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Research Article

Inner automorphisms of Clifford monoids

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Abstract

An automorphism ϕ of a monoid S is called *inner* if there exists g in U_S, the group of units of S, such that $\phi(s) = gsg^{-1}$ for all s in S; we call S *nearly complete* if all of its automorphisms are inner. In this paper, first we prove several results on inner automorphisms of a general monoid and subsequently apply them to Clifford monoids. For certain subclasses of the class of Clifford monoids, we give necessary and sufficient conditions for a Clifford monoid to be nearly complete. These subclasses arise from conditions on the structure homomorphisms of the Clifford monoids: all being either bijective, surjective, injective, or image trivial.

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1. Introduction

An automorphism of a group is termed *inner* if it can be expressed as conjugation by an element of the group. This notion naturally extends to monoids by choosing the conjugating element to be in the group of units of the monoid S; we let $\mathrm{Inn}(S)$ denote the inner automorphisms of S. We then define a monoid to be *nearly complete* if every automorphism is inner.

There are a number of key differences between the inner automorphisms of a group and of a monoid which will be explored further in Section 3. If G is a group, then $\mathrm{Inn}(G)$ is isomorphic to $\frac{G}{Z_G}$, where Z_G is the center of G, and thus $\mathrm{Inn}(G)$ can be seen as a measure for how 'close' G is to being abelian. On the other hand, for a monoid S we will show that $\mathrm{Inn}(S)$ gives a measure for how close the group of units of S is to being central in S, that is, commute with every element of S. In particular, while a group G is commutative if and only if $\mathrm{Inn}(G)$ is trivial, the same does not hold good for monoids (see Theorem 3.2). Thus the group $\mathrm{Inn}(S)$ gives more insight about the structure of a commutative monoid S as compared to the case when S is a commutative group.

There are numerous papers concerning the inner automorphisms of monoids of (partial) maps. For example, Schreier [17] and Mal'cev [10] proved that the automorphisms of $\mathcal{T}(X)$,

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the semigroup of all mappings of a set X to itself, are inner automorphisms induced by the elements of $\mathcal{S}(X)$, the group of all permutations on X. Similar results have been obtained by Sutov [20] and Magill [9] for $\mathcal{P}(X)$, the semigroup of all partial maps on X, and for $\mathcal{I}(X)$, the symmetric inverse semigroup of all partial one-to-one transformations by Liber [8]. More examples are provided among others, by Gluskin [3] and Symons [21]. Sullivan [19] and Levi [5] later generalized the above results to the class of $\mathcal{S}(X)$-normal semigroups, the transformation semigroups closed under conjugations by permutations. Recently, Mir et.al [11] generalized the results of Sullivan [19] for $\mathcal{P}_{\mathcal{M}}(X)$, the posemigroup of all partial monotone transformations on a poset X. For a general monoid the study of its inner automorphism group has not recieved much attention. Araújo et.al [1, 2] proved some general theorems on inner automorphisms of transformation semigroups and provided an algorithm for computing inner automorphisms of some classes of semigroups.

Note that our notion of inner automorphisms for a general monoid differs from the one used in the above cited work on inner automorphisms of (partial) transformation monoids. We carry the group theoretic notion simply by choosing the conjugating elements in the group of units of the given monoid, whereas in case of transformation monoids the conjugation is chosen in $\mathcal{S}(X)$, so that the conjugating elements may be outside of the semigroup. However, for all of the explicit examples given above (such as $\mathcal{T}(X)$), the group of units is equal to $\mathcal{S}(X)$, and hence the two notions intersect. We refer the reader to a further discussion on this in Section 3 and 4.

Much of the focus of this paper is investigating the inner automorphisms of Clifford semigroups, focusing particularly on nearly complete Clifford semigroups. A Clifford semigroup is an inverse semigroup S in which the idempotents are central. For a Clifford semigroup S there exists a semilattice Ω, groups S_α ($\alpha \in \Omega$) and structure homomorphisms $\psi_{\alpha,\beta}\colon S_\alpha \to S_\beta$ for each $\alpha \geq \beta$ such that $S = \bigcup_{\alpha\in\Omega} S_\alpha$ and with the structure homomorphisms defining the multiplication in S. Clifford semigroups play an important role in inverse semigroup theory since every element of a Clifford semigroup is contained in some subgroup, that is, Clifford semigroups are precisely the inverse completely regular semigroups. Importantly for our investigation, not only do Clifford semigroups have a relatively simple structure theory, but their automorphisms can also be described in terms of automorphisms of the semilattice Ω and isomorphisms between the groups S_α; similarly for their endomorphisms. This has allowed for a good understanding of the endomorphism monoid of a Clifford semigroup (see, e.g., Samman and Meldrum [14]) and certain model-theoretic properties such as homogeneity and ω-categoricity by Quinn-Gregson [12, 13].

In Section 2, we give an outline of the semigroup theory required in this paper. In Section 3, we prove some results on inner automorphisms of a general monoid and generalize a number of well-known results on inner automorphisms of groups. Our chief result is that $\mathrm{Inn}(S)$ is isomorphic to $\frac{U_S}{(Z_S \cap U_S)}$, where U_S is the group of units of S and Z_S is the set of central elements of S. The section ends with a comparison between our definition of inner automorphisms and with the one in [8] for transformation semigroups. In Section 4, we classify nearly complete Clifford monoids in the cases where the structure homomorphisms are either all bijective, surjective, injective, or have a trivial image. The key to the first three cases is that all automorphisms of S depend only on automorphisms of Ω and automorphisms of a single group (the group of units in the first two cases, and some quotient of S in the third).

2. Preliminaries

In this section, we fix some notations and gather various facts that we will need in our investigation; we refer the reader to [4] for an extensive study on semigroup theory. Throughout this section, S will denote a semigroup.

Given a pair of semigroups S and T, a map $\theta\colon S \to T$ is called a *homomorphism* if $\theta(xy) = \theta(x)\theta(y)$ for each $x, y \in S$; an *endomorphism* of S is a homomorphism of S to itself. A bijective homomorphism [endomorphism] is called an *isomorphism* [*automorphism*]. The set of all automorphisms of S is denoted by $\mathrm{Aut}(S)$ and forms a group with respect to composition. The identity automorphism of S is denoted by id_S.

A semigroup with an identity element is called a *monoid*. If S is not a monoid, then we may adjoin an identity element by taking $1 \notin S$ and extending the multiplication of S to $1^2 = 1$ and $1x = x1 = x$ for all $x \in S$; if S is a monoid, then let $S^1 = S$. Note that every automorphism of a monoid must fix the identity element.

For $a \in S$, we say that $a' \in S$ is an *inverse* of a if $a = aa'a$ and $a' = a'aa'$. We call S *regular* if every element has an inverse in S, and call S *inverse* if every element has a unique inverse in S. Every group is an inverse semigroup, but the class of inverse semigroups is far broader than the class of groups. The set of all idempotents of S will be denoted by E_S. If S is inverse, then E_S forms a semilattice, that is, a commutative idempotent semigroup. Each semilattice comes equipped with a partial order $\leq$ given by $e \leq f$ if and only if $ef = e$.

Given a semilattice Ω, we may construct an inverse semigroup S with $E_S \cong \Omega$ as follows. For each $\alpha \in \Omega$ let S_α be a group and assume that $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$. For each pair $\alpha \geq \beta$ in Ω, let $\psi_{\alpha,\beta}\colon S_\alpha \to S_\beta$ be a homomorphism such that:

(i) $\psi_{\alpha,\alpha} = \mathrm{id}_{S_\alpha}$ for any $\alpha \in \Omega$.
(ii) The homomorphisms are transitive: For any $\alpha, \beta, \gamma \in \Omega$ with $\alpha \geq \beta \geq \gamma$, $\psi_{\beta,\gamma}\ \psi_{\alpha,\beta}\ = \psi_{\alpha,\gamma}$.

On $S = \bigcup_{\alpha\in\Omega} S_\alpha$ define a multiplication $*$ where for $s \in S_\alpha$ and $t \in S_\beta$,

$$s * t = \psi_{\alpha,\alpha\beta}(s)\psi_{\beta,\alpha\beta}(t). \tag{1}$$

Then S forms an inverse semigroup, denoted $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$, known as a *strong semilattice of groups*. The homomorphisms $\psi_{\alpha,\beta}$ are called *structure homomorphisms*, Ω is called the *structure semilattice of* S, and the groups S_α are called the *components of* S. Unless stated otherwise the identity element of a component S_α will be denoted e_α, so that $E_S = \{e_\alpha : \alpha \in \Omega\}$.

An inverse semigroup S is *Clifford* if E_S is central, that is, if idempotents commute with every element of S. We let $\mathcal{J}$ denote the *Green's J-relation* on a semigroup S, where $a\,\mathcal{J}\,b$ if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$. The property of being Clifford has a number of alternative statements, to which we give only a few; we refer the reader to [4, Theorem 4.2.1] for a wider study.

Theorem 2.1. *Let S be a semigroup. Then the following statements are equivalent:*

(i) *S is a Clifford semigroup.*
(ii) *S is a strong semilattice of groups.*
(iii) *S is regular and each $\mathcal{J}$-class is a group.* □

In this paper, motivated by Theorem 2.1, we shall take the definition of a Clifford semigroup as a strong semilattice of groups. The next result describes all homomorphisms (respectively isomorphisms) of Clifford semigroups.

Theorem 2.2. [16, Theorem 2.1]**.** *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ and $T = [\Lambda; T_\alpha, \phi_{\alpha,\beta}]$ be two Clifford semigroups, $\pi\colon \Omega \to \Lambda$ be a semilattice homomorphism, for each $\alpha \in \Omega$ let*

$\chi_\alpha\colon S_\alpha \to T_{\pi(\alpha)}$ be a group homomorphism, and assume that for any $\alpha \geq \beta$, the diagram

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\chi_\alpha} & T_{\pi(\alpha)} \\ {\scriptstyle \psi_{\alpha,\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{\pi(\alpha),\pi(\beta)}} \\ S_\beta & \xrightarrow{\chi_\beta} & T_{\pi(\beta)} \end{array} \tag{2}$$

commutes. Then the map χ from S to T given by $\chi(s) = \chi_\alpha(s)$ if $s \in S_\alpha$ is a homomorphism, denoted $[\pi, \chi_\alpha]$. Moreover χ is an isomorphism if and only if π and all χ_α are isomorphisms. Conversely, every homomorphism (respectively isomomorphism) of S into T can be constructed in this way. □

Given an isomorphism $\Theta\colon S \to T$ of Clifford semigroups, we shall let Θ_Ω denote the underlying semilattice isomorphism and, for each $\alpha \in \Omega$, denote Θ_α for the group isomorphism $\Theta|_{S_\alpha}\colon S_\alpha \to T_{\Theta_\Omega(\alpha)}$. Hence $\Theta = [\Theta_\Omega, \Theta_\alpha]$.

In the next result we find the relationship between the images and kernels of the structure isomorphisms, respectively. The result is a simple application of Theorem 2.2 or, alternatively, the direct inclusion follows from [14, Corollary 1.5], where the reverse inclusion holds by applying the same result to the isomorphism Θ^{-1}.

Corollary 2.3. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ and $T = [\Lambda; T_\alpha, \phi_{\alpha,\beta}]$ be two Clifford semigroups and let $\Theta\colon S \to T$ be an isomorphism. If $\alpha, \beta \in \Omega$ with $\alpha \geq \beta$, then:*

(1) $\Theta_\beta(im\ \psi_{\alpha,\beta}) = im\ \phi_{\Theta_\Omega(\alpha),\Theta_\Omega(\beta)}$;
(2) $\Theta_\alpha(ker\ \psi_{\alpha,\beta}) = ker\ \phi_{\Theta_\Omega(\alpha),\Theta_\Omega(\beta)}$. □

3. Inner automorphism of semigroups.

Given a monoid S with identity 1 then the *group of units* of S, denoted U_S, is the set of $g \in S$ such that there exists $h \in S$ with $gh = hg = 1$.

Following [22], an endomorphism ϕ of S is called *inner* if there exists $a, b \in S$ with $\phi(x) = axb$ for all $x \in S$. By [22, Theorem 1], the map ϕ forms an automorphism of S if and only if S is a monoid with identity 1 and $ab = ba = 1$; in particular $a \in U_S$ with $b = a^{-1}$. We denote such an automorphism by ϕ_a^S. The set of all inner automorphisms of S is denoted by $\mathrm{Inn}(S)$ and forms a subgroup of $\mathrm{Aut}(S)$. Hence for a monoid S we have

$$\mathrm{Inn}(S) = \{\phi_g^S : g \in U_S\},$$

and if S is a group then we recover the usual definition of inner group automorphisms. A non-inner automorphism of S is called *outer*, and $\mathrm{Out}(S)$ denotes the set of all outer automorphisms of S (and may be empty).

Note also that if $x \in S$ and $g \in U_S$ then $x \mathcal{J} gxg^{-1}$ since $x = g^{-1}(gxg^{-1})g$, and it follows that inner automorphisms must setwise fix the $\mathcal{J}$-classes:

Lemma 3.1. *Let S be a monoid and $g \in U_S$. Then ϕ_g^S setwise fixes the $\mathcal{J}$-classes of S, that is, $x \mathcal{J} \phi_g^S(x)$ for each $x \in S$.* □

The center of a semigroup S, denoted Z_S, is the set of elements of S that commute with every element of S. That is, $Z_S = \{z \in S : \ zs = sz\ \forall s \in S\}$. For a group G, it is well known that $\mathrm{Inn}(G)$ is isomorphic to $\frac{G}{Z_G}$. Our first result is an extension of this to monoids.

Theorem 3.2. *Let S be a monoid and U_S be the group of units of S. Then $Inn(S)$ is isomorphic to $\frac{U_S}{(Z_S \cap U_S)}$.*

Proof. Define the map $\theta : \frac{U_S}{(Z_S \cap U_S)} \to \mathrm{Inn}(S)$ by

$$\theta[g(Z_S \cap U_S)] = \phi_g^S \quad \text{for all } g \in U_S.$$

Then for any $g_1, g_2 \in U_S$,

$$\begin{aligned} g_1(Z_S \cap U_S) = g_2(Z_S \cap U_S) &\Leftrightarrow g_2^{-1}g_1 \in Z_S \cap U_S \\ &\Leftrightarrow g_2^{-1}g_1 s = s g_2^{-1} g_1 \text{ for all } s \in S \\ &\Leftrightarrow g_1 s g_1^{-1} = g_2 s g_2^{-1} \text{ for all } s \in S \\ &\Leftrightarrow \phi^S_{g_1} = \phi^S_{g_2} \\ &\Leftrightarrow \theta[g_1(Z_S \cap U_S)] = \theta[g_1(Z_S \cap U_S)]. \end{aligned}$$

Hence θ is well-defined and injective, and is clearly surjective. Finally we have

$$\begin{aligned} \theta[g_1(Z_S \cap U_S) g_2(Z_S \cap U_S)] &= \theta[g_1 g_2 (Z_S \cap U_S)] \\ &= \phi^S_{g_1 g_2} \\ &= \phi^S_{g_1}\phi^S_{g_2} \\ &= \theta[g_1(Z_S \cap U_S)]\ \theta[g_2(Z_S \cap U_S)], \end{aligned}$$

and thus θ is an isomorphism. □

Note that unlike groups, Theorem 3.2 does not imply that S is commutative if and only if $\mathrm{Inn}(S)$ is a trivial group. Thus it is worth investigating $\mathrm{Inn}(S)$ even if S is commutative.

Theorem 3.3. *If S is a monoid then the map $Inn(S) \to Inn(U_S)$ defined by $\phi^S_g \mapsto \phi^{U_S}_g$ for each $g \in U_S$ is an isomorphism if and only if $Z_{U_S} = Z_S \cap U_S$.*

Proof. Let $\Upsilon\colon \mathrm{Inn}(S) \to \mathrm{Inn}(U_S)$ be the map defined by $\phi^S_g \mapsto \phi^{U_S}_g$. Suppose first that Υ is an isomorphism. Clearly $Z_S \cap U_S \subseteq Z_{U_S}$, so suppose conversely that $g \in Z_{U_S}$. Then $g \in U_S$ and

$$\begin{aligned} xg = gx\,(\forall x \in U_S) &\Rightarrow g^{-1}xg = x\,(\forall x \in U_S) \\ &\Rightarrow \phi^{U_S}_g = \phi^{U_S}_1 \\ &\Rightarrow \phi^S_g = \phi^S_1 \text{ (as } \Upsilon \text{ is injective)} \\ &\Rightarrow g^{-1}xg = x\,(\forall x \in S) \\ &\Rightarrow xg = gx\,(\forall x \in S) \\ &\Rightarrow g \in Z_S. \end{aligned}$$

Hence $g \in Z_S \cap U_S$, and thus $Z_{U_S} = Z_S \cap U_S$.

Conversely, suppose $Z_{U_S} = Z_S \cap U_S$. Then for any $g, h \in U_S$, we have

$$\begin{aligned} \Upsilon(\phi^S_g) = \Upsilon(\phi^S_h) &\Leftrightarrow \phi^{U_S}_g = \phi^{U_S}_h \\ &\Leftrightarrow h^{-1}g \in Z_{U_S} \\ &\Leftrightarrow h^{-1}gs = sh^{-1}g \text{ for all } s \in S \text{ (as } Z_{U_S} = Z_S \cap U_S) \\ &\Leftrightarrow gsg^{-1} = hsh^{-1} \text{ for all } s \in S \\ &\Leftrightarrow \phi^S_g = \phi^S_h. \end{aligned}$$

Hence Υ is well-defined and injective, and is clearly surjective. Now for any $\phi^S_g, \phi^S_h \in \mathrm{Inn}(S)$, we have $\Upsilon(\phi^S_g\phi^S_h) = \Upsilon(\phi^S_{gh}) = \phi^{U_S}_{gh} = \phi^{U_S}_g\phi^{U_S}_h = \Upsilon(\phi^S_g)\Upsilon(\phi^S_h)$. That is, Υ is homomorphism, and thus an isomorphism. □

A group G is called *complete* if every automorphism of G is inner and Z_G is trivial. This is equivalent to the map $G \to \mathrm{Aut}(G)$ defined by $g \mapsto \phi^G_g$ being an isomorphism. We naturally extend this by calling a monoid S *nearly complete* if every automorphism of S is inner, and *complete* if further $Z_S \cap U_S$ is trivial. The equivalent definition of a complete group extends to monoids as follows:

Proposition 3.4. *A monoid S is complete if and only if the map $U_S \to Aut(S)$ defined by $g \mapsto \phi_g^S$ is an isomorphism.*

Proof. Suppose first that S is complete. Then as $Z_S \cap U_S = \{1\}$ the proof of Theorem 3.2 shows that the map $U_S \to \mathrm{Inn}(S)$ defined by $g \mapsto \phi_g^S$ $(g \in U_S)$ is an isomorphism. Since Inn(S)=Aut(S) we are done.

Conversely, as the map is surjective every automorphism of S is inner. Moreover, as the map is injective we have $\phi_g^S \neq \mathrm{id}_S$ for any $g \neq 1$, so that $g \notin Z_S \cap U_S$. □

Example 3.5. The following are nearly complete:

(1) Any monoid with trivial automorphism group.
(2) The symmetric group $\mathcal{S}(X)$ for $|X| \neq 6$ [18].
(3) The transformation monoid $\mathcal{T}(X)$ [17].
(4) The partial transformation monoid $\mathcal{P}(X)$ [3] (the monoid of all partial maps $X \to X$).
(5) The symmetric inverse semigroup $\mathcal{I}(X)$ [8] (the inverse submonoid of $\mathcal{P}(X)$ of all injective partial maps).

The examples (2)-(5) above are well-known examples of (partial) transformation semigroups, that is, subsemigroups of $\mathcal{T}(X)$ ($\mathcal{P}(X)$).

A transformation semigroup S is called $\mathcal{S}(X)$-*normal* if $gSg^{-1} \subseteq S$ for every $g \in \mathcal{S}(X)$. For such semigroups an alternative definition of inner automorphisms has been widely studied (see, for example, [2, 5, 6, 8]), where inner automorphisms arise by conjugating by any permutation $g \in S(X)$ (not necessarily in U_S). We call automorphisms of this form *inner automorphisms of S with respect to $\mathcal{T}(X)$*[†], and if all automorphisms of S are inner with respect to $\mathcal{T}(X)$ then we say that S is *nearly complete with respect to* $\mathcal{T}(X)$. This concept easily generalizes to subsemigroups of any monoid.

The properties of being nearly complete and nearly complete with respect to $\mathcal{T}(X)$ are not, in general, equivalent. For example, let $S = \{1, c_x : x \in X\}$ be the submonoid of $\mathcal{T}(X)$ where $c_x(s) = x$ for all $s \in X$ is the constant map to x. Then $U_S = \{1\}$, so S has only one inner automorphism, namely the identity map. Any permutation π of the constant maps gives rise to an automorphism π' of S by also fixing 1, so S is not nearly complete if $|X| > 1$. However, such an automorphism is an inner automorphism with respect to $\mathcal{T}(X)$ by taking $g \in \mathcal{S}(X)$ such that $g(x) = y$ if and only if $\pi(c_x) = c_y$, so that

$$gc_xg^{-1} = gc_x = c_{g(x)} = \pi(c_x) = \pi'(c_x).$$

On the other hand, if S is a (partial) transformation monoid which contains $\mathcal{S}(X)$ (such as $\mathcal{T}(X), \mathcal{P}(X)$ and $\mathcal{I}(X)$) then our two definitions are clearly equal. Further examples of nearly complete submonoids of $\mathcal{P}(X)$ for X infinite can be found in [7] (where we note that submonoids which contain $\mathcal{S}(X)$ are trivially $\mathcal{S}(X)$-normal). In particular, the following is immediate from [7, Theorem 3.18]:

Corollary 3.6. *Let X be an infinite set and let S be a submonoid of $\mathcal{P}(X)$ in which $\mathcal{S}(X) \subseteq S$. Then S is nearly complete.* □

To find further examples of nearly complete monoids, and without being limited to transformation semigroups, we return our attention to Clifford monoids for the remainder of the article. We note first that Corollary 3.6 is of no use here, since in $\mathcal{P}(X)$ no idempotent outside of $\mathcal{S}(X)$ commutes with every element of $\mathcal{S}(X)$, so that a submonoid of $\mathcal{P}(X)$ (X infinite) which properly contains $\mathcal{S}(X)$ is not Clifford.

Given that every inverse semigroup S embeds into the symmetric inverse semigroup $\mathcal{I}(S)$ (Cayley's Theorem for inverse semigroups), it is more natural to study the inner automorphisms with respect to $\mathcal{I}(S)$. However, the required restriction to $\mathcal{S}(X)$-normal

[†]These are also known as *quasi-inner* automorphisms

Clifford subsemigroups of $\mathcal{I}(S)$ is too great as the following result dictates. For this, we let $\mathfrak{P}_X$ denote the power set of X and use a number of basic facts about $\mathcal{I}(X)$ which can be found in [4, Chapter 5].

Proposition 3.7. *Let X be a set and let S be an inverse $\mathcal{S}(X)$-normal subsemigroup of $\mathcal{I}(X)$ not contained in $\mathcal{S}(X)$. Then the following are equivalent:*

(1) *S is Clifford;*
(2) *S is a semilattice;*
(3) *$S = \{id_A : A \in Y\}$ for some $Y \subseteq \mathfrak{P}_X$ which is closed under both intersection and by $\mathcal{S}(X)$, that is, if $A \in Y$ and $g \in \mathcal{S}(X)$ then $g(A) \in Y$.*

Moreover, if $|X| = n$ is finite, then these are equivalent to:

(4) *there exists $r < n$ such that $S = \{1\} \cup \{id_A : |A| \leq r\}$.*

Proof. Note that $E_{\mathcal{I}(X)} = \{\mathrm{id}_A : A \in \mathfrak{P}_X\}$. Moreover, as S is $\mathcal{S}(X)$-normal if $g \in \mathcal{S}(X)$ then $g \circ \mathrm{id}_A \circ g^{-1} = \mathrm{id}_{g(A)} \in S$ for any $A \in \mathfrak{P}_X$. It follows that $E_S = \{\mathrm{id}_A : A \in Y\}$ for some $Y \subseteq \mathfrak{P}_X$ which is closed under $\mathcal{S}(X)$. Since $\mathrm{id}_A \circ \mathrm{id}_B = \mathrm{id}_{A\cap B}$ and E_S forms a subsemigroup of S it is clear that Y is closed under intersection.

Hence to prove the equivalence of (1)-(3), it suffices to show (1) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2) Let S be Clifford. Then for any $h \in S$ and $A \subseteq Y$ we have $h \circ \mathrm{id}_A = \mathrm{id}_A \circ h$ and so

$$A \cap \mathrm{dom}(h) = h^{-1}(A). \tag{3.1}$$

Since S is not contained in $\mathcal{S}(X)$, there exists a proper subset A of X with $e = \mathrm{id}_A \in E_S$. Let $h \in S$ be contained in the maximal subgroup of S with identity e. Then $h \in \mathcal{S}(A)$ and we suppose, seeking a contradiction, that $h \neq e$ so $h(a) \neq a$ for some $a \in A$. Let $x \in X \setminus A$ and let $g \in \mathcal{S}(X)$ fix a and map x to $h(a)$. Then $a \notin h^{-1}(g(A))$ but $a \in g(A) \cap \mathrm{dom}(h)$, which contradicts (3.1) (substituting $g(A)$ for A). Hence any non-identity idempotent has trivial maximal subgroup, so that $S = H \cup E$ for some $H \subseteq \mathcal{S}(X)$ and $E \subseteq E_S$. Now for any $k \in H$ we have from (3.1) that k preserves A, that is $k(A) = A$. Let $h \in H$ and suppose, seeking a contradiction, that $h(a) \neq a$ for some $a \in X$. Let $h \in \mathcal{S}(X)$ fix a and map $x \in X \setminus A$ to $h(a)$. Then $g^{-1}hg \in S$ but $g^{-1}hg(a) = g^{-1}h(a) = x$ does not preserve A, a contradiction.

The final equivalence is clear by taking r to be the maximal size of the domains of the non-identity idempotents of S, so that by closing under $\mathcal{S}(X)$ and then intersections, we obtain all domains of equal or smaller size. □

4. Inner automorphism of Clifford semigroups

Given the limited scope of inner automorphisms of Clifford semigroups with respect to $\mathcal{I}(X)$ that arose in the last section, we shall only consider in this section our original concept of inner automorphisms. The second benefit of keeping with our definition for Clifford semigroups is that it allows us to use the structure theorem and isomorphism theorem (Theorem 2.2); such luxuries have added caveats inside $\mathcal{I}(X)$. Our first result, which is well-known and immediate from definitions, describes the structure of Clifford monoids:

Lemma 4.1. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be a Clifford semigroup. Then S is a monoid if and only if Ω has maximal element say, η. Moreover, in this case $1 = e_\eta$ and $U_S = S_\eta$.* □

The following simple lemma allows many of the results of the last section to be translated into the Clifford semigroup setting.

Lemma 4.2. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be a Clifford monoid where η is the maximum element of Ω. Then $Z_S \cap U_S$ consists of precisely the elements of Z_{U_S} in which the structure homomorphisms preserve centrality, that is,*

$$Z_S \cap U_S = \{g \in Z_{S_\eta} : \psi_{\eta,\alpha}(g) \in Z_{S_\alpha} \text{ for each } \alpha \in \Omega\}.$$

Proof. By Lemma 4.1 $U_S = S_\eta$ and so $Z_{U_S} = Z_{S_\eta}$. If $g \in S_\eta$ and $x \in S_\alpha$ then $gx = xg$ if and only if $\psi_{\eta,\alpha}(g) \cdot x = x \cdot \psi_{\eta,\alpha}(g)$, from which the result follows. □

Due to the unwieldy nature of the structure homomorphisms of a Clifford semigroup, it seems unlikely that a meaningful description of all nearly complete Clifford monoids is possible. We study a number of restrictions on the Clifford monoids which allows for such a description; the first being the semilattice case.

Lemma 4.3. *The following are equivalent for a semilattice monoid* Ω*:*

(1) Ω *is nearly complete;*
(2) Ω *is complete;*
(3) $Aut(\Omega)$ *is trivial.*

Proof. Since Ω is commutative it is nearly complete if and only if it is complete. Moreover, as U_Ω is trivial this is equivalent to $\mathrm{Aut}(\Omega)$ being trivial. □

For example, every finite chain is nearly complete. However, the property of being nearly complete does not necessarily pass from the Clifford semigroup to its structure semilattice, as the following example illustrates; a nearly complete Clifford monoid can have underlying semilattice with non-trivial automorphisms.

Example 4.4. Let $\Omega = \{0, 1, \alpha, \beta\}$ be the diamond semilattice, with maximum element 1, minimum element 0, and $\alpha \wedge \beta = 0$. Then Ω has a single non-trivial automorphism which swaps α and β. Now let $S_1 \cong S_\alpha \cong \mathbb{Z}_2$ and $S_\beta \cong S_0$ be trivial groups. Form a Clifford monoid $S = S_0 \cup S_1 \cup S_\alpha \cup S_\beta$ by taking $\psi_{1,\alpha}$ as an isomorphism, and all other structure homomorphism having trivial image. Since $S_\alpha \not\cong S_\beta$, every automorphism of S has trivial underlying semilattice automorphism. Moreover, each component has a trivial automorphism group and it follows that $\mathrm{Aut}(S)$ is also trivial, and in particular S is nearly complete.

Motivated by this example, we generalize the property of Ω having a trivial automorphism group as follows. For a Clifford semigroup $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ we let $\mathrm{Aut}(\Omega)_S$ denote the set of all automorphisms π of Ω in which $\pi = \Theta_\Omega$ for some $\Theta_\Omega \in \mathrm{Aut}(S)$. If $\mathrm{Aut}(\Omega)_S$ is trivial, then we say that Ω has trivial automorphism group with respect to S.

Since the $\mathcal{J}$-classes of a Clifford semigroup are the components by Theorem 2.1 and as inner automorphisms preserve $\mathcal{J}$-classes by Lemma 3.1 we obtain:

Corollary 4.5. *If* $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ *is a Clifford monoid then* $(\phi_g^S)_\Omega = id_\Omega$ *for any* $g \in U_S$*. In particular, if* S *is nearly complete then* Ω *has trivial automorphism group with respect to* S*.* □

Finally, inner automorphisms of Clifford monoids restrict to inner automorphisms of their components, with restrictions on conjugating element:

Lemma 4.6. *Let* $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ *be a Clifford monoid with* Ω *having maximum element* η*, and let* $g \in S_\eta$*. Then for each* $\alpha \in \Omega$*, the inner automorphism* ϕ_g^S *restricts to the inner automorphism* $\phi_x^{S_\alpha}$ *of* S_α *for any* $x \in S_\alpha$ *such that* $\psi_{\eta,\gamma}(g) \cdot x^{-1} \in Z_{S_\alpha}$*. In particular,* $\phi_g^S = [id_\Omega, \phi^{S_\alpha}_{\psi_{\eta,\alpha}(g)}]$*.*

Proof. For $a \in S_\alpha$ we have

$$\phi_g^S(a) = gag^{-1} = \psi_{\eta,\alpha}(g) \cdot a \cdot \psi_{\eta,\alpha}(g^{-1}) = \psi_{\eta,\alpha}(g) \cdot a \cdot \psi_{\eta,\alpha}(g)^{-1},$$

and so ϕ_g^S restricts to the inner automorphism $\phi^{S_\alpha}_{\psi_{\eta,\alpha}(g)}$ of S_α, to which the first result follows. The final result is then immediate by Corollary 4.5. □

4.1. The bijective case

In this section, we give a characterization of the inner automorphisms of Clifford monoids in which all the structure homomorphisms are bijective. Our results will be simple consequences of the following pair of results from [15] and [12, Subsection 4.2].

Lemma 4.7. *Let S be a Clifford monoid with all the structure homomorphisms bijective. Then, for any $\alpha \in \Omega$, we have $S \cong \Omega \times S_\alpha$.* □

Theorem 4.8. *Let Ω be a semilattice and G be a group. Then $Aut(\Omega \times G) = Aut(\Omega) \times Aut(G)$.* □

Corollary 4.9. *Let $S = \Omega \times G$ for some semilattice monoid Ω and group G. Then $\phi \in Aut(S)$ is inner if and only if it is of the form $\phi(\alpha, s) = (\alpha, \phi_g^G(s))$ for $g \in G$.*

Proof. Let η be the maximum element of Ω, so that $U_S = \{(\eta, x) : x \in G\}$. Then for any $g = (\eta, g) \in U_S$ and $(\alpha, s) \in S$ we have

$$\phi_g^S(\alpha, s) = (\eta, g)(\alpha, s)(\eta, g)^{-1} = (\eta, g)(\alpha, s)(\eta, g^{-1}) = (\alpha, gsg^{-1}).$$

Hence $\phi_g^S = (\mathrm{id}_\Omega, \phi_g^G)$.

The converse is immediate by Theorem 4.8. □

The following result is then immediate from Theorem 4.8 and the previous corollary:

Corollary 4.10. *Let $S = \Omega \times G$ for some semilattice monoid Ω and group G. Then S is nearly complete if and only if Ω and G are nearly complete.* □

4.2. The surjective case

In the following two subsections, we relax our condition of all the structure homomorphisms being bijective in two ways. First, we consider the case where all the structure morphisms are taken to be surjective; we call such semigroups *surjective Clifford semigroups*. One motivation for studying such a generalization is that central units behave well:

Lemma 4.11. *If S is a surjective Clifford monoid, then $Z_S \cap U_S = Z_{U_S}$. In particular, the map $Inn(S) \to Inn(U_S)$ defined by $\phi_g^S \mapsto \phi_g^{U_S}$ for each $g \in U_S$ is an isomorphism.*

Proof. Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ and let η be the maximum element of Ω. Let $x \in Z_{U_S} = Z_{S_\eta}$ and $y \in S_\alpha$ for some $\alpha \in \Omega$. Then as $\psi_{\eta,\alpha}$ is surjective there exists $z \in S_\eta$ with $\psi_{\eta,\alpha}(z) = y$. Hence,

$$xy = \psi_{\eta,\alpha}(x) \cdot y = \psi_{\eta,\alpha}(x) \cdot \psi_{\eta,\alpha}(z) = \psi_{\eta,\alpha}(xz) = \psi_{\eta,\alpha}(zx) = \psi_{\eta,\alpha}(z) \cdot \psi_{\eta,\alpha}(x) = yx,$$

where the fourth equality is from x being central in S_η. Hence $Z_{U_S} \subseteq Z_S \cap U_S$, and the reverse inclusion is immediate.

The final result follows from Theorem 3.3. □

We are now interested in classifying the nearly complete surjective Clifford monoids. Our task is significantly simplified by the following result, which states that automorphisms of such semigroups arise uniquely from automorphisms of the structure semilattice and the group of units:

Theorem 4.12. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be a surjective Clifford monoid with $U_S = S_\eta$ and let $\Theta_\eta \in Aut(S_\eta)$ and $\pi \in Aut(\Omega)$. Then the following are equivalent:*

(i) *Θ_η extends to an automorphism Θ of S with $\Theta_\Omega = \pi$;*

(ii) *for each $\alpha \in \Omega$ we have $\Theta_\eta(ker\ \psi_{\eta,\alpha}) = ker\ \psi_{\eta,\pi(\alpha)}$;*

(iii) *for each $\alpha \in \Omega$ the map $\psi_{\eta,\pi(\alpha)}\Theta_\eta\psi_{\eta,\alpha}^{-1}$ is an isomorphism from S_α to $S_{\pi(\alpha)}$.*

Moreover, in this case the automorphism Θ extending Θ_η is such that $\Theta_\alpha = \psi_{\eta,\pi(\alpha)}\Theta_\eta\psi_{\eta,\alpha}^{-1}$ for each $\alpha \in \Omega$.

Proof. (i) $\Rightarrow$ (ii) Immediate from Corollary 2.3.

(ii) $\Rightarrow$ (iii) We first show that the map $\Theta_\alpha =: \psi_{\eta,\pi(\alpha)}\Theta_\eta\psi_{\eta,\alpha}^{-1}$ is well-defined and injective, so suppose $g_1, g_2 \in S_\alpha$, say $\psi_{\eta,\alpha}(h_i) = g_i$ $(i = 1, 2)$. Then

$$\begin{aligned}\Theta_\alpha(g_1) = \Theta_\alpha(g_2) &\Leftrightarrow \psi_{\eta,\pi(\alpha)}\Theta_\eta(h_1) = \psi_{\eta,\pi(\alpha)}\Theta_\eta(h_2)\\ &\Leftrightarrow \Theta_\eta(h_1h_2^{-1}) \in \ker\,\psi_{\eta,\pi(\alpha)}\\ &\Leftrightarrow h_1h_2^{-1} \in \ker\,\psi_{\eta,\alpha}\\ &\Leftrightarrow g_1 = g_2,\end{aligned}$$

where the third equality is from property (ii). Hence Θ_α is well-defined and injective, and it is a surjective homomorphism since Θ_η and each structure homomorphism are.

(iii)$\Rightarrow$ (i) and the final statement are immediate from Theorem 2.2, in particular diagram (2). □

It follows that for each $(\pi, \phi) \in \text{Aut}(\Omega) \times \text{Aut}(U_S)$, there exists at most one automorphism of S extending ϕ and with structure semilattice automorphism π.

Recall from Corollary 4.5 that a nearly complete Clifford semigroup has underlying semilattice with trivial automorphism group with respect to S. For surjective Clifford semigroups, by using Theorem 4.12 we may easily determine when this occurs from studying how automorphisms of U_S interplay with the kernels of the structure homomorphisms:

Corollary 4.13. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be a surjective Clifford monoid. Then Ω has a trivial automorphism group with respect to S if and only if for every non-identity $\pi \in Aut(\Omega)$ and every automorphism ϕ of $U_S = S_\eta$, there exists $\alpha \in \Omega$ with $\phi(ker\ \psi_{\eta,\alpha}) \neq ker\ \psi_{\eta,\pi(\alpha)}$.* □

Theorem 4.14. *Let S be surjective Clifford monoid. Then S is nearly complete if and only if Ω has trivial automorphism group with respect to S and every outer-automorphism of U_S does not preserve the kernel of some structure homomorphism.*

Proof. Let the semilattice Ω have maximum element η, so that $U_S = S_\eta$.

Suppose S is nearly complete, so that $\text{Aut}(\Omega)_S$ is trivial by Corollary 4.5. Let Θ_η be an outer-automorphism of S_η. Then Θ_η cannot be extended to an automorphism of S by Lemma 4.6, to which the result follows by Theorem 4.12 (with π taken to be id_Ω).

Conversely, let $\Phi = \bigcup_{\alpha\in\Omega}\Phi_\alpha$ be an automorphism of S. If Φ_η is outer, then it does not preserve the kernels of all of the structure homomorphisms which contradicts Theorem 4.12. Hence Φ_η is inner, say $\phi_g^{S_\eta}$. Then for any $x \in S_\alpha$ we have

$$\Phi(x) = \Phi_\alpha(x) = \psi_{\eta,\alpha}\Phi_\eta\psi_{\eta,\alpha}^{-1}(x) = \psi_{\eta,\alpha}(g \cdot \psi_{\eta,\alpha}^{-1}(x) \cdot g^{-1}) = \psi_{\eta,\alpha}(g) \cdot x \cdot \psi_{\eta,\alpha}(g^{-1}) = gxg^{-1}$$

and so $\Phi = \phi_g^S$ is inner. □

Corollary 4.15. *Let S be surjective Clifford monoid. If the semilattice Ω and U_S are nearly complete, then so is S.* □

If the structure homomorphisms are also injective, then the kernels of the structure homomorphisms are trivial and each component is isomorphic, so Corollary 4.10 follows from Theorem 4.14.

Note also that both Theorem 4.14 and its corollary do not hold if we drop surjectivity, as is illustrated below.

Example 4.16. Let $\Omega = \{\eta, \alpha\}$ be the chain $\eta > \alpha$, let $S_\eta = \{1\}$ be trivial and let S_α be any non-trivial group. Then $S = S_\eta \cup S_\alpha$ forms a Clifford semigroup with structure homomorphism $\psi_{\eta,\alpha}$ having trivial image. Moreover, both $U_S = \{1\}$ and the chain Ω are nearly complete, while Inn(S) is trivial. However, any automorphism of S_α extends to an automorphism of S by Theorem 2.2, so S is not nearly complete if Aut(S_α) is non-trivial. We will expand upon this example in Subsection 4.4.

The converse of Corollary 4.15 is also not true since clearly the semigroup in Example 4.4 is a nearly complete surjective Clifford monoid with Ω being not nearly complete.

4.3. The injective case

In this section, we consider the second generalization of Subsection 4.1: where all the structure homomorphisms are injective. A Clifford semigroup S has injective structure homomorphisms if and only it is E-unitary [4, Exercise 5.20], that is, for all $e \in E_S$ and $s \in S$, if $es \in E_S$ then $s \in E_S$.

We first restrict our attention to the case where the structure semilattice has a least element. The component of S corresponding to the minimum element is of particular importance for E-unitary Clifford semigroups, much in the way that the maximum component (the group of units) was in the surjective case:

Lemma 4.17. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be an E-unitary Clifford monoid in which Ω has minimum element γ. Then $Z_S \cap U_S = \psi^{-1}_{\eta,\gamma}(Z_{S_\gamma})$ where η is the maximum element of Ω. Consequently, Inn(S) is isomorphic to $S_\eta/\psi^{-1}_{\eta,\gamma}(Z_{S_\gamma})$.*

Proof. Recall that $U_S = S_\eta$. If $g \in Z_S \cap S_\eta$ then for every $x \in S_\gamma$ we have

$$xg = gx \Rightarrow x \cdot \psi_{\eta,\gamma}(g) = \psi_{\eta,\gamma}(g) \cdot x$$

and so $g \in \psi^{-1}_{\eta,\gamma}(Z_{S_\gamma})$. Hence $Z_S \cap S_\eta \subseteq \psi^{-1}_{\eta,\gamma}(Z_{S_\gamma})$. Conversely, let $h \in S_\eta$ be such that $\psi_{\eta,\gamma}(h) = y \in Z_{S_\gamma}$. For any $\alpha \in \Omega$ and $a \in S_\alpha$ we have $ha \in S_\alpha$ and, as the structure homomorphisms are transitive,

$$\begin{aligned}\psi_{\alpha,\gamma}(ha) &= \psi_{\alpha,\gamma}(\psi_{\eta,\alpha}(h) \cdot a) = \psi_{\eta,\gamma}(h) \cdot \psi_{\alpha,\gamma}(a) = y \cdot \psi_{\alpha,\gamma}(a)\\ &= \psi_{\alpha,\gamma}(a) \cdot y = \psi_{\alpha,\gamma}(a) \cdot \psi_{\eta,\gamma}(h) = \psi_{\alpha,\gamma}(a \cdot \psi_{\eta,\alpha}(h)) = \psi_{\alpha,\gamma}(ah)\end{aligned}$$

where the fourth equality is because y is central in S_γ. Hence $ah = ha$ since $\psi_{\alpha,\gamma}$ is injective, so that $h \in Z_S \cap S_\eta$.

The final result is immediate from Theorem 3.2. □

Theorem 4.18. *Let S be an E-unitary Clifford monoid in which Ω has minimum element γ. Let $\Theta_\gamma \in Aut(S_\gamma)$ and $\pi \in Aut(\Omega)$. Then the following are equivalent:*

(i) *Θ_γ extends to an automorphism of S with underlying semilattice automorphism π;*
(ii) *for each $\alpha \in \Omega$ we have $\Theta_\gamma(im\ \psi_{\alpha,\gamma}) = im\ \psi_{\pi(\alpha),\gamma}$;*
(iii) *for each $\alpha \in \Omega$ the map $\psi^{-1}_{\pi(\alpha),\gamma}\Theta_\gamma\psi_{\alpha,\gamma}$ is an isomorphism from S_α to $S_{\pi(\alpha)}$;*

Moreover, in this case the automorphism Θ extending Θ_γ is such that $\Theta_\alpha = \psi^{-1}_{\pi(\alpha),\gamma}\Theta_\gamma\psi_{\alpha,\gamma}$ for each $\alpha \in \Omega$.

Proof. (i) ⇒ (ii) Immediate from Corollary 2.3.

(ii) ⇒ (iii) Note that the composition is possible as $\Theta_\gamma(\text{im}\ \psi_{\alpha,\gamma}) = \text{im}\ \psi_{\pi(\alpha),\gamma}$ by condition (ii). The map is a (well-defined) injective homomorphism since Θ_γ as the structure homomorphisms are. Finally, let $x \in S_{\pi(\alpha)}$, and let $y = \Theta_\gamma^{-1}(\psi_{\pi(\alpha),\gamma}(x))$. Then by condition (ii) we have that $y \in \text{im}\ \psi_{\alpha,\gamma}$, say $\psi_{\alpha,\gamma}(x') = y$. Hence $\psi^{-1}_{\pi(\alpha),\gamma}\Theta_\gamma\psi_{\alpha,\gamma}(x') = \psi^{-1}_{\pi(\alpha),\gamma}\Theta_\gamma(y) = x$, and the map is surjective.

(iii)⇒ (i) and the final statement are immediate from Theorem 2.2, in particular diagram (2). □

Corollary 4.19. *Let S be an E-unitary Clifford monoid in which Ω has minimum element γ. Then Ω has trivial automorphism group with respect to S if and only if for every non-identity $\pi \in Aut(\Omega)$ and every automorphism θ of S_γ, there exists $\alpha \in \Omega$ with $\theta(im\ \psi_{\alpha,\gamma}) \neq im\ \psi_{\pi(\alpha),\gamma}$.* □

Theorem 4.20. *Let S be an E-unitary Clifford monoid in which Ω has minimum element γ. Then S is nearly complete if and only if Ω has trivial automorphism group with respect to S and every automorphism*

$$\theta \in Out(S_\gamma) \cup \{\phi_g^{S_\gamma} : \text{there exists } \alpha \in \Omega \text{ such that } gx^{-1} \notin Z_{S_\gamma} \text{ for all } x \in im\ \psi_{\alpha,\gamma}\}$$

does not preserve the image of some structure homomorphism, that is, there exists $\alpha \in \Omega$ with $\theta(im\ \psi_{\alpha,\gamma}) \neq im\ \psi_{\alpha,\gamma}$.

Proof. Let the semilattice Ω have maximum element η, so that $U_S = S_\eta$.

The forward direction to the proof follows from Lemma 4.6 similarly to the proof of Theorem 4.14, noting that

$$\{\phi_g^{S_\gamma} : \text{there exists } \alpha \in \Omega \text{ such that } gx^{-1} \notin Z_{S_\gamma} \text{ for all } x \in \text{im } \psi_{\alpha,\gamma}\}$$

is precisely the set of inner automorphisms of S_γ which are not restrictions of inner automorphisms of S.

Conversely, let $\Phi = \bigcup_{\alpha\in\Omega} \Phi_\alpha$ be an automorphism of S. Then $\Phi_\gamma = \phi_x^{S_\gamma}$ for some $x \in \text{im}\psi_{\eta,\gamma}$, since otherwise by our hypothesis it would not preserve the images of all of the structure homomorphisms which contradicts Theorem 4.18. Let $x = \psi_{\eta,\gamma}(g)$. Then for any $\alpha \in \Omega$ and any $y \in S_\alpha$ we have by Theorem 4.18

$$\begin{aligned}\Phi_\alpha(y) &= \psi_{\alpha,\gamma}^{-1}\Phi_\gamma\psi_{\alpha,\gamma}(y) = \psi_{\alpha,\gamma}^{-1}\phi_x^{S_\gamma}\psi_{\alpha,\gamma}(y) = \psi_{\alpha,\gamma}^{-1}(x \cdot \psi_{\alpha,\gamma}(y) \cdot x^{-1}) \\ &= \psi_{\alpha,\gamma}^{-1}(\psi_{\eta,\gamma}(g) \cdot \psi_{\alpha,\gamma}(y) \cdot \psi_{\eta,\gamma}(g^{-1})).\end{aligned}$$

However since $\psi_{\alpha,\gamma}\psi_{\eta,\alpha} = \psi_{\eta,\gamma}$ and the structure homomorphisms are injective it follows that $\psi_{\alpha,\gamma}^{-1}\psi_{\eta,\gamma} = \psi_{\eta,\alpha}$, and so

$$\Phi_\alpha(y) = \psi_{\eta,\alpha}(g) \cdot \psi_{\alpha,\gamma}^{-1}\psi_{\alpha,\gamma}(y) \cdot \psi_{\eta,\alpha}(g^{-1}) = \psi_{\eta,\alpha}(g) \cdot y \cdot \psi_{\eta,\alpha}(g^{-1}) = gyg^{-1}.$$

Thus $\Phi = \phi_g^S$ is inner, and S is nearly complete. □

Our next aim is to show that if the structure semilattice of our E-unitary Clifford monoid does not contain a minimal element, then we may pass to one which does whilst preserving the property of being nearly complete. Our construction will require a number of key ideas from inverse semigroup theory, which can be found in [4].

Given an inverse semigroup S, we call an equivalence relation γ a *congruence* if whenever $(x, y), (x', y') \in \gamma$ then $(xx', yy') \in \gamma$. The equivalence class of γ containing an element $x \in S$ is denoted $[x]_\gamma$, or simply $[x]$ when no confusion may arise. Each congruence γ gives rise to an inverse semigroup, known as the *quotient of S by γ* and denoted S/γ, by taking $S/\gamma = \{[x]_\gamma : x \in S\}$ and product $[x]_\gamma \cdot [y]_\gamma = [xy]_\gamma$.

We let σ_S, or simply σ if no confusion may arise, denote the smallest congruence on S such that the quotient is a group. It can be easily verified that the congruence is given by

$$\sigma_S = \{(x, y) : \exists e \in E_S,\ ex = ey\}.$$

For an E-unitary Clifford semigroup $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ and $x \in S_\alpha, y \in S_\beta$ we have

$$(x, y) \in \sigma_S \Leftrightarrow \psi_{\alpha,\alpha\beta}(x) = \psi_{\beta,\alpha\beta}(y).$$

In particular if $x \in S_\alpha$ and $\alpha > \beta$ then $(x, \psi_{\alpha,\beta}(x)) \in \sigma_S$, and if $(a, b) \in \sigma_S \cap \mathcal{J}$ then $a = b$. Hence if Ω has a minimal element γ then $S/\sigma_S \cong S_\gamma$. On the other hand, if Ω does not have a minimum element, then we may adjoin a minimum element 0 by taking $\alpha \wedge 0 = 0 \wedge \alpha = 0$ for all $\alpha \in \Omega$; let Ω^0 be the resulting semilattice, and put $\Omega^0 = \Omega$ if Ω has a minimum element.

Construction: Given an E-unitary Clifford semigroup $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ with $\Omega \neq \Omega^0$, we extend S to a Clifford semigroup over Ω^0 as follows. Let $S_0 = S/\sigma$ and, for each $\alpha \in \Omega$, define the structure homomorphism $\psi_{\alpha,0}$ by $\psi_{\alpha,0}(g) = [g]_\sigma$. For any $\alpha > \beta > 0$ in Ω^0 we have $\psi_{\beta,0}\psi_{\alpha,\beta}(g) = [\psi_{\alpha,\beta}(g)]_\sigma = [g]_\sigma = \psi_{\alpha,0}(g)$. Hence $S \cup S_0 = [\Omega^0; S_\alpha, \psi_{\alpha,\beta}]$ forms a Clifford semigroup, denoted S^*, and is E-unitary as $\psi_{\alpha,0}(g) = \psi_{\alpha,0}(h)$ implies $g \mathcal{J} h$ and $[g]_\sigma = [h]_\sigma$, so that $g = h$. If Ω has a minimum element, then we take $S^* = S$.

Note that automorphisms of Ω^0 must fix the minimum element 0. It follows that every automorphism of Ω^0 is obtained by extending some automorphism of Ω by fixing 0.

Proposition 4.21. *Let S be an E-unitary Clifford semigroup. Then every automorphism Θ of S extends to an automorphism Θ^* of S^* by defining $\Theta^*([g]_\sigma) = [\Theta(g)]_\sigma$. Moreover, every automorphism of S^* is obtained in this way.*

Proof. The result is trivial if $S = S^*$, so assume instead that $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}] \neq S^*$ and let $S_0 = S/\sigma$. Suppose first that $\Theta = [\pi, \Theta_\alpha] \in \mathrm{Aut}(S)$. We first show that the map $\Theta_0^* \colon S_0 \to S_0$ defined by $\Theta_0^*([g]_\sigma) = [\Theta(g)]_\sigma$ is an automorphism of S_0. Let $x \in S_\alpha$, $y \in S_\beta$ and $\delta = \alpha\beta$, then using the fact that $\Theta \in \mathrm{Aut}(S)$ we have

$$\begin{aligned} &[x]_\sigma = [y]_\sigma \\ \Leftrightarrow\ &\psi_{\alpha,\delta}(x) = \psi_{\beta,\delta}(y) \\ \Leftrightarrow\ &\Theta_\delta\psi_{\alpha,\delta}(x) = \Theta_\delta\psi_{\beta,\delta}(y) \\ \Leftrightarrow\ &\psi_{\pi(\alpha),\pi(\delta)}\Theta_\alpha(x) = \psi_{\pi(\beta),\pi(\delta)}\Theta_\beta(y) \\ \Leftrightarrow\ &[\Theta_\alpha(x)]_\sigma = [\Theta_\beta(y)]_\sigma \\ \Leftrightarrow\ &[\Theta(x)]_\sigma = [\Theta(y)]_\sigma \\ \Leftrightarrow\ &\Theta_0^*([x]) = \Theta_0^*([y]). \end{aligned}$$

Hence Θ_0^* is well-defined and injective. Moreover, it is a surjective homomorphism since Θ is, and thus Θ_0^* is an automorphism as required. To show that Θ^* is an automorphism of S^* we must show that diagram (2) holds, and for this it clearly suffices to show that $\Theta_0^*\psi_{\alpha,0} = \psi_{\pi(\alpha),0}\Theta_\alpha$ for any $\alpha \in \Omega$. For any $x \in S_\alpha$ we have

$$\Theta_0^*\psi_{\alpha,0}(x) = \Theta_0^*([x]_\sigma) = [\Theta(x)]_\sigma = \psi_{\pi(\alpha),0}\Theta_\alpha(x).$$

Conversely, suppose $\Phi = [\rho, \Phi_\alpha] \in \mathrm{Aut}(S^*)$. Since $\rho(\Omega) = \Omega$ it follows that $\Phi(S) = S$. Hence Φ extends the automorphism $\Phi' = \Phi|_S = [\rho|_\Omega, \Phi_\alpha]$ of S. Moreover, if $[x]_\sigma \in S_0$, say $x \in S_\alpha$, then

$$\Phi([x]_\sigma) = \Phi_0([x]_\sigma) = \psi_{\rho(\alpha),0}\Phi_\alpha\psi_{\alpha,0}^{-1}([x]_\sigma) = \psi_{\rho(\alpha),0}\Phi_\alpha(x) = [\Phi_\alpha(x)]_\sigma = [\Phi'(x)]_\sigma.$$

Hence $\Phi_0([x]_\sigma) = [\Phi'(x)]_\sigma$ as required. □

Theorem 4.22. *Let S be an E-unitary Clifford semigroup. Then S is nearly complete if and only if S^* is nearly complete.*

Proof. The result is trivial if $S = S^*$, so assume instead that $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}] \neq S^*$ and let $S_0 = S/\sigma$.

Let S be nearly complete, and let $\Theta = [\pi, \Theta_\alpha] \in \mathrm{Aut}(S^*)$. Then Θ restricts to an inner automorphism ϕ_g^S of S, so π restricts to the identity on Ω by Corollary 4.5. In particular, π is the identity on Ω^0. Moreover, by Proposition 4.21 we have for any $[x] \in S_0$ ($x \in S_\alpha$),

$$\Theta_0([x]_\sigma) = [\phi_g^S(x)]_\sigma = [gxg^{-1}]_\sigma = [g]_\sigma \cdot [x]_\sigma \cdot [g^{-1}]_\sigma = g \cdot [x]_\sigma \cdot g^{-1}$$

and so $\Theta = \phi_g^{S^*}$.

Conversely, let S^* be nearly complete. Then by Proposition 4.21 any automorphism of S extends to one of S^*, which must be inner. Any inner automorphism of S^* restricts to an inner automorphism of S, and the result follows. □

4.4. The image-trivial case

The aim of this section is to show that in general one cannot tell if a Clifford monoid is nearly complete by only studying the automorphisms of its semilattice and how the automorphisms of its group of units and (if it exists) group S_γ for minimum γ interacts with the kernels and images of the structure homomorphisms, respectively.

For this, we will study the class of Clifford monoids where each structure homomorphism has trivial image; we call such Clifford monoids *image-trivial*. The automorphisms are easily constructed in this case, and the proof of the following result can be found in [13, Proposition 4.4]:

Corollary 4.23. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be an image-trivial Clifford monoid and let π be an automorphism of Ω such that $S_\alpha \cong S_{\pi(\alpha)}$ for each $\alpha \in \Omega$. Then for any collection of isomorphisms $\Theta_\alpha\colon S_\alpha \to S_{\pi(\alpha)}$, the map $\Theta = \bigcup_{\alpha\in\Omega} \Theta_\alpha$ is an automorphism of S. Moreover, every automorphism of S can be constructed this way.* □

Theorem 4.24. *Let $S = [\Omega; S_\alpha, \psi_{\alpha,\beta}]$ be an image-trivial monoid with $U_S = S_\eta$ and let $S' = S \setminus S_\eta$. Then $Z_{S_\eta} = Z_S \cap S_\eta$ and*

$$Inn(S) = \{\phi_g^{S_\eta} \cup id_{S'} : g \in S_\eta\} \cong Inn(S_\eta).$$

Moreover, S is nearly complete if and only if the following hold:

(1) *for each non-identity automorphism π of Ω, there exists $\alpha \in \Omega$ with $S_\alpha \ncong S_{\pi(\alpha)}$;*
(2) *S_η is nearly complete;*
(3) *for $\alpha \neq \eta$, the component S_α has trivial automorphism group.*

Proof. Let $g \in Z_{S_\eta}$ and $x \in S_\alpha$. Then

$$gx = \psi_{\eta,\alpha}(g) \cdot x = e_\alpha x = x = xe_\alpha = x \cdot \psi_{\eta,\alpha}(g) = xg,$$

and so $Z_{S_\eta} = Z_S \cap S_\eta$. Hence $\mathrm{Inn}(S)$ is isomorphic to $\mathrm{Inn}(S_\eta)$ by Theorem 3.3. Moreover, if ϕ_h^S is an inner automorphism of S, then for $x \in S_\alpha$ a similar calculation gives $\phi_h^S(x) = hxh^{-1} = x$, and so ϕ_h^S is the identity on S'.

Now let S be nearly complete. Then by Corollary 4.23 any automorphism of Ω which preserves the isomorphism types of the components of S extends to an automorphism of S, and (1) follows. Moreover, every automorphism of a connected component can extend to an automorphism of S with underlying semilattice automorphism being the identity, to which (2) and (3) follow.

Conversely, suppose (1)-(3) holds and let $\Theta = [\pi, \Theta_\alpha] \in \mathrm{Aut}(S)$. Then π is the identity by (1), $\Theta_\eta = \phi_g^{S_\eta}$ for some $g \in S_\eta$ by (2), and $\Theta_\alpha = \mathrm{id}_{S_\alpha}$ for $\alpha < \eta$ by (3). Hence $\Theta = \phi_g^S$. □

Corollary 4.25. *There exists a Clifford monoid S with Ω having maximum element η and minimum element γ in which:*

(1) *S is not nearly complete;*
(2) *Ω has trivial automorphism group with respect to S;*
(3) *every outer automorphism of S_η does not preserve the kernel of some structure morphism;*
(4) *every automorphism in*

$$Out(S_\gamma) \cup \{\phi_g^{S_\gamma} : \textit{there exists } \alpha \in \Omega \textit{ such that } gx^{-1} \notin Z_{S_\gamma} \textit{ for all } x \in im\ \psi_{\alpha,\gamma}\}$$

does not preserve the image of some structure morphism.

Proof. Let $\Omega = \{\eta, \alpha, \gamma\}$ with $\eta > \alpha > \gamma$. Let $S_\eta = \{e_\eta\}, S_\gamma = \{e_\gamma\}$ and let S_α be any group with non-trivial automorphism group. Then the image-trivial Clifford monoid

$S = S_\eta \cup S_\alpha \cup S_\gamma$ is not nearly complete by the previous theorem and Ω has a trivial automorphism group, so (1)-(2) holds. Moreover, as S_η and S_γ are trivial groups, properties (3)-(4) are immediate. □

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References

[1] J. Araújo and J. Konieczny, *General Theorems on automorphisms of semigroups and their applications,* J. Austral Math. Soc. **87**, 1-17, 2009.

[2] J. Araújo, P. V. Bünau, J. D. Mitchell and M. Neunhöffer, *Computing automorphisms of semigroups,* J. Symbolic Comput. **45**, 373-392, 2010.

[3] L. M. Gluskin, *Semigroups and rings of endomorphisms of linear spaces-I*, Transc. of Amer. Math. Soc. **45**, 105-137, 1965.

[4] J. M. Howie, *Fundamentals of semigroup theory*, volume 12 of London Mathematical Society Monographs, New series, The Clarendon Press, Oxford University Press, New York, Oxford Science Publications, 1995.

[5] I. Levi, *Automorphisms of normal transformation semigroups*, Proc. Edinburg Math. Soc. **2**, 185-205, 1985.

[6] I. Levi, *On the inner automorphisms of finite transformation semigroups*, Proc. Edinburg Math. Soc. **39**, 27-30, 1996.

[7] I. Levi, *Automorphisms of normal partial transformation semigroups*, Glasgow Math. J. **29**, 149-157, 1987.

[8] A. E. Liber, *On symmetric generalised groups*, Math. Sbornik N. S. **33**, 531-544, 1953.

[9] K. D. Magill, *Automorphisms of semigroup of all relations on a set*, Canad. Math. Bull. **9**, 73-77, 1966.

[10] A. I. Mal'cev, *Symmetric Groupoids,* Math. Sbornik N. S. **31**, 146-151, 1952.

[11] D. J. Mir, A. H. Shah and, S. A. Ahanger, *On automorphisms of monotone transformation posemigroups*, Asian-Eur. J. of Math, **15**, 2250032, 2022.

[12] T. Quinn-Gregson, *Homogeneity of Inverse semigroups*, Internat. J. Algebra Comput. **28**, 837-875, 2018.

[13] T. Quinn-Gregson, $\aleph_0$*-category of semigroups II,* Semigroup Forum, **102**, 809-841, 2021.

[14] M. Samman and J. D. P. Meldrum, *On Endomorphisms of Semilattices of Groups,* Algebra Colloq. **12**, 93-100, 2005.

[15] A. H. Shah, and D. J. Mir, *On Automorphisms of Strong Semilattices of* π*-Groups*, Int. J. Nonlinear Anal. Appl. **13**, 3031-3036, 2022.

[16] A. H. Shah, D. J. Mir and N. M. Khan, *On automorphisms of Strong Semilattices of groups*, Mathematics Open **1**, 2250002, 2022.

[17] I. Schreier, *Uber Abbildungen einer absrakten Menge Auf ihre Telimengen*, Fund, Math. **28**, 261-264, 1936.

[18] J. Schreier and S. Ulam, *Über die Automorphismen der Permutations-gruppe der natürlichen Zahlenfolge,* Fundam. Math. **28**, 258-260, 1937.

[19] R. P. Sullivan, *Automorphisms of Transformation semigroups,* J. Austral Math. Soc. **20**, 77-84, 1975.

[20] E. G. Sutov, *Homomorphisms of the semigroups of all partial transformations,* Izv. Vyss Ucebn Zaved Mat. **3**, 177-184, 1961.

[21] J. S. V. Symons, *Normal transformation semigroups*, J. Austral Math. Soc.(A), **22**, 385-390, 1976.

[22] M. L. Vitanza, *Mappings of Semigroups Associated with Ordered Pairs*, Amer. Math. Monthly **73**, 1078-1082, 1966.