Kütahya Dumlupınar University
Institute of Graduate Studies

Journal of Scientific Reports-A
E-ISSN: 2687-6167
Number 53, June 2023

RESEARCH ARTICLE

ITERATION SCHEME FOR APPROXIMATING FIXED POINTS OF G −NONEXPANSIVE MAPS ON BANACH SPACES VIA A DIGRAPH

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Receive Date:30.01.2023 *Accepted Date: 23.03.2023*

ABSTRACT

In this writing, an influential modified multistep iterative process for finding a common fixed point of G −nonexpansive maps is presented. Some convergence theorems are constructed by *Property P* for the recommended schema on Banach spaces by which digraph. Two numerical examples are given to illustrate the convergence behavior and the validity of the process. The achieved conclusions enlarge, generalise and complement some well-known fixed point results from the literature.

Keywords: *Digraph,G −nonexpansiveness, Property P, Fixed Point*

1. INTRODUCTION and PRELIMINARIES

Khan et al. [1] expressed the nouvelle iterative schema contains the modified Mann and Ishikawa, Noor iteration algorithm for a finite family. Yıldırım and Ozdemir [2] considered multi-step iteration schema for a finite family of non-self asymptotically nonexpansive maps on a uniformly convex Banach space (shortly, UCBS). Kettapun et al. [3] inspired and motivated by [1], and thus they acquainted a novel iteration technic for solving a common fixed point. Gürsoy et al. [18] modified a multistep iterative procees presented by [2]. They also testified several convergence results of this iterative procees and S −iteration for contractive-like operators. Ahmad et al. [31] presented some convergence results on Picard-Krasnoselskii hybrid iterative process in CAT(0) spaces. More recently, El Kouch and Mouline [32] studied convergence of Mann and Ishikawa iterative processes for some contractions in convex generalized metric space.

Jachymski [4] established the conception of G −contraction, and unified two notions of graph and fixed point theories. Since then, varied authors have widely probed fixed point theorems in metric space, Banach and Hilbert via graph (see [6], [19-25]). Aleomraninejad et al. [5] achieved several iterative method consequences for G −nonexpansiveness and G −contractive maps on graphs. Tripak [7] studied two-step iteration method to approach common fixed point of G −nonexpansiveness. Suparatulatorn et al. [8] evidenced some convergence theorems for the modified S −iterative method

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of G −nonexpansiveness in UCBS with a directed graph. Subsequently Hunde et al. [9] studied an explicit iterative algorithm for various common fixed point of a family of G −nonexpansiveness, further gave some convergence results without supposing the Opial's condition. Recently, Sridarat et al. [12] considered SP −iterative schema for common fixed point of G −nonexpansiveness. They further parallel the rate of convergence between Noor and SP −iteration.
Motivated by [3], [8] and [9], we present a novel iteration technic for solving a common fixed point of a finite family of G −nonexpansiveness as noted below:
For $x_0 \in C$ and $k \geq 2$, let the sequence $\{x_n\}$ identified as

$$\begin{aligned} x_{n+1} &= (1-\mu_n^k)y_n^{k-1} + \mu_n^k g_k y_n^{k-1}, \\ y_n^{k-1} &= (1-\mu_n^{k-1})y_n^{k-2} + \mu_n^{k-1} g_{k-1} y_n^{k-2}, \\ y_n^{k-2} &= (1-\mu_n^{k-2})y_n^{k-3} + \mu_n^{k-2} g_{k-2} y_n^{k-3}, \\ &\vdots \\ y_n^2 &= (1-\mu_n^2)y_n^1 + \mu_n^2 g_2 y_n^1, \\ y_n^1 &= (1-\mu_n^1)x_n + \mu_n^1 g_1 x_n, \end{aligned} \tag{1}$$

where for $n \geq 1$ and $i = \overline{1,k}$, $x_n = y_n^0$, $\{\mu_n^i\} \in [0,1]$.

Goal of the present writing is to attain some convergence deductions for the iteration algorithms Eq. 1 of a finite family of G −nonexpansiveness on UCBS through a digraph.

Next, we present some lemmas, definitions and remark which are favourable to the main results in the manuscript.

Let $G = (V(G), E(G))$ be digraph, where $V(G)$ is the set of vertices of graph, $E(G)$ is the set of its edges which covers versal loops, i.e. $(x,x) \in E(G)$ for $\forall x \in V(G)$. G is mentioned to be transitive if, $x,y,z \in V(G)$; $(x,y),(y,z) \in E(G) \Longrightarrow (x,z) \in E(G)$.

Fixed point set of g is indicated by $g_{fix} = \{x \in C: gx = x\}$. If $g_{fix} \neq \emptyset$, then g is said

(I) G −nonexpansive [6] if it provides (i) $(x,y) \in E(G) \Rightarrow (gx,gy) \in E(G)$ (g preserves edges of G), (ii) $(x,y) \in E(G) \Rightarrow \|gx - gy\| \leq \|x - y\|$;
(II) G −continuous [4] if for any given $\omega \in X, \{\omega_n\} \subseteq X$, $\omega_n \longrightarrow \omega$ and $(\omega_n, \omega_{n+1}) \in E(G)$ imply $g\omega_n \rightarrow g\omega$;
(III) *semicompact* [17] if for $\{x_n\} \subseteq C$ with $\|x_n - gx_n\| \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow s_* \in C$.

Let $X \supseteq C \neq \emptyset$, $\{g_i\}_{i=1}^k: C \rightarrow C$ supply $Condition\ (A'')$ [3] if there is a nondecreasing function $f: [0,\infty) \rightarrow [0,\infty)$ with $f(t) > 0$ for $\forall t \in (0,\infty)$, $f(0) = 0$ such that $\|x - g_l x\| \geq f\left(d(x, g_{fix})\right)$ for $\forall x \in C$ and $1 \leq l \leq k$, here $d(x, g_{fix}) = inf\{\|x - s_*\|: s_* \in g_{fix} := \cap_{i=1}^k g_{fix}(g_i) \neq \emptyset\}$.

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Definition 1. [8] Let $x_0 \in V(G)$ and $V(G) \supseteq \Theta$. We call that (i) Θ is dominated by x_0 if $(x_0, x) \in E(G)$ for $\forall x \in \Theta$, (ii) Θ dominates x_0 if for each $x \in \Theta$, $(x_0, x) \in E(G)$.

Definition 2. [8] Let $C \neq \emptyset \subseteq X$, $g: C \longrightarrow X$ be a map. Then is called to be $G-demiclosed$ at $y \in X$ if, for any $\{x_n\} \subseteq C$ such that $\{x_n\} \rightharpoonup x \in C$, $\{gx_n\} \to y$ and $(x_n, x_{n+1}) \in E(G)$ imply $gx = y$.

Definition 3. [13] Let $C \neq \emptyset \subseteq X$, $G = (V(G), E(G))$ be digraph such that $V(G) = C$. Then C is called to own *Property P* if for each $\{x_n\} \subseteq C$ such that $\{x_n\} \rightharpoonup x \in C$, $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $(x_{n_l}, x) \in E(G)$ for $\forall l \in N$.

Remark 1. [9] If G is transitive, then *Property P* is equal to the feature: If $\{x_n\} \subseteq C$ with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_l}\}$ of $\{x_n\}$ converging weakly to x in X, then $(x_n, x) \in E(G)$ for $\forall n \in N$.

Lemma 1. [14] Let X be $UCBS$. Supposing that $n \geq 1$, $1 > c \geq t_n \geq b > 0$. Let $\{u_n\}, \{w_n\} \subseteq X$ be such that $limsup_{n\to\infty}\|u_n\| \leq a$, $limsup_{n\to\infty}\|w_n\| \leq a$, $\|(1-t_n)w_n + t_n u_n\| \to a \geq 0$ as $n \to \infty$. Then $\|u_n - w_n\| \to 0$ as $n \to \infty$.

Lemma 2. [9] Let $C \neq \emptyset$ be a closed convex subset of UCBS X. Assume that C own *Property P*. Let $\{g_i\}_{i=1}^k$ be $G-$nonexpansive maps in C. Then $I - g_i$ are $G-demiclosed$ at 0.

Lemma 3. [12] Let $C \neq \emptyset$ be a closed convex subset of a normed space, and let $G = (V(G), E(G))$ be digraph which is transitive with $V(G) = C$. Let $g: C \to C$ be $G-$nonexpansive map. If C own the *Property P*, then g is $G-$continuous.

2. MAIN RESULTS

Henceforward, $C \neq \emptyset$ express a subset of UCBS X involving $G = (V(G), E(G))$ such that convexness of $E(G)$, $V(G) = C$ and transitive of G. The maps $\{g_i\}_{i=1}^k : C \to C$ are $G-$nonexpansiveness with $g_{fix} := \cap_{i=1}^k g_{fix}(g_i) \neq \emptyset$. For $x_0 \in C$, let the sequence $\{x_n\}$ identified by Eq. 1.

Proposition 1. Let $c_0 \in g_{fix}$ be such that $(x_0, c_0), (c_0, x_0) \in E(G)$. Then (x_n, c_0), (c_0, x_n), (x_n, y_n^i), (y_n^i, x_n), (c_0, y_n^i), (y_n^i, c_0), $(x_n, x_{n+1}) \in E(G)$ for $i = \overline{1, k-1}$.

Proof. Using mathematical inductive, we shall show our results. Let $(x_0, c_0) \in E(G)$. Due to edge-preserving of g_1, $(g_1 x_0, c_0) \in E(G)$. Due to convexness of $E(G)$, we own $(y_0^1, c_0) \in E(G)$. From edge-preserving of g_2, $(g_2 y_0^1, c_0) \in E(G)$, by virtue of convexness to $E(G)$, we get $(1-\mu_0^2)(y_0^1, c_0) + \mu_0^2(g_2 y_0^1, c_0) = \left((1-\mu_0^2)y_0^1 + \mu_0^2 g_2 y_0^1, c_0\right) = (y_0^2, c_0) \in E(G)$. Suppose that $\left(y_0^j, c_0\right) \in E(G)$ for $j = \overline{1, k-2}$. Due to edge-preserving of g_{j+1}, $\left(g_{j+1} y_0^j, c_0\right) \in E(G)$, by virtue of convexness to $E(G)$, we furnish $\left(y_0^{j+1}, c_0\right) \in E(G)$. Consequently for $i = \overline{1, k-1}$; $\left(y_0^i, c_0\right) \in E(G)$. We enjoy $(y_0^{k-1}, c_0) \in E(G)$. Owing to edge-preserving of g_k, $(g_k y_0^{k-1}, c_0) \in E(G)$. Because $E(G)$ is convexness, we possess $(1-\mu_0^k)(y_0^{k-1},c_0)+\mu_0^k(g_k y_0^{k-1},c_0)=\left((1-\mu_0^k)y_0^{k-1}+\mu_0^k g_k y_0^{k-1},c_0\right)=(x_1,c_0)\in E(G)$. Renewing prior procedure for (x_1,c_0) instead of (x_0,c_0), we acquire $\left(y_1^i,c_0\right)\in E(G)$ and $(x_2,c_0)\in E(G)$. Assume that $(x_l,c_0)\in E(G)$ for $l\geq 1$. Due to edge-preserving of g_1, $(g_1x_l,c_0)\in E(G)$. Since $E(G)$ is convexness, we possess $(y_l^1,c_0)\in E(G)$. From edge-preserving of g_2, $(g_2y_l^1,c_0)\in E(G)$, as $E(G)$ is convexness, we belong $(y_l^2,c_0)\in E(G)$. Repeating the algorithm, we conclude that $\left(y_l^i,c_0\right)\in E(G)$ and $(x_{l+1},c_0)\in E(G)$. Prolong the argument againward for $(x_{l+1},c_0)\in E(G)$, we attain $\left(y_{l+1}^i,c_0\right)\in E(G)$. From induction, we deduce that $(x_n,c_0)\in E(G)$, we gain $(y_n^i,c_0)\in E(G)$ for $n\geq 1$ and $i=\overline{1,k-1}$. Using an analog argument, we can indicate (c_0,x_n), $(c_0,y_n^i)\in E(G)$ for $n\geq 1$ and $i=\overline{1,k-1}$, under the hypothesis that $(c_0,x_0)\in E(G)$. As the graph G is transitive, we hold (x_n,y_n^i), (y_n^i,x_n), $(x_n,x_{n+1})\in E(G)$ for $n\geq 1$ and $i=\overline{1,k-1}$.

Lemma 4. If $C\neq\emptyset$ is a closed convex subset of UCBS X, $\{\mu_n^i\}_{i=1}^k\subset[\varrho,\varsigma]$, where $0<\varrho<\varsigma<1$ and $(x_0,c_0),(c_0,x_0)\in E(G)$ for $x_0\in C$ and $c_0\in g_{fix}$, then for $n\geq 1$ and $i=\overline{1,k}$;

(i) $\|x_{n+1}-c_0\|\leq\|x_n-c_0\|$, and hence $\lim_{n\to\infty}\|x_n-c_0\|$ exists;
(ii) $\lim_{n\to\infty}\left\|x_n-g_i y_n^{i-1}\right\|=0$;
(iii) $\lim_{n\to\infty}\|x_n-g_i x_n\|=0$.

Proof. (i) Let $c_0\in g_{fix}$. It follows from Eq. 1, Proposition 1 and G −nonexpansiveness of g_1, we have

$$\begin{aligned}\|y_n^1-c_0\|&=\|(1-\mu_n^1)x_n+\mu_n^1 g_1x_n-c_0\|\\&\leq(-\mu_n^1+1)\|-c_0+x_n\|+\mu_n^1\|-c_0+g_1x_n\|\\&\leq(1-\mu_n^1)\|-c_0+x_n\|+\mu_n^1\|-c_0+x_n\|\\&=\|x_n-c_0\|.\end{aligned}\tag{2}$$

Using an analogue way, allied to Eq. 2, we have

$$\begin{aligned}\|y_n^2-c_0\|&=\|(1-\mu_n^2)y_n^1+\mu_n^2 g_2y_n^1-c_0\|\\&\leq(-\mu_n^2+1)\|-c_0+y_n^1\|+\mu_n^2\|-c_0+g_2y_n^1\|\\&\leq(1-\mu_n^2)\|-c_0+y_n^1\|+\mu_n^2\|-c_0+y_n^1\|\\&=\|y_n^1-c_0\|\leq\|x_n-c_0\|.\end{aligned}\tag{3}$$

By induction, it follows from Eq. 1 and G −nonexpansiveness of $\{g_i\}_{i=1}^{k-1}$, we have

$$\left\|y_n^j-c_0\right\|\leq\|x_n-c_0\|\tag{4}$$

for $j=1,2,\cdots,k-1$. Hence, it follows from Eq. 1, Eq. 4 and G −nonexpansiveness of g_k, we have

$$\begin{aligned}\|x_{n+1}-c_0\|&=\|(1-\mu_n^k)y_n^{k-1}+\mu_n^k g_k y_n^{k-1}-c_0\|\\&\leq(1-\mu_n^k)\|y_n^{k-1}-c_0\|+\mu_n^k\|g_k y_n^{k-1}-c_0\|\\&\leq(1-\mu_n^k)\|y_n^{k-1}-c_0\|+\mu_n^k\|y_n^{k-1}-c_0\|\end{aligned}\tag{5}$$

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$$= \|y_n^{k-1} - c_0\| \leq \|x_n - c_0\|.$$

Hence, $\lim_{n\to\infty}\|x_n - c_0\|$ exists.

(ii) From hypothesis (i), we get that $\{x_n\}$ is bounded. In turn there is a real numbers $\gamma \geq 0$ such that

$$\|x_n - c_0\| \to \gamma \;\; as\, n \to \infty. \tag{6}$$

By Eq. 4, we have $\left\|y_n^j - c_0\right\| \leq \|x_n - c_0\|, for\ i = \overline{1, k-1}$.
Getting $lim\ sup$ on both aspects of the hereinabove inequality, we have $j = \overline{1, k-1}$;

$$limsup_{\,n\to\infty}\left\|y_n^j - c_0\right\| \leq \gamma. \tag{7}$$

We further write down that

$$\|x_{n+1} - c_0\| = \|(1-\mu_n^k)(y_n^{k-1} - c_0) + \mu_n^k(g_k y_n^{k-1} - c_0)\|$$
$$\leq (1-\mu_n^k)\|y_n^{k-1} - c_0\| + \mu_n^k\|y_n^{k-1} - c_0\|$$
$$\leq \|y_n^{k-1} - c_0\|$$
$$\vdots$$
$$\leq \left\|y_n^j - c_0\right\|, for\ j = \overline{1, k-1}.$$
$$\Rightarrow$$
$$liminf_{n\to\infty}\left\|y_n^j - c_0\right\| \geq \gamma, for\ j = \overline{1, k-1}. \tag{8}$$

By Eq. 7 and Eq. 8, we get $lim_{n\to\infty}\left\|y_n^j - c_0\right\| = \gamma, for\ j = \overline{1, k-1}$.

In other words, $\lim_{n\to\infty}\left\|(1-\mu_n^j)(y_n^{j-1} - c_0) + \mu_n^j(g_j y_n^{j-1} - c_0)\right\| = \gamma, for\ j = \overline{1, k-1}$.

Owing to G −nonexpansiveness of $\{g_i\}_{i=1}^{k-1}$, from Eq. 7, we possess $limsup_{n\to\infty}\left\|g_j y_n^{j-1} - c_0\right\| \leq \gamma, for\ j = \overline{1, k-1}$.

By Lemma 1, we have $for\ j = \overline{1, k-1}$

$$\lim_{n\to\infty}\left\|g_j y_n^{j-1} - y_n^{j-1}\right\| = 0. \tag{9}$$

For $j = k$, by Eq. 4 and G −nonexpansiveness of g_k, we own $\left\|g_j y_n^{j-1} - c_0\right\| \leq \left\|y_n^{j-1} - c_0\right\| \leq \|x_n - c_0\|$. Taking lim sup on both sides of the above term, we get $limsup_{n\to\infty}\|g_k y_n^{k-1} - c_0\| \leq \gamma$.
As $\lim_{n\to\infty}\|x_{n+1} - c_0\| = \gamma$, we have $\lim_{n\to\infty}\|(1-\mu_n^k)(y_n^{k-1} - c_0) + \mu_n^k(g_k y_n^{k-1} - c_0)\| = \gamma$.
By Eq. 7 and Lemma 1, we get

$$\lim_{n\to\infty}\|-g_k y_n^{k-1} + y_n^{k-1}\| = 0. \tag{10}$$

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Therefore, Eq. 9 and Eq. 10 we deduced that

$$\lim_{n\to\infty}\left\|g_j y_n^{j-1} - y_n^{j-1}\right\| = 0\,, j = \overline{1,k}. \tag{11}$$

By Eq. 1, we have for $i = \overline{1,k-1}$, $\|y_n^i - y_n^{i-1}\| = \mu_n^i \|g_i y_n^{i-1} - y_n^{i-1}\|$.

By Eq. 11, we own for $i = \overline{1,k-1}$

$$\lim_{n\to\infty}\left\|y_n^i - y_n^{i-1}\right\| = 0. \tag{12}$$

Using Eq. 12, we have for $i = \overline{1,k-1}$

$$\begin{aligned}\left\|y_n^i - x_n\right\| &\le \|-x_n + y_n^1\| + \cdots + \left\|-y_n^{i-2} + y_n^{i-1}\right\| \\ &+\left\|-y_n^{i-1} + y_n^i\right\| \to 0\ as\ n \to \infty.\end{aligned} \tag{13}$$

It implies from Eq. 11 and Eq. 13 that for $i = \overline{1,k}$

$$\begin{aligned}\left\|-x_n + g_i y_n^{i-1}\right\| &\le \left\|-y_n^{i-1} + g_i y_n^{i-1}\right\| + \left\|-x_n + y_n^{i-1}\right\| \\ &\to 0\ as\ n \to \infty.\end{aligned} \tag{14}$$

(iii) Due to the case $i = 1$, by (ii), we get

$$\|x_n - g_1 x_n\| \to 0\ as\ n \to \infty. \tag{15}$$

It follows from Eq. 13 and Eq. 14, by G –nonexpansiveness of $\{g_i\}_{i=2}^k$, we get

$$\begin{aligned}\|g_i x_n - x_n\| &\le \left\|g_i x_n - g_i y_n^{i-1}\right\| + \left\|g_i y_n^{i-1} - x_n\right\| \\ &\le \left\|x_n - y_n^{i-1}\right\| + \left\|g_i y_n^{i-1} - x_n\right\| \\ &\to 0\ as\ n \to \infty, for\ i = 2,3,\cdots,k.\end{aligned} \tag{16}$$

Hence, from Eq. 15 and Eq. 16, for $i = \overline{1,k}$ we deduce that

$$\|x_n - g_i x_n\| \to 0\ as\ n \to \infty. \tag{17}$$

Theorem 1. Let $C \neq \emptyset$ is a closed convex subset of UCBS X, $\{\mu_n^i\}_{i=1}^k \subset [\varrho, \varsigma]$, where $0 < \varrho < \varsigma < 1$. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. Given that C hold the *Property* P, $\{g_i\}_{i=1}^k$ satisfy the $Condition\ (A'')$, g_{fix} is dominated by x_0 and g_{fix} dominates x_0, then $\{x_n\} \to w^* \in g_{fix}$.

Proof. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. We know that $\|x_n - g_i x_n\| \to 0\ as\ n \to \infty$, for $i = \overline{1,k}$ with Lemma 4 (iii). By $Condition\ (A'')$, we can write

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$$\|x_n - g_l x_n\| \geq f\left(d(x_n, g_{fix})\right), for\ 1 \leq l \leq k. \quad (18)$$

Getting lim sup on both aspects of the hereinabove term, we hold

$$\lim_{n\to\infty} \|x_n - g_l x_n\| \geq \lim_{n\to\infty} f\left(d(x_n, g_{fix})\right), for\ 1 \leq l \leq k. \quad (19)$$

Namely, $f\left(d(x_n, g_{fix})\right) \to 0\ as\ n \to \infty$. Because of $Condition\ (A'')$, we get $d(x_n, g_{fix}) \to 0\ as\ n \to \infty$. Thus, we may receive a subsequence $\{x_{n_v}\}$ of $\{x_n\}$ and $\{w_v^*\} \subset g_{fix}$ such that $\|x_{n_v} - w_v^*\| < 2^{-v}$. For strong convergence implies weak convergence, by Remark 1, we get $(x_{n_v}, w_v^*) \in E(G)$. From the proof method of [15], we own $\|x_{n_{v+1}} - w_v^*\| \leq \|x_{n_v} - w_v^*\| < 2^{-v}$, thus

$$\|w_{v+1}^* - w_v^*\| \leq \|w_{v+1}^* - x_{n_{v+1}}\| + \|x_{n_{v+1}} - w_j^*\| \leq \frac{3}{2} 2^{-v} \quad (20)$$

We conclude that $\{w_v^*\}$ is a Cauchy sequence, so $w_j^* \to w^*$. Due to closed of g_{fix}, $w^* \in g_{fix}$. Then $x_{n_v} \to w^*$. Thereof Lemma 4 (i), $x_n \to w^* \in g_{fix}$.
The following two example illustrate which is inspired by Example 2.2 and Example 3.2 in [16] for fulfilling of Theorem $1-2$ which the *Condition* (A'') and *semicompact* are used to verify the convergence of iterative algorithm Eq. 1, resp.

Example 1. Let $C = [0,2]$ and $G = (V(G), E(G))$ be digraph via $E(G) = \{(x,y): x \in [0,1], y \in [0,2]\ with\ 1 \geq |x-y|\}$ and $V(G) = C$. For every $i = \overline{1,k}$, let $\{g_i\}_{i=1}^k: [0,2] \to [0,2]$ be defined by

$$\{g_i\}_{i=1}^k = \begin{cases} [(1-x)/2^i] + 1 & if\ x \in [0,1], \\ 5/2 & if\ x \in (1,2]. \end{cases} \quad (21)$$

Let $\{\mu_n^i\} = [(4^i - 1)(5^i - 1)].20^{-i}$ for $\forall i = \overline{1,k}$. Then $g_{fix} := \{1\}$ and $\{g_i\}_{i=1}^k: C \to C$ be G −nonexpansive maps for every $i = \overline{1,k}$.

Theorem 2. Let $C \neq \emptyset$ is a closed convex subset of UCBS X, $\{\mu_n^i\}_{i=1}^k \subset [\varrho, \varsigma]$, where $0 < \varrho < \varsigma < 1$. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. Supposing that C has the *Property P* and one of $\{g_i\}_{i=1}^k$ is *semicompact*, g_{fix} is dominated by x_0 and g_{fix} dominates x_0, then $\{x_n\} \to \kappa \in g_{fix}$.

Proof. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. Let g_l is semicompact for $1 \leq l \leq k$. We get $\|x_n - g_l x_n\| \to 0\ as\ n \to \infty$ by Lemma 4. On account of the fact that $\{x_n\}$ is bounded and g_l is semicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \kappa \in C$ as $j \to \infty$. As strong convergence implies weak convergence, by Remark 1, we get $(x_{n_j}, \kappa) \in E(G)$ It is apparent that $\kappa \in g_{fix}$. By Lemma 3 and Lemma 4 (iii), we obtain that

$$\| \kappa - g_l \, \kappa \| = \lim_{j \to \infty} \left\| x_{n_j} - g_l x_{n_j} \right\| = 0, \; for \; 1 \leq l \leq k. \tag{22}$$

Therefore $\kappa \in g_{fix}$ so that $\lim_{n\to\infty} \|x_n - \kappa\|$ exists. Hence, $x_n \to \kappa$ as $j \to \infty$.

Example 2. Let $G = (V(G), E(G))$ be digraph via

$$E(G) = \{(x, y): x \in [3,3.2], y \in [3,3.3] \; with \; |x - y| < 1\} \text{ and } V(G) = [3,3.3]. \tag{23}$$

For every $i = \overline{1,k}$, let $\{g_i\}_{i=1}^{k}: [3,3.3] \to [3,3.3]$ be defined by

$$\{g_i\}_{i=1}^{k} = \begin{cases} [x/2^i] + \left[\left((2^i - 1)/2^i\right)(22/7)\right] & if \; x \in [3,3.2], \\ 3 & if \; x \in (3.2,3.3]. \end{cases} \tag{24}$$

Let $\{\mu_n^i\} = (2^i - 1).6^{-i}$ for $\forall i = \overline{1,k}$. Then $g_{fix} \coloneqq \{22/7\}$ and $\{g_i\}_{i=1}^{k}$ be G –nonexpansive maps for every $i = \overline{1,k}$.

Table 1: The value of $\{x_n\}$ with initial value $x_0 = 3.0000$ and $n = 25$, resp.

n	$k = 5$ *in* Eq. 1	$k = 3$ *in* Eq. 1	$k = 2$ *in* Eq. 1	$k = 1$ *in* Eq. 1
1	3.0000	3.0000	3.0000	3.0000
2	3.0254	3.0236	3.0201	3.0119
3	3.0463	3.0432	3.0373	3.0228
4	3.0634	3.0596	3.0521	3.0328
5	3.0775	3.0733	3.0649	3.0420
6	3.0854	3.0848	3.0759	3.0504
7	3.0956	3.0943	3.0853	3.0581
8	3.1040	3.1023	3.0934	3.0652
9	3.1108	3.1090	3.1003	3.0717
10	3.1165	3.1145	3.1062	3.0776
11	3.1211	3.1192	3.1114	3.0830
12	3.1249	3.1232	3.1158	3.0880
13	3.1281	3.1264	3.1196	3.0926
14	3.1306	3.1291	3.1228	3.0968
15	3.1327	3.1313	3.1256	3.1006
16	3.1344	3.1333	3.1280	3.1041
17	3.1359	3.1348	3.1301	3.1073
18	3.1372	3.1362	3.1319	3.1103
19	3.1382	3.1374	3.1334	3.1130
20	3.1390	3.1383	3.1347	3.1155
21	3.1396	3.1391	3.1359	3.1178
22	3.1402	3.1397	3.1369	3.1199
23	3.1407	3.1403	3.1377	3.1218

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24	3.1411	3.1407	3.1384	3.1236
25	3.1413	3.1411	3.1391	3.1252

Theorem 3. Let $C \neq \emptyset$ is a closed convex subset of UCBS X, $\{\mu_n^i\}_{i=1}^k \subset [\varrho, \varsigma]$, where $0 < \varrho < \varsigma < 1$. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. Supposing that C has the *Property P*, g_{fix} is dominated by x_0 and g_{fix} dominates x_0, then $\{x_n\} \rightharpoonup c_0 \in g_{fix}$.

Proof. Let $c_0 \in g_{fix}$ such that $(x_0, c_0), (c_0, x_0) \in E(G)$ for $x_0 \in C$. Owing to Lemma 4 (i) and weaakly compact, there exists a subsequnce $\{x_{n_v}\}$ of $\{x_n\}$ such that $x_{n_v} \to \kappa^* \in C \; as \; v \to \infty$. It follows by Lemma 4, $\lim_{n\to\infty}\|x_{n_v} - c_0\| = 0$, $\lim_{n\to\infty}\|x_{n_v} - g_i y_{n_v}^{i-1}\| = 0$ and $\lim_{n\to\infty}\|x_{n_v} - g_i x_{n_v}\| \to 0 \; as \; n \to \infty$ for $i = \overline{1,k}$. Using Lemma 2, $I - g_i$ are G −demiclosed at 0. Afterward the remainder of proving follows as in the proof of Theorem 3.4 in [9].

Remark 2.

(i) Taking $\mathrm{k} = 3$ and $\mathrm{g}_1 = \mathrm{g}_2 = \mathrm{g}_3 = \mathrm{g}$ in Eq. 1, we acquire the generalized form of the SP −iteration scheme by Phuengrattana and Suantai [10].

(ii) Taking $\mathrm{k} = 2$ in Eq. 1, we have the two-step iterative schema by Thianwan [11] for a self-map.

(iii) Taking $\mathrm{k} = 1$ in Eq. 1, then we obtain some convergence theorems of Mann algorithm for G −nonexpansiveness in the frame of UCBS via graph.

(iv) Theorem $1 - 2$ widen and advance the concerning deductions of Khan et al. [1], Kettapun et al. [3] and Yildirim and Özdemir [2] for a self-map in a finite family of G −nonexpansiveness in UCBS via digraph.

(v)The iteration method (1.3) in [9] is replaced by the modified multistep iterative process of a finite family of G −nonexpansiveness, also additionally, we give strong convergence result under the Condition (A'').

(vi)Taking $\mathrm{k} = 3$ in Eq. 1, then Theorem $2 - 3$ extend and improve the outcomes of Theorem 3.7-3.8 in [12] without supposing the Opial's condition, resp.

3. CONCLUSION

Withinside the forthcoming scope of the sight, reader may verify the convergence theorems of the following iteration processes to a common fixed point of nonexpansiveness identified on UCBS via digraph.

Let $C \neq \emptyset$ is a closed convex subset of UCBS X with a digraph $\big(V(G), E(G)\big) = G$ such that convexness of $E(G)$, $V(G) = C$ and transitive of G. Let $\{h_i\}_{i=1}^k, \{g_i\}_{i=1}^k : C \to C$ are G −nonexpansive maps; supposing the existence of common fixed points of these operators, our results and proof procedure go along to this class of maps by using the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_{n+1} &= (1-\mu_n^k)h_k y_n^{k-1} + \mu_n^k g_k y_n^{k-1}, \\ y_n^{k-1} &= (1-\mu_n^{k-1})h_{k-1} y_n^{k-2} + \mu_n^{k-1} g_{k-1} y_n^{k-2}, \\ y_n^{k-2} &= (1-\mu_n^{k-2})h_{k-2} y_n^{k-3} + \mu_n^{k-2} g_{k-2} y_n^{k-3}, \\ &\vdots \\ y_n^2 &= (1-\mu_n^2)h_2 y_n^1 + \mu_n^2 g_2 y_n^1, \\ y_n^1 &= (1-\mu_n^1)h_1 x_n + \mu_n^1 g_1 x_n, \end{aligned} \tag{25}$$

or, in brief,

$$\begin{aligned} x_{n+1} &= (1-\mu_n^k)h_k y_n^{k-1} + \mu_n^k g_k y_n^{k-1}, \\ y_n^l &= (1-\mu_n^l)h_l y_n^{l-1} + \mu_n^l g_l y_n^{l-1}, for\ l = 2,3,\cdots,k-1 \\ y_n^1 &= (1-\mu_n^1)h_1 x_n + \mu_n^1 g_1 x_n, n \geq 1 \end{aligned} \tag{26}$$

where $x_n = y_n^0$, $\{\mu_n^i\} \in [0,1]$.

In image deblurring, many engineers seek to recover the original, sharp image by using a mathematical model of the blurring process [26]. Signal recovery presents potential problems for most researchers at one stage or another in an experiment. The most frequent problem here is either a very weak signal, or a very low signal to noise ratio [27]. Many mathematicians have been interested in simulated results for image deblurring and signal recovering problems in recent years, also see e.g. [28-30]. The reader on the other hand can apply our proposed method to solve image deblurring and signal recovering problems.

ACKNOWLEDGEMENT

This research received no specific grants from any funding agency in public, commercial or non-profit sectors.

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