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Research Article

Reconstruction of the nonlocal Sturm-Liouville operator with boundary conditions depending on the parameter

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Abstract

In the present paper, we consider the Sturm–Liouville equation with nonlocal boundary conditions depending polynomially on the parameter. We obtain a result and give an algorithm for the reconstruction of the coefficients of the problem using asymptotics of the nodal points.

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1. Introduction

One of the solution methods for the inverse problems of the Sturm-Liouville operators is to use the zeros of the eigenfunctions. These zeros are also called nodal points. Trying to reconstruct the coefficients of the operator from the asymptotic formula of the nodal points is known as inverse nodal problem. This problem for Sturm-Liouville operator was first investigated by McLaughlin in [20]. She succeeded in giving a uniqueness theorem for this type of inverse problems with Dirichlet boundary conditions. Some further numerical calculations for reconstruction of potential are given in [12]. In 1997, Yang [35] obtained a definite algorithm for the solution of inverse nodal problems with separated boundary conditions. Later, similar results for various boundary conditions were obtained in (see [1, 4–7, 10, 11, 14–19, 21, 28, 30–32, 36–38] and references therein). On the other hand, it can be said that the inverse nodal problem for nonlocal boundary conditions is a relatively new topic. Indeed, there exist only a few studies with these boundary conditions [8, 13, 26, 27, 33, 34].

Nonlocal boundary conditions first appeared in Bitsadze and Samarskii's paper which includes some results on elliptic equations [3]. These conditions, which cannot be measured exactly at the boundary, have various applications in fields such as biology and physics (see [9, 23]). Various spectral results for differential operators with boundary conditions of this type are obtained in [2, 22, 24, 29].

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In this study, we deal with the following boundary value problem $L = L(q, \alpha_i, \beta_i)$

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in (0,1) \tag{1.1}$$

$$B_1(y). = a(\lambda)y'(0) + b(\lambda)y(0) - \gamma_0(\lambda)y(\xi_0) = 0, \tag{1.2}$$

$$B_2(y). = c(\lambda)y'(1) + d(\lambda)y(1) - \gamma_1(\lambda)y(\xi_1) = 0, \tag{1.3}$$

where $\xi_i \in \mathbb{Q} \cap (0,1)$ for $i = 0,1$ and λ is the spectral parameter. $q(x)$ is a real valued continuously differentiable function; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are monic polynomials such that

$$a(\lambda) = \sum_{i=0}^{k} a_i \lambda^i,\ b(\lambda) = \sum_{i=0}^{k} b_i \lambda^i,$$

$$c(\lambda) = \sum_{i=0}^{r} c_i \lambda^i,\ d(\lambda) = \sum_{i=0}^{r} d_i \lambda^i.$$

Also,

$$\gamma_0(\lambda) = \sum_{i=0}^{k} \alpha_i \lambda^i,\ \gamma_1(\lambda) = \sum_{i=0}^{r} \beta_i \lambda^i.$$

The main purpose of the present paper is to solve inverse nodal problem for L. We obtain $q(x)$ which is the potential of operator L from asymptotics of the nodal points and give an algorithm for the reconstruction of coefficients α_k and β_r. Consequently, our main result is a kind of generalization of the first result in [25] in which the polynomials in the boundary conditions are constant. Moreover, it covers a wide class of nonlocal boundary conditions.

2. Main results

In this section, we will first give the asymptotics of the nodal points. Then with the help of this result we will obtain a uniqueness theorem and an algorithm. Let the eigenvalues of the problem (1.1)-(1.3) be $\{\lambda_n\}_{n\geq 0}$ and the eigenfunctions corresponding to these eigenvalues be $\{\varphi(x,\lambda_n)\}_{n\geq 0}$. The following theorems are the main results in this article.

Theorem 2.1. *$\varphi(x,\lambda_n)$ has $n-k-r$ nodal points in $(0,1)$ for sufficiently large n, namely x_n^j, $j = 0,1,2,...,n-k-r-1$, and the following asymptotic formula is valid*

$$x_n^j = \tfrac{j+1/2}{(n-k-r)} - (-1)^{n-m-r} \tfrac{[\beta_r \cos((n-k-r)\pi\xi_1) - \alpha_k \cos((n-k-r)\pi(1-\xi_0))]}{(n-k-r)^2\pi^2} \tfrac{(j+1/2)}{(n-k-r)} + \\ + \tfrac{\left(Q(x_n^j)-1\right)}{(n-k-r)^2\pi^2} + \tfrac{\alpha_k \cos((n-k-r)\pi\xi_0)}{(n-k-r)^2\pi^2} + o\left(\tfrac{1}{n^2}\right).$$

where $Q(x) = \frac{1}{2}\int_0^x q(s)ds$.

Proof. Let $C(x,\lambda)$ and $S(x,\lambda)$ be the solutions of (1.1) under the initial conditions

$$\begin{aligned} S(0,\lambda) &= 0,\ S'(0,\lambda) = 1 \\ C(0,\lambda) &= 1,\ C'(0,\lambda) = 0 \end{aligned}$$

respectively. From [13] and [39], the functions $C(x,\lambda)$ and $S(x,\lambda)$ satisfy the following asymptotic relations for $|\lambda| \to \infty$,

$$C(x,\lambda) = \cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}Q(x) + \frac{\cos\sqrt{\lambda}x}{\lambda}q_1(x) + O\left(\frac{1}{\lambda^{3/2}}\exp|\tau|\,x\right),$$

$$S(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}x}{\lambda}Q(x) + O\left(\frac{1}{\lambda^{3/2}}\exp|\tau|\,x\right),$$

where $q_1(x) = \frac{q(x)-q(0)}{4} - \frac{1}{8}\left(\int_0^x q(s)ds\right)^2$ and $\tau = \left|\mathrm{Im}\sqrt{\lambda}\right|$.

The characteristic function of L is

$$\Delta(\lambda)=\det\begin{pmatrix} B_1(C) & B_1(S) \\ B_2(C) & B_2(S)\end{pmatrix}. \tag{2.1}$$

Since $\Delta(\lambda)$ is an entire function, L has a discrete spectrum. Also, the eigenvalues of L are the zeros of the function $\Delta(\lambda)$. From (2.1), we have that

$$\begin{aligned}\Delta(\lambda) &= \left(b(\lambda)-\gamma_0(\lambda)C(\xi_0)\right)\left(c(\lambda)S'(1)+d(\lambda)S(1)-\gamma_1(\lambda)S(\xi_1)\right) \\ &\quad -\left(a(\lambda)-\gamma_0(\lambda)S(\xi_0)\right)\left(c(\lambda)C'(1)+d(\lambda)C(1)-\gamma_1(\lambda)C(\xi_1)\right).\end{aligned} \tag{2.2}$$

Using the asymptotics of $C(x,\lambda)$ and $S(x,\lambda)$ in (2.2), we find the following asymptotic expression for $\Delta(\lambda)$ as $\lambda\to\infty$:

$$\begin{aligned}\Delta(\lambda) &= a(\lambda)c(\lambda)\sqrt{\lambda}\sin\sqrt{\lambda}-c(\lambda)\gamma_0(\lambda)\sin(\sqrt{\lambda}\xi_0)\sin\sqrt{\lambda}-a(\lambda)d(\lambda)\cos\sqrt{\lambda} \\ &\quad +b(\lambda)c(\lambda)\cos\sqrt{\lambda}-c(\lambda)\gamma_0(\lambda)\cos(\sqrt{\lambda}\xi_0)\cos\sqrt{\lambda}+a(\lambda)\gamma_1(\lambda)\cos(\sqrt{\lambda}\xi_1) \\ &\quad +d(\lambda)\gamma_0(\lambda)\sin(\sqrt{\lambda}\xi_0)\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}}-a(\lambda)\gamma_1(\lambda)\sin(\sqrt{\lambda}\xi_0)\frac{\cos(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}} \\ &\quad +b(\lambda)d(\lambda)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-d(\lambda)\gamma_0(\lambda)\cos(\sqrt{\lambda}\xi_0)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-b(\lambda)\gamma_1(\lambda)\frac{\sin(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}} \\ &\quad +\gamma_0(\lambda)\gamma_1(\lambda)\frac{\cos(\sqrt{\lambda}\xi_0)\sin(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}}+O\left(\frac{\lambda^{k+r}}{\sqrt{\lambda}}\exp|\tau|\right)\end{aligned}$$

and so

$$\Delta(\lambda)=\lambda^{k+r}\left[\sqrt{\lambda}\sin\sqrt{\lambda}+\beta_r\cos(\sqrt{\lambda}\xi_1)-\alpha_k\cos\sqrt{\lambda}(1-\xi_0)+o(\exp|\tau|)\right]. \tag{2.3}$$

Let $G_n(\varepsilon)=\left\{\sqrt{\lambda}:\left|\sqrt{\lambda}-(n-k-r)\pi\right|<\varepsilon\right\}$ for $n=1,2,\ldots$. It follows from (2.3) that there exist some $M(\varepsilon)>0$ such that $|\Delta(\lambda)|\geq M(\varepsilon)\left|\sqrt{\lambda}\right|\exp|\tau|$ for sufficiently large $\left|\sqrt{\lambda}\right|$ in $G_n(\varepsilon)$. For sufficiently large n, we can see that λ_n must be real number.

It can be shown using classical methods in [25] that the sequence $\{\lambda_n\}_{n\geq 0}$ satisfies the following asymptotic expression for $n\to\infty$:

$$\begin{aligned}\sqrt{\lambda_n} &= (n-k-r)\pi \\ &\quad -(-1)^{n-k-r}\frac{\left[\beta_r\cos\left((n-k-r)\pi\xi_1\right)-\alpha_k\cos\left((n-k-r)\pi(1-\xi_0)\right)\right]}{(n-k-r)\pi} \\ &\quad +o\left(\frac{1}{n}\right).\end{aligned} \tag{2.4}$$

Let $\varphi(x,\lambda)$ be the solution of equation (1.1), satisfying the initial conditions $\varphi(0,\lambda)=a(\lambda)-\gamma_0(\lambda)S(\xi_0)$, $\varphi'(0,\lambda)=\gamma_0(\lambda)C(\xi_0)-b(\lambda)$.

Thus, we have that

$$\varphi(x,\lambda_n)=C(x,\lambda_n)U(S(x,\lambda_n))-S(x,\lambda_n)U(C(x,\lambda_n)). \tag{2.5}$$

From (2.4) and (2.5), we can see easily the following asymptotic formula

$$\begin{aligned}&\varphi(x,\lambda_n) \\ &=\lambda_n^k\left(\cos\sqrt{\lambda_n}x+\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\left(Q(x)-1\right)+\frac{\alpha_k}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\left(x-\xi_0\right)+o\left(\frac{1}{\sqrt{\lambda_n}}\right)\right),\end{aligned} \tag{2.6}$$

is valid for sufficiently large n. We get from (2.6) that $\varphi(x,\lambda_n)$ has exactly $n-k-r$ zeros in $(0,1)$.

Use the asymptotic formula (2.6) to get

$$\begin{aligned}0 &= \varphi(x_n^j,\lambda_n)\\ &= \lambda_n^m\left(\cos\sqrt{\lambda_n}x_n^j + \frac{\sin\sqrt{\lambda_n}x_n^j}{\sqrt{\lambda_n}}\left(Q(x_n^j)-1\right) + \frac{\alpha_k}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\left(x_n^j-\xi_0\right) + o\left(\frac{1}{\sqrt{\lambda_n}}\right)\right)\end{aligned}$$

and so

$$\tan\left(\sqrt{\lambda_n}x_n^j - \frac{\pi}{2}\right) = \frac{(Q(x_n^j)-1)}{\sqrt{\lambda_n}} + \frac{\alpha_k}{\sqrt{\lambda_n}}\frac{\sin\sqrt{\lambda_n}\,(x_n^j-\xi_0)}{\sin\sqrt{\lambda_n}x_n^j} + o\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

This yields

$$x_n^j = \frac{(j+1/2)\,\pi}{\sqrt{\lambda_n}} + \frac{(Q(x_n^j)-1)}{\lambda_n} + \frac{\alpha_k}{\lambda_n}\frac{\sin\sqrt{\lambda_n}\,(x_n^j-\xi_0)}{\sin\sqrt{\lambda_n}x_n^j} + o\left(\frac{1}{\lambda_n}\right). \tag{2.7}$$

Using $\sqrt{\lambda_n}x_n^j = (j+1/2)\,\pi + O(\frac{1}{n})$, $n\to\infty$ we can show

$$\frac{\sin\sqrt{\lambda_n}\,(x_n^j-\xi_0)}{\lambda_n\sin\sqrt{\lambda_n}x_n^j} = \frac{\cos\left((n-k-r)\pi\xi_0\right)}{(n-k-r)^2\pi^2} + o\left(\frac{1}{n^2}\right). \tag{2.8}$$

In addition, we obtain

$$\begin{aligned}\frac{1}{\sqrt{\lambda_n}} &= \frac{1}{(n-k-r)\pi}\\ &\times\left(1+(-1)^{n-k-r}\frac{[\beta_r\cos\left((n-k-r)\pi\xi_1\right)-\alpha_k\cos\left((n-k-r)\pi(1-\xi_0)\right)]}{(n-k-r)^2\pi^2} + o\left(\frac{1}{n^3}\right)\right),\end{aligned} \tag{2.9}$$

$$\frac{1}{\lambda_n} = \frac{1}{(n-k-r)^2\pi^2} + o\left(\frac{1}{n^3}\right) \tag{2.10}$$

using by (2.4).

Substituting (2.8), (2.9) and (2.10) in (2.7), it is concluded that

$$\begin{aligned}x_n^j &= \frac{j+1/2}{(n-k-r)}\\ &-(-1)^{n-k-r}\frac{(-1)^{n-k-r}\,[\beta_r\cos\left((n-k-r)\pi\xi_1\right)-\alpha_k\cos\left((n-k-r)\pi(1-\xi_0)\right)]}{(n-k-r)^2\pi^2}\\ &\times\frac{(j+1/2)}{(n-k-r)}\\ &+\frac{(Q(x_n^j)-1)}{(n-k-r)^2\pi^2} + \frac{\alpha_k\cos\left((n-k-r)\pi\xi_0\right)}{(n-k-r)^2\pi^2} + o\left(\frac{1}{n^2}\right).\end{aligned}$$

□

Let X_0 be a subsequence of the numbers x_n^j that is dense on $(0,1)$. According to above result, the existence of such a set is obvious.

Consider the problem $\widetilde{L} = L\left(\widetilde{q},\widetilde{\alpha}_i,\widetilde{\beta}_i\right)$ under the same assumptions with L. Without loss of generality, we assume that $\int_0^1\left(q(x)-\widetilde{q}(x)\right)dx = 0$.

Theorem 2.2. *If $X_0 = \widetilde{X}_0$ then $\alpha_k = \widetilde{\alpha}_k$, $\beta_r = \widetilde{\beta}_r$ and $q(x) = \widetilde{q}(x)$ a.e. in $(0,1)$. Thus, the coefficients α_k, β_r and the potential $q(x)$ are uniquely determined by X_0.*

Proof. Put $\xi_0 = \dfrac{p_0}{r_0}$ and $\xi_1 = \dfrac{p_1}{r_1}$, where $p_i, r_i \in \mathbb{Z}$ for $i = 0,1$. For each fixed $x\in[0,1]$, there exists a sequence $\left(x_n^j\right)$ converges to x. For $n_s = 2sr_0r_1+k+r$, $\;s\in\mathbb{Z}$, the subsequence (x_n^j) converges also to x. Therefore we get from the asymptotic in Theorem 2.1 the following limit is finite and given equality holds:

$$\begin{aligned}\lim_{s\to\infty}(n-k-r)^2\pi^2\left(x_n^j-\tfrac{j+1/2}{(n-k-r)}\right) &= g(x)=(\alpha_k-\beta_r)x+ \\ &+Q(x)-1+\alpha_k,\end{aligned} \tag{2.11}$$

Direct calculations in (2.11) yield

$$q(x)=2\left(g'(x)-g(1)+g(0)\right) \tag{2.12}$$

$$\begin{aligned}\alpha_k &= g(0)+1, \\ \beta_r &= 2g(0)-g(1)+1\end{aligned} \tag{2.13}$$

Since $X_0=\widetilde{X}_0$ then $g(x)=\widetilde{g}(x)$ and so $q(x)=\widetilde{q}(x)$, a.e. in $(0,1)$. □

3. Algorithm

Let X_0, $\xi_i=\dfrac{p_i}{r_i}$ for $i=0,1$ be given. Then $q(x)$, α_k and β_r can be reconstructed by the following algorithm:

i) Denote $n_s=2sr_0r_1+k+r,\ \ s\in\mathbb{Z}$;
ii) Find $q(x)$ by (2.12) ;
iii) Find α_k and β_r by the formulas (2.13).

Example 3.1. We consider the following nonlocal boundary value problem

$$\ell y:=-y''+q(x)y=\lambda y,\quad x\in\Omega=(0,1)$$

$$a(\lambda)y'(0)+b(\lambda)y(0)-\gamma_0(\lambda)y(\frac{2}{3})=0,$$

$$c(\lambda)y'(1)+d(\lambda)y(1)-\gamma_1(\lambda)y(\frac{5}{6})=0,$$

where $q(x)\in C^1[0,1]$; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$, $\gamma_0(\lambda)$ and $\gamma_1(\lambda)$ are unknown coefficients of the problem. For sufficiently large n, let the nodal points provide the following asymptotic

$$\begin{aligned}x_n^j &= \frac{j+1/2}{(n-k-r)}+ \\ &-(-1)^{n-k-r}\frac{[4\cos\left((n-k-r)\frac{5\pi}{6}\right)-3\cos\left((n-k-r)\frac{\pi}{3}\right)]}{(n-k-r)^2\pi^2}\frac{(j+1/2)}{(n-k-r)}+ \\ &+\frac{3\left(\cos\left((n-k-r)\frac{2\pi}{3}\right)-1/3\right)}{(n-k-r)^2\pi^2}-\frac{(j+1/2)}{6(n-k-r)^3\pi^2}+\frac{(j+1/2)^3}{6(n-k-r)^5\pi^2}+o\left(\frac{1}{n^2}\right).\end{aligned}$$

According to these data, we can calculate $q(x)$, α_k and β_r.
Let $n_s=36s{+}k{+}r,\ s\in\mathbb{Z}$.
One can calculate that
$\lim_{s\to\infty}(n-k-r)^2\pi^2\left(x_n^j-\frac{j+1/2}{(n-k-r)}\right)=g(x)=-x+2-\frac{x}{6}+\frac{x^3}{6}$.
By the formulas (2.12) and (2.13) ;

$$\begin{aligned}\alpha_k &= 3 \\ \beta_r &= 4\end{aligned}$$

$$q(x)=x^2-\frac{1}{3}.$$

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