \\ K_\nu(x) &\sim \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{4\nu^2-1}{8x}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!\,(8x)^2}\right), \quad \text{as } x\to\infty, \end{aligned}$$

the asymptotic behavior of $Q_\nu^{(n)}(x)$ for $n=0,1,2$ can be described as follows:

$$Q_\nu(x) \sim \begin{cases} \dfrac{1}{2(\nu-1)}x^2 & \text{if } \nu>1, \\ -x^2\ln x & \text{if } \nu=1, \\ 2^{1-2\nu}\dfrac{\Gamma(1-\nu)}{\Gamma(\nu)}x^{2\nu} & \text{if } \nu\in(0,1), \\ -\dfrac{1}{\ln x} & \text{if } \nu=0, \end{cases} \quad \text{as } x\to 0,$$

$$Q_\nu(x) \sim x-\nu+\frac{1}{2}+\frac{\nu^2-1/4}{2x}, \quad \text{as } x\to\infty.$$

$$Q'_\nu(x) \sim \begin{cases} \dfrac{1}{\nu-1}x & \text{if } \nu>1, \\ -x-2x\ln x & \text{if } \nu=1, \\ \dfrac{\nu 2^{2-2\nu}\Gamma(1-\nu)}{\Gamma(\nu)}x^{2\nu-1} & \text{if } \nu\in(0,1), \\ \dfrac{1}{x\ln^2 x} & \text{if } \nu=0, \end{cases} \quad \text{as } x\to 0,$$

$$Q'_\nu(x) \sim 1-\frac{\nu^2-1/4}{2x^2}, \quad \text{as } x\to\infty.$$

$$Q''_\nu(x) \sim \begin{cases} \dfrac{1}{\nu-1} & \text{if } \nu>1, \\ -2\ln x-3 & \text{if } \nu=1, \\ 2^{2-2\nu}\frac{\nu(2\nu-1)\Gamma(1-\nu)}{\Gamma(\nu)}x^{2\nu-2} & \text{if } \nu\in(0,1), \\ -\dfrac{\ln x+2}{x^2\ln^3 x} & \text{if } \nu=0, \end{cases} \quad \text{as } x\to 0,$$

$$Q''_\nu(x) \sim \frac{\nu^2-1/4}{4x^3}, \quad \text{as } x\to\infty.$$

It then follows that

$$G_\nu(x) = xQ'_\nu(x)-2Q_\nu(x) \to \begin{cases} 0 & \text{as } x\to 0, \\ -\infty & \text{as } x\to\infty. \end{cases} \tag{3.1}$$

$$G'_\nu(x) = xQ''_\nu(x)-Q'_\nu(x) \to \begin{cases} 0 & \text{if } \nu>\frac{1}{2} & \text{as } x\to 0, \\ -\infty & \text{if } \nu<\frac{1}{2} & \text{as } x\to 0, \\ -1 & & \text{as } x\to\infty. \end{cases} \tag{3.2}$$

$$\begin{aligned} h_\nu(x) &= xG'_\nu(x)-G_\nu(x) && (3.3)\\ &= x^2Q''_\nu(x)-2xQ'_\nu(x)+2Q_\nu(x) \to \begin{cases} 0 & \text{as } x\to 0, \\ 1-2\nu & \text{as } x\to\infty. \end{cases} && (3.4) \end{aligned}$$

We first prove the log-concavity of the function $Q_\nu(x)$.

Corollary 3.1. *Let $\nu\geq 0$. The function $Q_\nu(x)$ is log-concave on $(0,\infty)$.*

Proof. Differentiation yields

$$\begin{aligned} Q_\nu^2(x)\left[\ln Q_\nu(x)\right]'' &= Q_\nu(x)Q''_\nu(x)-Q'_\nu(x)^2 := \phi_\nu(x), \\ \phi'_\nu(x) &= Q_\nu(x)Q'''_\nu(x)-Q'_\nu(x)Q''_\nu(x). \end{aligned}$$

If $\nu>(<)\,1/2$, then $Q''_\nu(x)>(<)\,0$, $Q'''_\nu(x)<(>)\,0$, which together with $Q_\nu(x), Q'_\nu(x)>0$ yields that $\phi'_\nu(x)<(>)\,0$. It follows that, for $x>0$,

$$\begin{aligned} \phi_\nu(x) &< \lim_{x\to 0}\left[Q_\nu(x)Q''_\nu(x)-Q'_\nu(x)^2\right]=0 \text{ if } \nu>1/2, \\ \phi_\nu(x) &< \lim_{x\to\infty}\left[Q_\nu(x)Q''_\nu(x)-Q'_\nu(x)^2\right]=-1<0 \text{ if } \nu<1/2. \end{aligned}$$

This shows that $[\ln Q_\nu(x)]'' < 0$ for $x > 0$ if $\nu \ge 0$ with $\nu \ne 1/2$. Since $Q_{1/2}(x) = x$ is clearly log-concave on $(0,\infty)$, $Q_\nu(x)$ is log-concave on $(0,\infty)$ for all $\nu \ge 0$. This completes the proof. □

Remark 3.2. Corollary 3.1 is not true for $\nu < 0$. A counter example is

$$Q_{-3/2}(x) = \frac{x^2+3x+3}{x+1},$$

which satisfies

$$\left[\ln Q_{-3/2}(x)\right]'' = -\frac{x^4+4x^3+2x^2-6x-6}{\left(x^3+4x^2+6x+3\right)^2}.$$

The numerator of the above fraction is an NP-type polynomial, which has a unique zero. For the NP-type polynomials and its sign rule can be seen [25, p. 126], [26, Sec. 2], [21].

Yang and Tian [22, Theorem 2] proved that the function $G_\nu(x) = xQ'_\nu(x) - 2Q_\nu(x)$ is strictly decreasing on $(0,\infty)$ for $\nu \ge 0$. By Theorem 1.2 we further find $G''_\nu(x) = xQ'''_\nu(x) < (>)\, 0$ if $0 \le \nu > (<)\, 1/2$.

Corollary 3.3. *The function $G_\nu(x) = xQ'_\nu(x) - 2Q_\nu(x)$ is concave (convex) on $(0,\infty)$ if $0 \le \nu > (<)\, 1/2$.*

Remark 3.4. Since $G''_\nu(x) < (>)\, 0$ for $x > 0$ if $\nu > (<)\, 1/2$, by (3.2) we have that, for $x > 0$,

$$\begin{aligned} G'_\nu(x) &< \lim_{x\to 0} G'_\nu(x) = 0 \ \text{ if } \nu > \frac{1}{2}, \\ G'_\nu(x) &< \lim_{x\to \infty} G'_\nu(x) = -1 < 0 \ \text{ if } \nu < \frac{1}{2}. \end{aligned}$$

In view of $Q_{1/2}(x) = x$, we see that $G_{1/2}(x) = -x$, which is decreasing. This offers another proof of the decreasing property of $G_\nu(x)$ on $(0,\infty)$ for $\nu \ge 0$.

Corollary 3.5. *If $\nu > 1/2$, then $x \mapsto [Q'_\nu(x) - p]/x$ is increasing (decreasing) on $(0,\infty)$ if and only if $p \ge 1$ ($p \le 0$). If $0 \le \nu < 1/2$, then it is decreasing on $(0,\infty)$ if and only if $p \le 1$.*

Proof. Differentiation yields

$$\left[\frac{Q'_\nu(x) - p}{x}\right]' = \frac{xQ''_\nu(x) - Q'_\nu(x) + p}{x^2} = \frac{G'_\nu(x) + p}{x^2},$$

where $G_\nu(x)$ is defined in (3.1). Since $G''_\nu(x) = xQ'''_\nu(x) < (>)\, 0$ for $x > 0$ if $|\nu| > (<)\, 1/2$, the function in question is increasing (decreasing) if and only if

$$p \ge \sup_{x>0}\{-G'_\nu(x)\} = -\min\{G'_\nu(0), G'_\nu(\infty)\}$$

or

$$p \le \inf_{x>0}\{-G'_\nu(x)\} = -\max\{G'_\nu(0), G'_\nu(\infty)\}.$$

By (3.2) the required assertion follows. □

Let $h_\nu(x)$ be defined by (3.3). Differentiation yields

$$h'_\nu(x) = \left[x^2 Q''_\nu(x) - 2xQ'_\nu(x) + 2Q_\nu(x)\right]' = x^2 Q'''_\nu(x).$$

By (3.4) and Theorem 1.2 we have the following corollary.

Corollary 3.6. *Let $\nu \ge 0$. If $\nu > (<)\, 1/2$, then the function $h_\nu(x)$ defined by (3.3) is decreasing (increasing) from $(0,\infty)$ onto (α,β), where*

$$\alpha = \min\{0, 1-2\nu\} \quad \text{and} \quad \beta = \max\{0, 1-2\nu\}.$$

The following corollary is a consequence of Corollary 3.6, which gives an answer to [22, Conjecture 2].

Corollary 3.7. *Let* $\nu \geq 0$. *The function* $\xi_{p,\nu}(x) = [Q_\nu(x) - p]/x$ *is concave (or convex) on* $(0,\infty)$ *if and only if* $p \geq -\min\{0, \nu - 1/2\}$ *(or* $p \leq -\max\{0, \nu - 1/2\}$*).*

Proof. Differentiation yields

$$\begin{aligned} \xi'_{p,\nu}(x) &= \frac{xQ'_\nu(x) - Q_\nu(x) + p}{x^2}, \\ x^3\xi''_{p,\nu}(x) &= x^2Q''_\nu(x) - 2xQ'_\nu(x) + 2Q_\nu(x) - 2p := h_\nu(x) - 2p. \end{aligned}$$

By Corollary 3.6, $\xi''_{p,\nu}(x) \leq (\geq)\, 0$ for $x > 0$ if and only if

$$2p \geq \beta \quad \text{or} \quad 2p \leq \alpha,$$

that is,

$$p \geq -\min\{0, \nu - 1/2\} \quad \text{or} \quad p \leq -\max\{0, \nu - 1/2\}.$$

This completes the proof. □

Taking $p = 0$ in Corollary 3.7, the following corollary is immediate.

Corollary 3.8. *Let* $\nu \geq 0$. *The function* $\xi_{0,\nu}(x) = Q_\nu(x)/x$ *is strictly concave (convex) on* $(0,\infty)$ *if* $\nu > (<)\, 1/2$.

Corollary 3.9. *Let* $\nu \geq 0$ *and* $\eta_{p,\nu}(x) = x^p e^x K_\nu(x)$. *Then* $[\ln \eta_{p,\nu}(x)]''' \geq (\leq)\, 0$ *for* $x > 0$ *if and only if* $p \geq \max\{\nu, 1/2\}$ *(or* $p \leq \min\{\nu, 1/2\}$*).*

Proof. Differentiation yields

$$[\ln \eta_{p,\nu}(x)]' = \frac{p}{x} + 1 + \frac{K'_\nu(x)}{K_\nu(x)},$$

which, by using the relation (1.7), can be written as

$$[\ln \eta_{p,\nu}(x)]' = 1 + \frac{p - \nu - Q_\nu(x)}{x} = 1 - \xi_{p-\nu,\nu}(x).$$

By Theorem 1.3, we immediately deduce that $[\ln \eta_{p,\nu}(x)]''' \geq (\leq)\, 0$ for $x > 0$ if and only if

$$p - \nu \geq -\min\{0, \nu - 1/2\} \quad \text{or} \quad p - \nu \leq -\max\{0, \nu - 1/2\},$$

that is,

$$p \geq \max\{\nu, 1/2\} \quad \text{or} \quad p \leq \min\{\nu, 1/2\},$$

which completes the proof. □

4. Several functional inequalities for $Q_\nu(x)$

In this section, we present several functional inequalities involving $Q_\nu(x)$ using the monotonicity and concavity or convexity results given in previous section.

Differentiating for Riccati equation (1.4) we have

$$xQ''_\nu(x) + Q'_\nu(x) = 2Q_\nu(x)\,Q'(x) + 2\nu Q'_\nu(x) - 2x.$$

Applying Riccati equation (1.4) we obtain that

$$\begin{aligned} x^2Q''_\nu(x) &= 2Q_\nu(x)\,[xQ'(x)] + (2\nu - 1)\,[xQ'_\nu(x)] - 2x^2 \\ &= 2Q^3_\nu(x) + (6\nu - 1)\,Q^2_\nu(x) + \left(2\nu(2\nu - 1) - 2x^2\right)Q_\nu(x) - (2\nu + 1)\,x^2. \end{aligned}$$

Then

$$\begin{aligned} xG'_\nu(x) &= x^2Q''_\nu(x) - xQ'_\nu(x) \\ &= 2Q^3_\nu(x) + 2(3\nu - 1)\,Q^2_\nu(x) + 2\left(2\nu(\nu - 1) - x^2\right)Q_\nu(x) - 2\nu x^2, \end{aligned}$$

$$\begin{aligned} h_\nu(x) &= x^2 Q_\nu''(x) - 2xQ_\nu'(x) + 2Q_\nu(x) \\ &= 2Q_\nu^3(x) + 3(2\nu-1)Q_\nu^2(x) + 2\left((2\nu-1)(\nu-1) - x^2\right)Q_\nu(x) - (2\nu-1)x^2. \end{aligned}$$

Since $G_\nu''(x) = xQ_\nu'''(x)$, $h_\nu'(x) = x^2 Q_\nu'''(x)$, by Theorem 1.2 and those limit values of $G_\nu'(x)$ and $h_\nu(x)$ given in (3.2) and (3.3), we immediately get the following corollary.

Corollary 4.1. *Let $\nu \geq 0$. (i) If $\nu > 1/2$, then the double inequality*

$$-x < 2Q_\nu^3(x) + 2(3\nu-1)Q_\nu^2(x) + 2\left(2\nu(\nu-1) - x^2\right)Q_\nu(x) - 2\nu x^2 < 0,$$

holds for $x > 0$. If $\nu < 1/2$, then the left hand side inequality reverses.

(ii) If $\nu > (<)\, 1/2$, then the double inequality

$$\begin{aligned} 1-2\nu &< (>)\, 2Q_\nu^3(x) + 3(2\nu-1)Q_\nu^2(x) \\ &\quad +2\left((2\nu-1)(\nu-1) - x^2\right)Q_\nu(x) - (2\nu-1)x^2 < (>)\, 0 \end{aligned}$$

holds for $x > 0$.

Using the concavity or convexity of the function $G_\nu(x)$ on $(0,\infty)$ given in Corollary 3.1, we find that the function

$$x \mapsto g(x) = \frac{G_\nu(x) - G_\nu(0)}{x-0} = \frac{xQ_\nu'(x) - 2Q_\nu(x)}{x}$$

is decreasing (increasing) if $0 \leq \nu > (<)\, 1/2$. By (3.2) we see that

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} G_\nu'(x) = \begin{cases} 0 & \text{if } \nu > \frac{1}{2}, \\ -\infty & \text{if } \nu < \frac{1}{2}; \end{cases}$$

Also, as $x \to \infty$,

$$g(x) = \frac{xQ_\nu'(x) - 2Q_\nu(x)}{x} \sim \frac{x-2x}{x} = -1.$$

It follows that, for $x > 0$,

$$-1 < \frac{xQ_\nu'(x) - 2Q_\nu(x)}{x} < 0 \quad \text{if } \nu > 1/2, \tag{4.1}$$

$$-\infty < \frac{xQ_\nu'(x) - 2Q_\nu(x)}{x} < -1 \quad \text{if } \nu < 1/2. \tag{4.2}$$

By Riccati equation (1.4) we have

$$\begin{aligned} xQ_\nu'(x) - 2Q_\nu(x) &= Q_\nu(x)^2 + 2(\nu-1)Q_\nu(x) - x^2 \\ &= [Q_\nu(x) + \nu - 1]^2 - \left[x^2 + (\nu-1)^2\right]. \end{aligned}$$

Solving the inequalities (4.1) and (4.2), we can derive the following sharp bounds for $Q_\nu(x)$.

Corollary 4.2. *Let $\nu \geq 0$. (i) If $\nu > 1/2$, then the inequalities*

$$Q_\nu(x) < \sqrt{x^2 + (\nu-1)^2} - (\nu-1), \tag{4.3}$$

$$x^2 - x + (\nu-1)^2 < [Q_\nu(x) + \nu - 1]^2 \tag{4.4}$$

hold for $x > 0$. (ii) If $\nu < 1/2$, then the inequality

$$Q_\nu(x) < \sqrt{x^2 - x + (\nu-1)^2} - (\nu-1) \tag{4.5}$$

holds for $x > 0$.

Remark 4.3. The inequality (4.3) for $\nu \geq 0$ was due to [14, Eq. (75)]. Clearly, for $0 \leq \nu < 1/2$, the upper bound in (4.5) is better than in (4.3).

Remark 4.4. The inequality (4.4) implies that

$$\sqrt{x^2 - x + (\nu - 1)^2} - (\nu - 1) < Q_\nu (x)$$

for $x \geq 1$ and $\nu > 1/2$. This lower bound is weaker than in [22, (3.9)] (see also [4, (3.10)]).

Since $Q''_\nu (x) > (<) 0$ if $0 \leq \nu > (<) 1/2$ and $[\ln Q_\nu (x)]'' < 0$ for $x > 0$, we have the following corollary.

Corollary 4.5. *Let $\nu \geq 0$. The following inequalities hold for $x, y > 0$ with $x \neq y$:*

$$\sqrt{Q_\nu (x) Q_\nu (y)} < Q_\nu \left(\frac{x+y}{2} \right) < \frac{Q_\nu (x) + Q_\nu (y)}{2} \quad \text{if } \nu > \frac{1}{2},$$

$$\sqrt{Q_\nu (x) Q_\nu (y)} < \frac{Q_\nu (x) + Q_\nu (y)}{2} < Q_\nu \left(\frac{x+y}{2} \right) \quad \text{if } \nu < \frac{1}{2}.$$

A function $f : (a, \infty) \to \mathbb{R}$ is said to be superadditive if

$$f(x) + f(y) \leq f(x+y) \text{ for } x, y \in (a, \infty).$$

If $-f$ is superadditive, then f is called subadditive on (a, ∞) (see [13]). It is easy to see that every convex function $f : [0, \infty) \to \mathbb{R}$ satisfies a functional inequality

$$f(x) + f(y) \leq f(0) + f(x+y) \text{ for } x, y \in [0, \infty)$$

(see [12]). Now, according to Theorem 1.2, Corollaries 3.3 and 3.8, the functions $Q'_\nu (x)$, $G_\nu (x) = xQ'_\nu (x) - 2Q_\nu (x)$ and $Q_\nu (x) / x = K_{\nu-1} (x) / K_\nu (x)$ are concave on $(0, \infty)$ if $\nu > 1/2$ with

$$\lim_{x \to 0} Q'_\nu (x) = \lim_{x \to 0} G_\nu (x) = \lim_{x \to 0} \frac{Q_\nu (x)}{x} = 0.$$

Then we have

$$\begin{aligned} Q'_\nu (x) + Q'_\nu (y) &> Q'_\nu (x+y), \\ xQ'_\nu (x) - 2Q_\nu (x) + yQ'_\nu (y) - 2Q_\nu (y) &> (x+y) Q'_\nu (x+y) - 2Q_\nu (x+y), \\ \frac{Q_\nu (x)}{x} + \frac{Q_\nu (y)}{y} &> \frac{Q_\nu (x+y)}{x+y} \end{aligned}$$

for $x, y > 0$.

Corollary 4.6. *Let $\nu > 1/2$. The functions $Q'_\nu (x)$, $xQ'_\nu (x) - 2Q_\nu (x)$ and $Q_\nu (x) / x$ are subadditive on $(0, \infty)$.*

As shown in Corollary 3.9, the function $[\ln \eta_{p,\nu} (x)]' = [\ln (x^p e^x K_\nu (x))]'$ is convex (concave) on $(0, \infty)$ if and only if $p \geq \max \{\nu, 1/2\}$ $(p \leq \min \{\nu, 1/2\})$. Applying Hermite-Hadamard inequality yields that

$$\left[\ln \eta_{p,\nu} \left(\frac{x+y}{2} \right) \right]' < (>) \frac{\int_x^y [\ln \eta_{p,\nu} (t)]' \, dt}{y - x} < (>) \frac{[\ln \eta_{p,\nu} (x)]' + [\ln \eta_{p,\nu} (y)]'}{2}$$

for $x, y > 0$ with $x \neq y$ if $p \geq \max \{\nu, 1/2\}$ $(p \leq \min \{\nu, 1/2\})$. Then the following inequalities are immediate.

Corollary 4.7. *Let $\nu \geq 0$. If $p \geq \max \{\nu, 1/2\}$, then the double inequality*

$$\begin{aligned} \frac{p - \nu - Q_\nu ((x+y)/2)}{(x+y)/2} &< \frac{1}{x-y} \ln \left(\frac{x^p K_\nu (x)}{y^p K_\nu (y)} \right) \\ &< \frac{1}{2} \left[\frac{p - \nu - Q_\nu (x)}{x} + \frac{p - \nu - Q_\nu (y)}{y} \right] \end{aligned}$$

holds for $x, y > 0$ with $x \neq y$. It is reversed if $p \leq \min\{\nu, 1/2\}$. In particular, when $p = \nu$ we have

$$\frac{1}{2}\left[\frac{Q_\nu(x)}{x} + \frac{Q_\nu(y)}{y}\right] < (>) \frac{1}{y-x} \ln\left(\frac{x^\nu K_\nu(x)}{y^\nu K_\nu(y)}\right) < (>) \frac{2}{x+y} Q_\nu\left(\frac{x+y}{2}\right)$$

for $x, y > 0$ with $x \neq y$ and $\nu > (<) 1/2$.

5. Further results

We have proven that $Q_\nu(x)$ satisfies that $Q'_\nu(x) > 0$ and $(-1)^k Q_\nu^{(k)}(x) > (<) 0$ if $0 \leq \nu > (<) 1/2$ for $x > 0$ and $k = 2, 3$. In Section 1, we claim that (1.6) holds, which is needed a strict proof. First, as mentioned in Section 1, the fourth derivative of $Q_{5/2}(x)$ changes sign on $(0, \infty)$. In fact, when $\nu = 5/2$,

$$Q_{5/2}(x) = \frac{x^2(x+1)}{x^2+3x+3},$$

the fourth order derivative of which equals to

$$Q_{5/2}^{(4)}(x) = 72 \frac{x^5 + 10x^4 + 30x^3 + 30x^2 - 9}{(x^2+3x+3)^5}.$$

Clearly, the numerator of the above fraction is an NP-type polynomial, hence there is an $x_0 > 0$ such that $Q_{5/2}^{(4)}(x) < 0$ for $x \in (0, x_0)$ and $Q_{5/2}^{(4)}(x) > 0$ for $x \in (x_0, \infty)$. Second, we claim that $Q_\nu^{(n)}(x)$ for $|\nu| > 1/2$ and $n \geq 5$ also changes sign on $(0, \infty)$. If not, that is, $Q_\nu^{(n)}(x)$ does not changes sign on $(0, \infty)$, then $Q_\nu^{(n)}(x) > (<) 0$ for all $x \in (0, \infty)$. This yields

$$Q_\nu^{(n-1)}(x) < (>) \lim_{x\to\infty} Q_\nu^{(n-1)}(x) = 0$$

for all $x \in (0, \infty)$. In particular, $Q_\nu^{(4)}(x) < (>) 0$ for all $x > 0$ and $|\nu| > 1/2$, which yields a contradiction with that $Q_{5/2}^{(4)}(x)$ changes sign on $(0, \infty)$. This proves claim (1.6).

Remark 5.1. It should be pointed out that the claim (1.6) is valid only for $|\nu| > 1/2$, while $|\nu| < 1/2$, we guess that $Q_\nu^{(4)}(x)$ also changes sign on $(0, \infty)$. Then

$$O_{\text{incm}}\left[Q'_\nu(0, \infty)\right] = 2 \text{ for } |\nu| < 1/2.$$

Now, let us consider the high order monotonicity of the function $Q_\nu^{(n)}(x)/x^k$ for certain $k, n \in \mathbb{N}$. We have seen that $Q_\nu(x)$ is not a completely monotonic function and $Q'_\nu(x)$ is an incompletely monotonic function of x on $(0, \infty)$. However, the product of a completely monotonic function and a not completely monotonic function may be completely monotonic. In fact, we find that the functions $Q_\nu(x)/x^3$ and $Q'_\nu(x)/x^3$ are completely monotonic on $(0, \infty)$ for $\nu \in \mathbb{R}$. To prove this, the following lemma is needed.

Lemma 5.2. *For $t, x > 0$ and $k = 1, 2$, let*

$$f_k(x, t) = \frac{1}{x(x^2+t^2)^k}.$$

Then $f_1(x, t)$ is completely monotonic in x on $(0, \infty)$.

Proof. Note that

$$\frac{1}{x} = \int_0^\infty e^{-xu} du \quad \text{and} \quad \frac{x}{x^2+t^2} = \int_0^\infty \cos(tu) e^{-xu} du.$$

Then

$$f_1(x, t) = \frac{1}{x(x^2+t^2)} = \frac{1}{t^2}\left(\frac{1}{x} - \frac{x}{x^2+t^2}\right) = \frac{1}{t^2} \int_0^\infty (1 - \cos(tu)) e^{-xu} du,$$

which is clearly completely monotonic in x on $(0, \infty)$, and the proof is completed. □

Theorem 5.3. *Let $\nu \in \mathbb{R}$. The functions $x \mapsto Q_\nu(x)/x^3$ and $x \mapsto Q'_\nu(x)/x^3$ are completely monotonic on* $(0,\infty)$.

Proof. By the integral representation (1.3) we have

$$\begin{aligned}\frac{Q_\nu(x)}{x^3} &= \frac{4}{\pi^2}\int_0^\infty \frac{1}{x(x^2+t^2)}\Phi_\nu(t)\,dt = \frac{4}{\pi^2}\int_0^\infty f_1(x,t)\,\Phi_\nu(t)\,dt,\\ \frac{Q'_\nu(x)}{x^3} &= \frac{8}{\pi^2}\int_0^\infty \frac{t^2}{x^2(x^2+t^2)^2}\Phi_\nu(t)\,dt. = \frac{8}{\pi^2}\int_0^\infty f_1^2(x,t)\,t^2\Phi_\nu(t)\,dt.\end{aligned}$$

From Lemma 5.2, we see that $f_1(x,t)$ is completely monotonic in x on $(0,\infty)$, and so is $f_1^2(x,t)$. Then the desired results follow immediately. □

Next we present two incompletely monotonic functions having the form of $Q_\nu^{(n)}(x)/x^k$. The first example is the function $x \mapsto Q'_\nu(x)/x^2$. By (2.1) we have

$$\frac{Q'_\nu(x)}{x^2} = \frac{8}{\pi^2}\int_0^\infty \frac{t^2}{x(t^2+x^2)^2}\Phi_\nu(t)\,dt = \frac{8}{\pi^2}\int_0^\infty f_2(x,t)\,\Phi_\nu(t)\,dt.$$

It is easy to check that

$$f_2(x,1) = \frac{1}{x(x^2+1)^2} = \int_0^\infty \left(1-\frac{1}{2}u\sin u - \cos u\right)e^{-xu}du.$$

Since the function $1-(u\sin u)/2-\cos u$ infinitely changes sign on $(0,\infty)$, by Bernstein theorem [19, p. 161, Theorem 12b] we see that $f_2(x,t) = t^{-5}f_2(x/t,1)$ is not completely monotonic in x on $(0,\infty)$. But a computation by mathematical soft we find that

$$f_2^{(10)}(x,1) = \frac{3628\,800}{x^{11}(x^2+1)^{12}}P_{10}\left(x^2\right),$$

where

$$\begin{aligned}P_{10}(x) &= 1001x^{10} - 4004x^9 + 5148x^8 - 572x^7 + 1001x^6\\ &\quad +792x^5 + 495x^4 + 220x^3 + 66x^2 + 12x + 1,\end{aligned}$$

which is positive for $x > 0$ due to

$$\begin{aligned}P_{10}(x) &> 1001x^{10} - 4004x^9 + 5148x^8 - 572x^7 + 1001x^6\\ &= 1001x^8(x-2)^2 + \frac{143}{2}x^6(4x-1)^2 + \frac{1859}{2}x^6 > 0\end{aligned}$$

for $x > 0$. Then

$$\frac{\partial^{10}}{\partial x^{10}}f_2(x,t) = \frac{1}{t^5}\frac{\partial^{10}}{\partial(x/t)^{10}}f_2\left(\frac{x}{t},1\right) \times \frac{\partial^{10}(x/t)}{\partial x^{10}} > 0$$

for $x,t > 0$. This implies that $[Q'_\nu(x)/x^2]^{(10)} > 0$ for $x > 0$. Since

$$\frac{1}{x^2}Q'_\nu(x) \sim \frac{1}{x^2} \quad \text{as } x \to \infty,$$

we see that $[Q'_\nu(x)/x^2]^{(n)} \to 0$ as $x \to \infty$ for $0 \le n \le 9$. Then we have the following statement.

Theorem 5.4. *Let $\nu \in \mathbb{R}$. Then $(-1)^k[Q'_\nu(x)/x^2]^{(k)} > 0$ for $x > 0$ and $k = 1, 2, ..., 10$.*

Remark 5.5. Theorem 5.4 tells us that $x \mapsto Q'_\nu(x)/x^2$ is a 10-th order incompletely monotonic function on $(0,\infty)$, and hence

$$O_{\text{incm}}\left[\frac{Q'_\nu}{x^2}(0,\infty)\right] \ge 10.$$

Further, we guess that

$$O_{\text{incm}}\left[\frac{Q_\nu'}{x^2}(0,\infty)\right]=18.$$

The second incompletely monotonic function is: $x\mapsto Q_\nu''(x)/x$. By (2.2), we have

$$\frac{Q_\nu''(x)}{x}=\frac{8}{\pi^2}\int_0^\infty\frac{t^2(t^2-3x^2)}{x(t^2+x^2)^3}\Phi_\nu(t)\,dt:=\frac{8}{\pi^2}\int_0^\infty g_1(x,t)\,\Phi_\nu(t)\,dt.$$

Differentiation yields

$$\begin{aligned}\frac{\partial^4 g_1(x,t)}{\partial x^4} &= 24t^2\frac{t^{10}+7t^8x^2+21t^6x^4-105t^4x^6+630t^2x^8-210x^{10}}{x^5(t^2+x^2)^7}\\ &= 24\frac{t^2x^5}{(t^2+x^2)^7}g_2\left(\frac{t^2}{x^2}\right),\end{aligned}$$

where

$$g_2(y)=y^5+7y^4+21y^3-105y^2+630y-210.$$

Since

$$g_2'(y)=5y^4+28y^3+63y^2-210y+630>0$$

for $y>0$, with $g_2(0)=-210<0$ and $g_2(\infty)=\infty$, there is $y_0\in(0,\infty)$ such that $g_2(y)<0$ for $y\in(0,y_0)$ and $g_2(y)>0$ for $y\in(y_0,\infty)$. This means that $g_2(t^2/x^2)<0$ for $t\in(0,t_0)$ and $g_2(t^2/x^2)>0$ for $t\in(t_0,\infty)$, where $t_0=x\sqrt{y_0}$, and so is $\partial^4 g_1(x,t)/\partial x^4$. Note that

$$\int_0^\infty\frac{\partial^4 g_1(x,t)}{\partial x^4}dt=\left[-24\frac{t^3(t^8+6t^6x^2+15t^4x^4+70x^8)}{x^5(t^2+x^2)^6}\right]_{t\to0}^{t\to\infty}=0.$$

Using Lemmas 2.1 and 2.2 we immediately obtain that $[Q_\nu''(x)/x]^{(4)}>(<)\,0$ for $x>0$. From (3.1) it is seen that

$$\frac{1}{x}Q_\nu''(x)\sim\frac{\nu^2-1/4}{4x^4}\quad\text{as } x\to\infty,$$

which indicates that $[Q_\nu''(x)/x]^{(n)}\to0$ as $x\to\infty$ for $n\geq0$. We thus conclude that $(-1)^k[Q_\nu''(x)/x]^{(k)}>(<)\,0$ for $x>0$ and $k=1,2,3,4$. Then we have the following statement.

Theorem 5.6. *Let $\nu\in\mathbb{R}$. If $|\nu|>(<)\,1/2$, then $(-1)^k[Q_\nu''(x)/x]^{(k)}>(<)\,0$ for $x>0$ and $k=1,2,3,4$.*

Remark 5.7. Theorem 5.6 shows that $x\mapsto Q_\nu''(x)/x^2$ for $|\nu|>1/2$ is a 4-th order incompletely monotonic function on $(0,\infty)$, and hence

$$O_{\text{incm}}\left[\frac{Q_\nu''}{x}(0,\infty)\right]\geq4\text{ for }|\nu|>1/2.$$

Further, elaborate computations support the following guess:

$$O_{\text{incm}}\left[\frac{Q_\nu''}{x}(0,\infty)\right]=8\text{ for }|\nu|>1/2.$$

6. Conclusions

In this paper, we proved that $Q_\nu'''(x) < (>) 0$ for $x > 0$ if $|\nu| > (<) 1/2$, which gives an affirmative answer to the guess (1.5) for $n = 3$. This together with $Q_{-\nu}(x) = Q_\nu(x) + 2\nu$ yields some monotonicity and concavity (convexity) results, including,

- $Q_\nu(x)$ is log-concave on $(0, \infty)$ for $\nu \geq 0$;
- $G_\nu(x) = xQ_\nu'(x) - 2Q_\nu(x)$ is concave (convex) on $(0, \infty)$ if $|\nu| > (<) 1/2$;
- $h_\nu(x) = x^2 Q_\nu''(x) - 2xQ_\nu'(x) + 2Q_\nu(x)$ is decreasing (increasing) on $(0, \infty)$ for $\nu \in \mathbb{R}$;
- for $\nu \geq 0$, $\xi_{p,\nu}(x) = [Q_\nu(x) - p]/x$ is concave (or convex) on $(0, \infty)$ if and only if $p \geq -\min\{0, \nu - 1/2\}$ (or $p \leq -\max\{0, \nu - 1/2\}$).

Furthermore, using the integral representation (1.3) and Lemmas 2.1 and 2.2 we established several high order monotonicity results, for example,

- the functions $x \mapsto Q_\nu(x)/x^3$ and $Q_\nu'(x)/x^3$ for $\nu \in \mathbb{R}$ are completely monotonic on $(0, \infty)$;
- for $\nu \in \mathbb{R}$, $(-1)^k \left[Q_\nu'(x)/x^2\right]^{(k)} > 0$ for $x > 0$ and $k = 1, 2, ..., 10$;
- if $|\nu| > (<) 1/2$, then $(-1)^k \left[Q_\nu''(x)/x\right]^{(k)} > (<) 0$ for $x > 0$ and $k = 1, 2, 3, 4$.

Moreover, consider another important ratio

$$W_\nu(x) = \frac{xI_\nu(x)}{I_{\nu+1}(x)},$$

where I_ν is the modified Bessel functions of the first of order ν. Simpson and Spector [15] showed that the ratio $W_\nu(x)$ is convex in x on $(0, \infty)$ for all $\nu \geq 0$. Baricz [3, p. 591] conjectured the function $W_\nu(x)$ is strictly convex on $(0, \infty)$ for all $\nu > -1$. Very recently, Yang and Tian [23, Theorem 3] have proved that $W_\nu(x)$ is strictly convex on $(0, \infty)$ if and only if $\nu \geq -1/2$. Due to the relations

$$y_\nu(x) = \frac{xI_\nu'(x)}{I_\nu(x)} = W_{\nu-1}(x) - \nu$$

(see [28, Eq. (15)]), (1.7) and Wronskian recurrence relation

$$\frac{xI_\nu'(x)}{I_\nu(x)} - \frac{xK_\nu'(x)}{K_\nu(x)} = \frac{1}{I_\nu(x)K_\nu(x)},$$

we have

$$\frac{1}{I_\nu(x)K_\nu(x)} = W_{\nu-1}(x) + Q_\nu(x). \tag{6.1}$$

Since $Q_\nu''(x), W_{\nu-1}''(x) > 0$ for $x > 0$ and $\nu > 1/2$, we conclude that

Theorem 6.1. *The function $[I_\nu(x)K_\nu(x)]^{-1}$ is strictly convex in x on $(0, \infty)$ if $\nu > 1/2$.*

Remark 6.2. The above theorem gives an answer to the guess given in [22, Conjecture 3].

Finally, since the similarity between the ratios $W_\nu(x)$ and $Q_\nu(x)$, inspired by that $Q_\nu'''(x) < 0$ for $x > 0$ and $\nu > 1/2$, we propose the following problem.

Problem 6.3. If $\nu > -1/2$ then $W_\nu'''(x) < 0$ for $x > 0$?

Remark 6.4. If the above problem is solved, then by (6.1) we find that

$$\left[\frac{1}{I_\nu(x)K_\nu(x)}\right]''' < 0 \quad \text{for } x > 0 \text{ and } \nu > 1/2.$$

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