CONSTRUCTIVE MATHEMATICAL ANALYSIS
6 (2023), No. 2, pp. 77-89
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Existence and uniqueness of viscosity solutions to the infinity Laplacian relative to a class of Grushin-type vector fields

THOMAS BIESKE AND ZACHARY FORREST*

ABSTRACT. In this paper, we pose the ∞-Laplace equation as a Dirichlet Problem in a class of Grushin-type spaces whose vector fields are of the form

$$X_k(p) := \sigma_k(p)\frac{\partial}{\partial x_k}$$

and σ_k is not a polynomial for indices $m+1 \leq k \leq n$. Solutions to the ∞-Laplacian in the viscosity sense have been shown to exist and be unique in [3], when σ_k is a polynomial; we extend these results by exploiting the relationship between Grushin-type and Euclidean second-order jets and utilizing estimates on the viscosity derivatives of sub- and supersolutions in order to produce a comparison principle for semicontinuous functions.

Keywords: ∞-Laplace equation, viscosity solution, Grushin-type spaces.

2020 Mathematics Subject Classification: Primary 53C17, 35D40, 35J94; Secondary 35H20, 22E25, 17B70.

1. INTRODUCTION

In [3] the author considers the Dirichlet Problem

$$\begin{cases} \Delta_\infty w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and establishes conditions under which a viscosity solution (see Section 3) to (1.1) exists and is unique when the problem is posed in a wide variety of Grushin-type spaces. The goal of the current paper is to extend the existence/uniqueness results of [3] to a more general class of Grushin-type spaces.

The spaces under consideration in [3] are defined by Lie Algebras consisting of vector fields of the form

$$Y_k(p) := P_k(p)\frac{\partial}{\partial x_k} \text{ for } k \leq n, \tag{1.2}$$

where P_k is a polynomial in the variables x_i ($i \leq k-1$) and $P_1 \equiv 1$. The current paper considers the situation when the vector fields are of the form

$$X_k(p) := \sigma_k(p)\frac{\partial}{\partial x_k} \text{ for } k \leq n, \tag{1.3}$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ need not be a polynomial when $k > m \geq 1$. Grushin-type spaces defined by vector fields as in (1.2) are known to possess certain desirable properties – e.g. it is known

Received: 04.02.2023; Accepted: 27.04.2023; Published Online: 02.05.2023
*Corresponding author: Zachary Forrest; zachary9@usf.edu
DOI: 10.33205/cma.1245581

that the vector fields Y_j and their commutators

$$[Y_j, Y_k], [Y_j, [Y_k, Y_\ell]], [Y_j, [Y_k, [Y_\ell, Y_m]]], \ldots$$

span $\mathbb{R}^n$ and hence we may apply Chow's Theorem to conclude that points of the related Grushin-type space may be connected by appropriately smooth curves. Spaces defined by vector fields as in (1.3), however, can not be treated this way and require modified techniques.

The article will proceed as follows. In Section 2 we will define the spaces of interest and consider notions of geometry and calculus. The trappings of viscosity theory are introduced in Section 3, and a lemma relating Euclidean and Grushin second-order jets is presented. We conclude with Section 4 in which we produce results necessary to establish a comparison principle for sub- and supersolutions and existence of solutions – the culmination of these results is the theorem below.

Main Theorem. *Let $\mathbb{G}$ be a Grushin-type space whose Lie Algebra consists of vectors fields as defined in the forthcoming section. Then there exists a unique solution to the Dirichlet Problem* (1.1).

2. The Grushin-Type Environment $\mathbb{G}$

Let $n \geq 2$ and $1 \leq m < n$ be given. Fixing any $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$, consider the frame $\{X_i, X_j\}$ containing the vector fields

$$X_i(p) := \frac{\partial}{\partial x_i} \ (1 \leq i \leq m) \tag{2.4}$$

and

$$X_j(p) := \sigma(p)\frac{\partial}{\partial x_j} \ (m+1 \leq j \leq n), \tag{2.5}$$

where we will assume that:

(1) $\sigma(p) = \sigma(x_1, \ldots, x_m)$. That is, $\sigma(p)$ is independent of $x_{m+1}, \ldots, x_n$.
(2) σ is Euclidean C^2 (denoted C^2_{eucl} for what follows).
(3) The set of zeroes for σ is given by $Z \times \mathbb{R}^{n-m}$, where Z is a discrete subset of $\mathbb{R}^m$.

In the case that σ is a polynomial the frame $\{X_i, X_j\}$ defines a generalized Grushin space such as the ones under consideration in [3]; otherwise $\{X_i, X_j\}$ corresponds to a member of a more general class of Grushin-type spaces.

The Lie Algebra $\mathfrak{g} := \operatorname{span}\{X_i, X_j\}$ may be endowed with an inner-product $\langle \cdot, \cdot \rangle$ which is singular on $Z \times \mathbb{R}^{n-m}$ and makes $\{X_i, X_j\}$ an orthonormal basis otherwise. Defining the space $\mathbb{G}$ to be the image of $\mathfrak{g}$ under the exponential map, note that points of $\mathbb{G}$ are also n-tuples $p = (x_1, \ldots, x_n)$ and that the tangent space to $\mathbb{G}$ at any point p is $\mathfrak{g}(p)$. One consequence of this definition is that $\mathbb{G}$ is not a group: Indeed, the dimension of the tangent space to $\mathbb{G}$ at p is $\dim \mathfrak{g}(p)$ which equals m if $p \in Z \times \mathbb{R}^{n-m}$ and otherwise equals n.

The natural metric to impose upon $\mathbb{G}$ is the *Carnot-Carathéodory* (or CC) metric

$$d_{CC}(p, q) := \inf_{\gamma \in \Gamma} \int_0^1 \|\gamma'(t)\| dt, \tag{2.6}$$

where Γ is the collection of all curves γ satisfying **(i)** $\gamma(0) = p, \gamma(1) = q$ and **(ii)** $\gamma' \in \mathfrak{g}$. Because $X_j \equiv 0$ on $Z \times \mathbb{R}^{n-m}$, Chow's Theorem (see, for example, [5]) does not apply. However, since the vector fields X_i are nonzero, points of $\mathbb{G}$ can always be connected by concatenating curves – so $\Gamma \neq \varnothing$ and $d_{CC}(\cdot, \cdot)$ is an honest metric.
We may therefore define balls in $\mathbb{G}$ by

$$B(p_0, r) := \{p \in \mathbb{G} : d_{CC}(p_0, p) < r\}$$

and consider notions of bounded domains, which we shall typically denote by $\Omega \Subset \mathbb{G}$.

Given a smooth function $u : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O} \subseteq \mathbb{G}$ is open, the gradient of u in $\mathbb{G}$ is defined by

$$\nabla_{\mathbb{G}} u := (X_1 u, \ldots, X_n u)$$

and the second derivative matrix $(D^2u)^\star$ is the symmetric $n \times n$ matrix whose entries are given by

$$[(D^2u)^\star]_{k\ell} := \frac{1}{2}\left(X_\ell X_k u + X_k X_\ell u\right).$$

We also have notions of regularity.

Definition 2.1. *A function $u : \mathcal{O} \to \mathbb{R}$ is said to be $C^1_{\mathbb{G}}(\mathcal{O})$ if $X_k u$ is continuous for each $1 \le k \le n$. The function u is $C^2_{\mathbb{G}}(\mathcal{O})$ if $X_\ell X_k u$ is continuous for each $1 \le k, \ell \le n$.*

Finally, given $1 \le \mathrm{p} \le \infty$, we also may define the function spaces $L^{\mathrm{p}}(\mathcal{O}), L^{\mathrm{p}}_{\text{loc}}(\mathcal{O}), W^{1,\mathrm{p}}(\mathcal{O})$ and $W^{1,\mathrm{p}}_{\text{loc}}(\mathcal{O})$ in the obvious way.

3. Jets & Viscosity Solutions

With the appropriate definitions of derivatives and function spaces introduced in the previous section, we turn our attention to homogeneous PDEs of the form

$$H(p, \eta, X) = 0 \tag{3.7}$$

for $\eta \in \mathbb{R}^n$ and symmetric $n \times n$ matrices X (frequently denoted $X \in \mathcal{S}^n$). The operators H will be continuous and *proper* in the sense of [6]: That is, for $X \le Y$ we will have $H(p,\eta,Y) \le H(p,\eta,X)$. Specifically, assuming that w is smooth, we will have interest in the ∞-Laplace operator

$$\Delta_\infty w := -\left\langle \left(D^2 w\right)^\star \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} w \right\rangle;$$

the related p-Laplace operators (for $1 < \mathrm{p} < \infty$)

$$\begin{aligned}\Delta_{\mathrm{p}} w &:= -\operatorname{div}\left(\| \nabla_{\mathbb{G}} w\|^{\mathrm{p}-2} \nabla_{\mathbb{G}} w\right)\\ &= -\| \nabla_{\mathbb{G}} w\|^{\mathrm{p}-2} \sum_{a=1}^{n} X_a X_a w + (\mathrm{p}-2)\| \nabla_{\mathbb{G}} w\|^{\mathrm{p}-4} \Delta_\infty w;\end{aligned}$$

and Jensen's Auxiliary Functions (see [7])

$$\mathcal{F}^\varepsilon(p, \nabla_{\mathbb{G}} w, \left(D^2 w\right)^\star) := \min\left\{\| \nabla_{\mathbb{G}} w\|^2 - \varepsilon^2, \Delta_\infty w\right\}$$

and

$$\mathcal{G}^\varepsilon(p, \nabla_{\mathbb{G}} w, \left(D^2 w\right)^\star) := \max\left\{\varepsilon^2 - \| \nabla_{\mathbb{G}} w\|^2, \Delta_\infty w\right\},$$

where $\varepsilon \in \mathbb{R}$ will be given. In what follows, we will use H to represent any of the four operators above.

In order to introduce the machinery of viscosity solutions to $Hw = 0$, we first must consider the following classes of test functions which "touch" the function $u : \mathcal{O} \to \mathbb{R}$. Given an open set $\mathcal{O} \subseteq \mathbb{G}$, a point $p_0 \in \mathcal{O}$, and a function $u : \mathcal{O} \to \mathbb{R}$, we have the so-called "touching above" functions

$$\mathcal{TA}(u, p_0) := \left\{\varphi \in C^2_{\mathbb{G}}(\Omega) : \ 0 = \varphi(p_0) - u(p_0) < \varphi(p) - u(p) \text{ near } p_0\right\};$$

we have also the "touching below" functions at p_0 defined by

$$\mathcal{TB}(u, p_0) := \left\{\varphi \in C^2_{\mathbb{G}}(\Omega) : \ 0 = u(p_0) - \varphi(p_0) < u(p) - \varphi(p) \text{ near } p_0\right\}.$$

Comparisons between the derivatives of smooth functions w and the touching functions φ, and between the operations $Hw, H\varphi$ then lead us to make the following definition.

Definition 3.2. *Let $\Omega \Subset \mathbb{G}$ be a domain and let $u \in \mathrm{USC}(\Omega)$. We say that u is a viscosity subsolution to (3.7) in Ω if the following is satisfied: For every $p \in \Omega$ and each $\varphi \in \mathcal{TA}(u,p)$,*

$$H(p, \nabla_{\mathbb{G}}\, \varphi(p), \big(D^2\varphi\big)^\star(p)) \leq 0.$$

We say that $v \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution to Equation (3.7) if $-v$ is a viscosity subsolution to Equation (3.7). We say that $w \in C(\Omega)$ is a viscosity solution to Equation (3.7) if it is both a viscosity sub- and supersolution.

When convenient, we may also speak in terms of "jets" for a function u at a point p_0.

Definition 3.3. *Given $u : \mathcal{O} \to \mathbb{R}$, we define the second-order upper jet for u by*

$$J^{2,+}\, u(p_0) := \Big\{ \big(\nabla_{\mathbb{G}}\, \varphi(p_0), \big(D^2\varphi\big)^\star(p_0)\big) \in \mathbb{R}^n \times \mathcal{S}^n \,:\, \varphi \in \mathcal{TA}(u,p_0) \Big\}$$

and the second-order lower jet for u by $J^{2,-}\, u(p_0) := -\, J^{2,+}[-u](p_0)$. We say that the ordered pair $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}^n$ belongs to the closure of the upper jet, written $(\eta, X) \in \overline{J}^{2,+}\, u(p_0)$, if there exists $(p_k) \subseteq \mathcal{O}$ and jet entries $(\eta_k, X_k) \in J^{2,+}\, u(p_k)$ so that

$$(p_k, u(p_k), \eta_k, X_k) \to (p_0, u(p_0), \eta, X);$$

the definition for $\overline{J}^{2,-}\, u(p_0)$ is similar.

Remark 3.1. *Definition 3.2 above can also be stated equivalently through the lens of the jet closures: $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution if for every $p \in \Omega$*

$$H(p, \eta, X) \leq 0$$

for each $(\eta, X) \in \overline{J}^{2,+}\, u(p)$. Similar restatements can be made for viscosity supersolutions and viscosity solutions.

Remark 3.2. *If it should happen that $H = \Delta_{\mathrm{p}}$, then we will call solutions to (3.7) p-harmonic; if $H = \Delta_\infty$, then we call solutions to (3.7) infinite harmonic.*

The jets for $\mathbb{G}$ can be related to Euclidean jets via the following lemma, which is an application of [4, Corollary 3.2].

Lemma 3.1 (The $\mathbb{G}$ Twisting Lemma). *Let $\mathcal{O} \subseteq \mathbb{G}$ be open, let $u : \mathcal{O} \to \mathbb{R}$, and let $p_0 \in \mathcal{O}$. Suppose that we know $(\eta, X) \in J^{2,+}_{\text{eucl}}(u, p_0)$: Then*

$$\Big(\boldsymbol{A}(p_0) \cdot \eta, \boldsymbol{A}(p_0) \cdot X \cdot \boldsymbol{A}^{\mathrm{T}}(p_0) + \boldsymbol{M}(\eta, p_0) \Big) \in J^{2,+}(u, p_0), \tag{3.8}$$

where

$$\big(\boldsymbol{A}(p_0)\big)_{k\ell} = \begin{cases} 1, & k = \ell \leq m \\ \sigma(p_0), & m+1 \leq k = \ell \leq n \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

and

$$\big(\boldsymbol{M}(\eta, p_0)\big)_{k\ell} = \begin{cases} \dfrac{1}{2} \cdot \dfrac{\partial \sigma}{\partial x_k}(p_0)\eta_\ell, & 1 \leq k \leq m < \ell \leq n \\ \dfrac{1}{2} \cdot \dfrac{\partial \sigma}{\partial x_\ell}(p_0)\eta_k, & 1 \leq \ell \leq m < k \leq n \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Proof. The result in (3.8) is known (see [4, Corollary 3.2] and [1, Lemma 3]); we shall restrict our attention to verifying Equations (3.9) and (3.10). The $n \times n$ matrix $\boldsymbol{A}$ is defined by [4] as $\boldsymbol{A}(p) := (A_{k\ell}(p))$, where

$$X_k(\cdot) = \sum_{\ell=1}^{n} A_{k\ell}(\cdot)\frac{\partial}{\partial x_\ell}.$$

The definitions (2.4) and (2.5) imply:

(1) $A_{k\ell} \equiv 0$ if $k \neq \ell$;
(2) $A_{kk} \equiv 1$ if $k \leq m$ and $A_{kk} = \sigma$ if $m+1 \leq k \leq n$.

This justifies (3.9).

To verify (3.10), recall the definition of $\boldsymbol{M}(\eta, p_0)$ in [4]:

$$(\boldsymbol{M}(\eta,p_0))_{k\ell} := \begin{cases} \dfrac{1}{2}\displaystyle\sum_{r=1}^{n}\sum_{s=1}^{n}\left(A_{ks}(p_0)\frac{\partial A_{\ell r}}{\partial x_s}(p_0) + A_{\ell s}(p_0)\frac{\partial A_{kr}}{\partial x_s}(p_0)\right)\eta_r, & k \neq \ell \\ \displaystyle\sum_{r=1}^{n}\sum_{s=1}^{n} A_{ks}(p_0)\frac{\partial A_{kr}}{\partial x_s}(p_0)\eta_r, & k = \ell. \end{cases}$$

Because $A_{rs} \equiv 0$ whenever $r \neq s$, we may simplify the equation above:

$$\begin{aligned}(\boldsymbol{M}(\eta,p_0))_{k\ell} =& \frac{1}{2}\sum_{r=1}^{n}\left(\left(A_{kk}(p_0)\frac{\partial A_{\ell r}}{\partial x_k}(p_0) + 0\right) + \left(0 + A_{\ell\ell}(p_0)\frac{\partial A_{kr}}{\partial x_\ell}(p_0)\right)\right)\eta_r \\ =& \frac{1}{2}\left(A_{kk}(p_0)\frac{\partial A_{\ell\ell}}{\partial x_k}(p_0)\eta_\ell + A_{\ell\ell}(p_0)\frac{\partial A_{kk}}{\partial x_\ell}(p_0)\eta_k\right) \text{ if } k \neq \ell,\end{aligned} \tag{3.11}$$

and

$$(\boldsymbol{M}(\eta,p_0))_{kk} = \sum_{r=1}^{n} A_{kk}(p_0)\frac{\partial A_{kr}}{\partial x_k}(p_0)\eta_r = A_{kk}(p_0)\frac{\partial A_{kk}}{\partial x_k}(p_0)\eta_k \text{ if } k = \ell. \tag{3.12}$$

First consider Equation (3.12). If $k = 1, \ldots, m$, then $\partial A_{kk}/\partial x_k \equiv 0$. If $k = m+1, \ldots, n$ we also have $\partial A_{kk}/\partial x_k \equiv 0$ because σ is independent of the variables $x_{m+1}, \ldots, x_n$. Hence, $(\boldsymbol{M}(\eta,p_0))_{kk} = 0$ for all $k \leq n$.

Turning our attention to Equation (3.11), we reduce the expression utilizing Item 2 and the definition of σ:

- If $k, \ell \leq m$, then $A_{kk} \equiv 1 \equiv A_{\ell\ell}$ and hence
$$(\boldsymbol{M}(\eta,p_0))_{k\ell} = \frac{1}{2}\left(1\cdot 0\cdot\eta_\ell + 1\cdot 0\cdot\eta_k\right) = 0.$$
- If $k \leq m < \ell \leq n$, then $A_{kk} \equiv 1$ and $A_{\ell\ell} = \sigma$. Since σ is constant with respect to $x_{m+1}, \ldots, x_n$,
$$\begin{aligned}(\boldsymbol{M}(\eta,p_0))_{k\ell} =& \frac{1}{2}\left(1\cdot\frac{\partial\sigma}{\partial x_k}(p_0)\eta_\ell + \sigma(p_0)\cdot 0\cdot\eta_k\right) \\ =& \frac{1}{2}\cdot\frac{\partial\sigma}{\partial x_k}(p_0)\eta_\ell.\end{aligned}$$
- If $\ell \leq m < k \leq n$, then work similar to the above shows
$$(\boldsymbol{M}(\eta,p_0))_{k\ell} = \frac{1}{2}\cdot\frac{\partial\sigma}{\partial x_\ell}(p_0)\eta_k.$$
- If $m < k, \ell \leq n$, then $A_{kk} = \sigma = A_{\ell\ell}$ and so
$$(\boldsymbol{M}(\eta,p_0))_{k\ell} = \frac{1}{2}\left(\sigma(p_0)\cdot 0\cdot\eta_\ell + \sigma(p_0)\cdot 0\cdot\eta_k\right) = 0.$$

We conclude from the above that the matrix given by (3.10) is indeed $M(\eta, p_0)$. □

4. Uniqueness of Infinite Harmonic Functions

It is standard knowledge (see, for example, [2] and [8]) that there exist solutions to the Equation (3.7), so we turn our attention to uniqueness of these solutions. This will be achieved by proving uniqueness for the operators $\mathcal{F}^\varepsilon$ and $\mathcal{G}^\varepsilon$, and will rely upon the properties of jet entries.

4.1. **Iterated Maximum Principle & Estimates on Derivatives.**

The focus of this subsection is Lemma 4.4, which requires the Iterated Maximum Principle of [3]. As we shall show in Lemma 4.4, the Iterated Maximum Principle gives conditions for finding points possessing nonempty jet closures for viscosity sub- and supersolutions – this will enable us to produce necessary estimates on the "viscosity derivatives". As in [6], we will have need for a "penalty function"; specifically, we make use of the function

$$\varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p,q) = \varphi_{\vec{\tau}}(p,q) := \frac{1}{2}\sum_{k=1}^{n} \tau_k (x_k - y_k)^2,$$

where the entries of $\vec{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, \tau_n)$ are positive, real numbers. The use of n real parameters as opposed to the one employed by [6] allows us to take the set $Z \times \mathbb{R}^{n-m}$ into account.

Lemma 4.2 (The Iterated Maximum Principle). *Let $\Omega \Subset \mathbb{G}$ be a domain, $u \in \mathrm{USC}(\Omega)$, and $v \in \mathrm{LSC}(\Omega)$; assume that there exists some $p_0 \in \Omega$ so that*

$$u(p_0) - v(p_0) > 0.$$

Let $\vec{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \in \mathbb{R}^n$ have positive coordinates and, for each pair of points in $\mathbb{G}$ $p = (x_1, x_2, x_3, \ldots, x_n), q = (y_1, y_2, y_3, \ldots, y_n)$ define the functions

$$\begin{aligned}
\varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=1}^{n} \tau_k (x_k - y_k)^2 \\
\varphi_{\tau_2,\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=2}^{n} \tau_k (x_k - y_k)^2 \\
\varphi_{\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=3}^{n} \tau_k (x_k - y_k)^2 \\
&\vdots \\
\varphi_{\tau_n}(p,q) &:= \frac{1}{2}\tau_n (x_n - y_n)^2.
\end{aligned}$$

Appealing to the compactness of $\overline{\Omega}$ and to upper semicontinuity, we may also define

$$\begin{aligned} M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}(p,q)\} \\ &= u(p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) - v(q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) - \varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) \\ M_{\tau_2,\tau_3,\dots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_2,\tau_3,\dots,\tau_n}(p,q) : x_1 = y_1\} \\ &= u(p_{\tau_2,\tau_3,\dots,\tau_n}) - v(q_{\tau_2,\tau_3,\dots,\tau_n}) - \varphi_{\tau_2,\tau_3,\dots,\tau_n}(p_{\tau_2,\tau_3,\dots,\tau_n}, q_{\tau_2,\tau_3,\dots,\tau_n}) \\ M_{\tau_3,\dots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_3,\dots,\tau_n}(p,q) : x_k = y_k,\ k = 1,2\} \\ &= u(p_{\tau_3,\dots,\tau_n}) - v(q_{\tau_3,\dots,\tau_n}) - \varphi_{\tau_3,\dots,\tau_n}(p_{\tau_3,\dots,\tau_n}, q_{\tau_3,\dots,\tau_n}) \\ &\ \vdots \\ M_{\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_3,\dots,\tau_n}(p,q) : x_k = y_k,\ k = 1,\dots,n-1\} \\ &= u(p_{\tau_n}) - v(q_{\tau_n}) - \varphi_{\tau_n}(p_{\tau_n}, q_{\tau_n}). \end{aligned}$$

Then

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_3\to\infty}\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty} M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n} = u(p_0) - v(p_0)$$

and

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_3\to\infty}\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) = 0.$$

Additionally, the first ℓ coordinates of $p_{\tau_{\ell+1},\dots,\tau_n}$ and $q_{\tau_{\ell+1},\dots,\tau_n}$ are identical – that is,

$$x_k^{\tau_{\ell+1},\dots,\tau_n} = y_k^{\tau_{\ell+1},\dots,\tau_n},\ k = 1,\dots,\ell.$$

The proof of the Iterated Maximum Principle leads immediately to the following results which permit us to take the parameters $\tau_k \to \infty$ in any order, and to speak of the full limit as $\tau_{k_1}, \tau_{k_2}, \dots, \tau_{k_n} \to \infty$.

Corollary 4.1 (cf. [3, Corollary 4.4])**.** *Under the conditions of Lemma 4.2, each iterated limit of $M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}$ exists and is equal to $u(p_0) - v(p_0)$ – in other words,*

$$\lim_{\tau_{k_1}\to\infty}\cdots\lim_{\tau_{k_{n-2}}\to\infty}\lim_{\tau_{k_{n-1}}\to\infty}\lim_{\tau_{k_n}\to\infty} M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n} = u(p_0) - v(p_0).$$

Consequently,

$$\lim_{\tau_{k_1}\to\infty}\cdots\lim_{\tau_{k_{n-2}}\to\infty}\lim_{\tau_{k_{n-1}}\to\infty}\lim_{\tau_{k_n}\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) = 0.$$

Lemma 4.3 (cf. [3, Lemma 4.5])**.** *Under the conditions of Lemma 4.2, the full limit of $M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}$ exists and is equal to $u(p_0) - v(p_0)$ – more precisely,*

$$\lim_{\tau_n,\dots,\tau_3,\tau_2,\tau_1\to\infty} M_{\tau_1,\tau_2,\tau_3,\dots,\tau_n} = u(p_0) - v(p_0).$$

In addition,

$$\lim_{\tau_n,\dots,\tau_3,\tau_2,\tau_1\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}) = 0.$$

Remark 4.3. *Owing to Lemma 4.3, there is no ambiguity in relabeling the intermediate points $p_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}$, $q_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}$ and function $\varphi_{\tau_1,\tau_2,\tau_3,\dots,\tau_n}$ as $p_{\vec{\tau}}$, $q_{\vec{\tau}}$, and $\varphi_{\vec{\tau}}$. We will also denote the coordinates of $p_{\vec{\tau}}$, $q_{\vec{\tau}}$ as $x_k^{\vec{\tau}}$, $y_k^{\vec{\tau}}$ respectively.*

Applying the results above and [6, Theorem 3.2], we have the following estimates.

Lemma 4.4. *Let $u, v, \varphi_{\vec{\tau}}$, and $(p_{\vec{\tau}}, q_{\vec{\tau}})$ be as in Lemma 4.2 and assume additionally that at least one of u, v is locally $\mathbb{G}$-Lipschitz. Then:*

(A) There exist $(\eta^+_{\vec{\tau}}, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+} u(p_{\vec{\tau}})$ and $(\eta^-_{\vec{\tau}}, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-} v(q_{\vec{\tau}})$.

(B) Define $(p \diamond q)_k$ to be the point whose k-th coordinate coincides with q and whose other coordinates coincide with p – in other words,

$$(p \diamond q)_k = (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n).$$

Then for each index k,

$$\tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 \lesssim d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right). \tag{4.13}$$

For the indices $i \leq m$,

$$\tau_i \left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right| = O(1) \text{ as } \tau_i \to \infty. \tag{4.14}$$

(C) The vector estimate

$$\left| \left\| \eta^+_{\vec{\tau}} \right\|^2 - \left\| \eta^-_{\vec{\tau}} \right\|^2 \right| = o(1) \ \text{ as } \tau_k \to \infty \text{ for all } k \leq n \tag{4.15}$$

holds.

(D) The matrix estimate

$$\left\langle \mathcal{X}^{\vec{\tau}} \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}} \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle = o(1) \ \text{ as } \tau_k \to \infty \text{ for all } k \leq n \tag{4.16}$$

holds.

Proof. For clarity, we split the proof between the items above.

Item (A).

[6, Theorem 3.2] guarantees the existence of elements in the Euclidean jet closures: In particular, for each fixed $\delta > 0$ we will have

$$\left(D_p \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), X^{\vec{\tau}}\right) \in \overline{J}^{2,+}_{\text{eucl}} u(p_{\vec{\tau}}) \text{ and } \left(-D_q \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), Y^{\vec{\tau}}\right) \in \overline{J}^{2,-}_{\text{eucl}} v(q_{\vec{\tau}}).$$

Applying the $\mathbb{G}$ Twisting Lemma (Lemma 3.1) produces the members $(\eta^+_{\vec{\tau}}, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+} u(p_{\vec{\tau}})$ and $(\eta^-_{\vec{\tau}}, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-} v(q_{\vec{\tau}})$.

Item (B).

By the definition of the points $p_{\vec{\tau}}, q_{\vec{\tau}}$, for all points $p, q \in \Omega$ the inequality

$$u(p) - v(q) - \varphi_{\vec{\tau}}(p, q) \leq u(p_{\vec{\tau}}) - v(q_{\vec{\tau}}) - \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}})$$

is satisfied. Hence assuming (without loss of generality) that u is $\mathbb{G}$-Lipschitz, decreeing $p := (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k$ and $q := q_{\vec{\tau}}$, and recollecting terms, we obtain

$$\begin{aligned} \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 &= \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) - \varphi_{\vec{\tau}}\left((p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k, q_{\vec{\tau}}\right) \\ &\leq u(p_{\vec{\tau}}) - u\left((p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right) \\ &\leq K\, d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right), \end{aligned} \tag{4.17}$$

where K is the Lipschitz constant for u. This is Inequality (4.13), so to complete 4.4 we turn our attention to to the expression $\tau_i \left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right|$ $(i \leq m)$. If $x_i^{\vec{\tau}} \neq y_i^{\vec{\tau}}$ then (4.17) shows

$$\tau_i \left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right| = \tau_i (x_i^{\vec{\tau}} - y_i^{\vec{\tau}})^2 \cdot \frac{1}{\left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right|} \leq \frac{K\, d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_i\right)}{\left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right|} \text{ as } \tau_1, \cdots, \tau_n \to \infty. \tag{4.18}$$

Note that

$$d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_i\right) \leq \left| x_i^{\vec{\tau}} - y_i^{\vec{\tau}} \right| \tag{4.19}$$

as a consequence of [5, Theorem 7.34]. Combining (4.18) and (4.19) proves Equation (4.14) and completes the proof of 4.4.

Item (C).

Observe that

$$\frac{\partial}{\partial x_k}\varphi(p_{\vec{\tau}}, q_{\vec{\tau}}) = \tau_k(x_k^{\vec{\tau}} - q_k^{\vec{\tau}}) = -\frac{\partial}{\partial y_k}\varphi(p_{\vec{\tau}}, q_{\vec{\tau}});$$

consequently, referring back to the definition of the matrix $\boldsymbol{A}$, the coordinates of $\eta_{\vec{\tau}}^+$ and $\eta_{\vec{\tau}}^-$ are

$$[\eta_{\vec{\tau}}^+]_k = \begin{cases} \tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), & \text{if } k \leq m \\ \tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}})\sigma(p_{\vec{\tau}}), & \text{if } m+1 \leq k \leq n \end{cases}$$

and

$$[\eta_{\vec{\tau}}^-]_k = \begin{cases} \tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), & \text{if } k \leq m \\ \tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}})\sigma(q_{\vec{\tau}}), & \text{if } m+1 \leq k \leq n. \end{cases}$$

Fixing $\vec{\tau}$ for the moment, this leads to the estimate

$$\left| \|\eta_{\vec{\tau}}^+\|^2 - \|\eta_{\vec{\tau}}^-\|^2 \right| \leq \sum_{k=m+1}^{n} \left|\sigma^2(p_{\vec{\tau}}) - \sigma^2(q_{\vec{\tau}})\right| \cdot \tau_k^2 \left(x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right)^2. \tag{4.20}$$

The values τ_i for $i \leq m$ are not present in Inequality (4.20). Taking the iterated limits of (4.20) as $\tau_i \to \infty$, recalling that $\sigma(p)$ depends only upon the first m coordinates of p, and applying the Iterated Maximum Principle yields

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} \left| \|\eta_{\vec{\tau}}^+\|^2 - \|\eta_{\vec{\tau}}^-\|^2 \right| = 0.$$

The above implies

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_{m+1} \to \infty} \lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} \left| \|\eta_{\vec{\tau}}^+\|^2 - \|\eta_{\vec{\tau}}^-\|^2 \right| = 0,$$

concluding Item **(C)**.

Item (D).

[6, Theorem 3.2] and the Twisting Lemma imply

$$\left\langle \mathcal{X}^{\vec{\tau}} \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}} \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle = I_1 + I_2,$$

where we define

$$I_1 := \left\langle \left(\boldsymbol{A}(p_{\vec{\tau}}) \cdot X^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})\right) \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \left(\boldsymbol{A}(q_{\vec{\tau}}) \cdot Y^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})\right) \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle$$

and

$$I_2 := \left\langle \boldsymbol{M}(D_p \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}}) \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \boldsymbol{M}(D_q \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}}) \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle. \tag{4.21}$$

Writing $\widetilde{\epsilon} := \boldsymbol{A}(p_{\vec{\tau}}) \cdot \epsilon, \widetilde{\kappa} := \boldsymbol{A}(q_{\vec{\tau}}) \cdot \kappa$ to mean the twisting of $\epsilon, \kappa \in \mathbb{R}^n$ according to the Twisting Lemma,

$$\begin{aligned} \left\langle \boldsymbol{A}(p_{\vec{\tau}}) \cdot X^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})\epsilon, \epsilon \right\rangle - \left\langle \boldsymbol{A}(q_{\vec{\tau}}) \cdot Y^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})\kappa, \kappa \right\rangle &= \left\langle X^{\vec{\tau}} \cdot \widetilde{\epsilon}, \widetilde{\epsilon} \right\rangle - \left\langle Y^{\vec{\tau}} \cdot \widetilde{\kappa}, \widetilde{\kappa} \right\rangle \\ &\leq \left\langle \mathcal{C} \cdot \Upsilon, \Upsilon \right\rangle, \end{aligned}$$

where $\Upsilon := (\widetilde{\epsilon}, \widetilde{\kappa})$ and $\mathcal{C}$ is a $2n \times 2n$ block matrix resulting from [6, Theorem 3.2] of the form

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and

$$[B]_{ab} = \begin{cases} \tau_a + 2\delta\tau_a^2, & a = b \\ 0, & a \neq b. \end{cases}$$

(Recall that δ is a consequence of [6, Theorem 3.2].) Choosing $\epsilon := \eta_{\vec{\tau}}^{+}$ and $\kappa := \eta_{\vec{\tau}}^{-}$, the above shows

$$\begin{aligned} I_1 &\leq \left\langle B \cdot \left(\widetilde{\eta_{\vec{\tau}}^{+}} - \widetilde{\eta_{\vec{\tau}}^{-}} \right), \widetilde{\eta_{\vec{\tau}}^{+}} - \widetilde{\eta_{\vec{\tau}}^{-}} \right\rangle \\ &= \sum_{k=m+1}^{n} (\tau_k + 2\delta\tau_k^2)(\sigma^2(p_{\vec{\tau}}) - \sigma^2(q_{\vec{\tau}}))^2 \cdot \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2. \end{aligned} \tag{4.22}$$

The right-hand side of Relation (4.22) is free of the τ_i for $i \leq m$, so proceeding as in the proof of Item **(C)**, we find

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} I_1 = 0$$

so that

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_{m+1} \to \infty} \lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} I_1 = 0. \tag{4.23}$$

For the term I_2, let us begin by simplifying the notation for the matrix $\boldsymbol{M}(\cdot,\cdot)$. Appealing to Equation (3.10) in the Twisting Lemma, we see that

$$\boldsymbol{M}(D_p\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}}) = \begin{pmatrix} 0 & S(p_{\vec{\tau}}) \\ S(p_{\vec{\tau}})^T & 0 \end{pmatrix}$$

and

$$\boldsymbol{M}(D_q\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}}) = \begin{pmatrix} 0 & S(q_{\vec{\tau}}) \\ S(q_{\vec{\tau}})^T & 0 \end{pmatrix},$$

where, permitting t to represent either the point $p_{\vec{\tau}}$ or $q_{\vec{\tau}}$, the $m \times (n-m)$ matrix $S(t)$ is defined by

$$[S(t)]_{rs} := \frac{1}{2} \cdot \frac{\partial \sigma}{\partial x_r}(t) \cdot \tau_s (x_s^{\vec{\tau}} - y_s^{\vec{\tau}}).$$

Calculations with (4.21) show

$$\begin{aligned} I_2 &= \sum_{\ell=m+1}^{n} \sum_{r=1}^{m} \frac{\partial \sigma}{\partial x_r}(p_{\vec{\tau}}) \cdot \tau_r (x_r^{\vec{\tau}} - y_r^{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \sigma(p_{\vec{\tau}}) \\ &- \sum_{\ell=m+1}^{n} \sum_{r=1}^{m} \frac{\partial \sigma}{\partial x_r}(q_{\vec{\tau}}) \cdot \tau_r (x_r^{\vec{\tau}} - y_r^{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \sigma(q_{\vec{\tau}}). \end{aligned}$$

We adopt the notation

$$T_{r\ell} := \tau_r (x_r^{\vec{\tau}} - y_r^{\vec{\tau}}) \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right)(p_{\vec{\tau}}) - \tau_r (x_r^{\vec{\tau}} - y_r^{\vec{\tau}}) \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right)(q_{\vec{\tau}})$$

for the (r, ℓ)-term of I_2. Since the Iterated Maximum Principle implies

$$p_{\vec{\tau}} \to (x_1^0, \ldots, x_i^0, x_{i+1}^{\vec{\tau}}, \ldots, x_n^{\vec{\tau}}) \text{ and } q_{\vec{\tau}} \to (x_1^0, \ldots, x_i^0, y_{i+1}^{\vec{\tau}}, \ldots, y_n^{\vec{\tau}})$$

as $\tau_1, \ldots \tau_i \to \infty$ $(i \le m)$, and since $1 \le r \le m < \ell \le n$ and $\sigma \in C^2_{\text{eucl}}$, we obtain the iterated limit

$$\lim_{\tau_i\to\infty}\cdots\lim_{\tau_1\to\infty} T_{r\ell} = \tau_r(x_r^{\vec\tau}-y_r^{\vec\tau})\tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_i^0,x_{i+1}^{\vec\tau},\ldots,x_n^{\vec\tau})$$
$$-\tau_r(x_r^{\vec\tau}-y_r^{\vec\tau})\tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_i^0,y_{i+1}^{\vec\tau},\ldots,y_n^{\vec\tau})$$

if $i < r$; if $r \le i$ we may apply Item 4.4, Inequality (4.14), and arrive at

$$\lim_{\tau_i\to\infty}\cdots\lim_{\tau_1\to\infty} T_{r\ell} \approx \tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_i^0,x_{i+1}^{\vec\tau},\ldots,x_n^{\vec\tau})$$
$$-\tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_i^0,y_{i+1}^{\vec\tau},\ldots,y_n^{\vec\tau}).$$

This second limit in particular implies that

$$\begin{aligned}\lim_{\tau_m\to\infty}\cdots\lim_{\tau_1\to\infty} T_{r\ell} &\approx \tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_m^0,x_{m+1}^{\vec\tau},\ldots,x_n^{\vec\tau})\\ &\quad-\tau_\ell^2(x_\ell^{\vec\tau}-y_\ell^{\vec\tau})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(x_1^0,\ldots,x_m^0,y_{m+1}^{\vec\tau},\ldots,y_n^{\vec\tau})\end{aligned} \tag{4.24}$$

for all $r \le m$. Since $\sigma, \partial\sigma/\partial x_r$ depend only upon the first m coordinates of points p, (4.24) implies

$$\lim_{\tau_m\to\infty}\cdots\lim_{\tau_1\to\infty} I_2 = 0$$

and hence

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_{m+1}\to\infty}\lim_{\tau_m\to\infty}\cdots\lim_{\tau_1\to\infty} I_2 = 0. \tag{4.25}$$

Equation (4.16) then follows from (4.23) and (4.25). □

4.2. **A Comparison Principle & Uniqueness.** With the completion of Lemma 4.4, we prove a comparison principle for viscosity solutions to the Dirichlet problems

$$\begin{cases}\mathcal{F}^\varepsilon\left(p,\nabla_{\mathbb{G}}w(p),\left(D^2w\right)^\star(p)\right) = \min\left\{\|\nabla_{\mathbb{G}}w(p)\|^2-\varepsilon^2,\Delta_\infty w(p)\right\} = 0, & p\in\Omega\\ w(p)=g(p), & p\in\partial\Omega\end{cases} \tag{4.26}$$

and

$$\begin{cases}\mathcal{G}^\varepsilon\left(p,\nabla_{\mathbb{G}}w(p),\left(D^2w\right)^\star(p)\right) = \max\left\{\varepsilon^2-\|\nabla_{\mathbb{G}}w(p)\|^2,\Delta_\infty w(p)\right\} = 0, & p\in\Omega\\ w(p)=g(p), & p\in\partial\Omega\end{cases} \tag{4.27}$$

in order to prove the uniqueness of solutions to

$$\begin{cases}\Delta_\infty w(p) = \left\langle D^2w(p)\cdot\nabla_{\mathbb{G}}w(p),\nabla_{\mathbb{G}}w(p)\right\rangle = 0, & p\in\Omega\\ w(p)=g(p), & p\in\partial\Omega\end{cases} \tag{4.28}$$

As before, Ω is a bounded domain; we also assume $g \in C(\partial\Omega)$. In the interest of maintaining clear notation, we establish the following convention.

Definition 4.4. *A viscosity supersolution to Problems (4.26), (4.27), or (4.28) is a viscosity supersolution v to the equations $\mathcal{F}^\varepsilon = 0$, $\mathcal{G}^\varepsilon = 0$, or $\Delta_\infty = 0$ (respectively) such that $v \ge g$ on $\partial\Omega$; a viscosity supersolution u to Problems (4.26), (4.27), or (4.28) is defined similarly. A viscosity solution to any of the above three Dirichlet problems is both a viscosity sub- and supersolution to the problem in the above sense.*

Theorem 4.1. *Suppose that u, v are sub- and supersolutions to Problem (4.26) or Problem (4.27) such that at least one of the functions is locally $\mathbb{G}$-Lipschitz in Ω. Then $u \leq v$ in Ω.*

Proof. We will complete the proof for Problem (4.26) and note that the proof for Problem (4.27) is similar. Suppose, to the contrary of the theorem, that there exists some $p_0 \in \Omega$ such that

$$u(p_0) - v(p_0) = \max_{\overline{\Omega}}(u - v) > 0.$$

Appealing to Lemma 5.1 and Theorem 5.3 in [2], we may assume that v is a strict viscosity supersolution to $\mathcal{F}^{\varepsilon} = 0$ – that is, there exists $\mu(p) > 0$ so that

$$\mathcal{F}^{\varepsilon}\left(\nabla_{\mathbb{G}}\, v(p), \left(D^2 v\right)^{\star}(p)\right) = \mu(p) > 0$$

holds in the viscosity sense for each $p \in \Omega$. Applying Lemma 4.4 to produce the sequence of ordered pairs $(p_{\vec{\tau}}, q_{\vec{\tau}}) \in \Omega \times \Omega$, we have

$$\begin{aligned} 0 < \mu(q_{\vec{\tau}}) &\leq \mathcal{F}^{\varepsilon}\left(\eta^-_{\vec{\tau}}, \mathcal{Y}^{\vec{\tau}}\right) - \mathcal{F}^{\varepsilon}\left(\eta^+_{\vec{\tau}}, \mathcal{X}^{\vec{\tau}}\right) \\ &= \min\left\{\left\|\eta^-_{\vec{\tau}}\right\|^2 - \varepsilon^2, -\left\langle \mathcal{Y}^{\vec{\tau}}\eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\} - \min\left\{\left\|\eta^+_{\vec{\tau}}\right\|^2 - \varepsilon^2, -\left\langle \mathcal{X}^{\vec{\tau}}\eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle\right\} \\ &\leq \max\left\{\left\|\eta^-_{\vec{\tau}}\right\|^2 - \left\|\eta^+_{\vec{\tau}}\right\|^2, \left\langle \mathcal{X}^{\vec{\tau}}\eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}}\eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\}. \end{aligned} \tag{4.29}$$

[2, Lemma 5.1], [2, Theorem 5.3], Lemma 4.4, and Lemma 4.2 imply

$$\mu(q_{\vec{\tau}}) \to \mu(p_0) > 0 \tag{4.30}$$

and that

$$\max\left\{\left\|\eta^-_{\vec{\tau}}\right\|^2 - \left\|\eta^+_{\vec{\tau}}\right\|^2, \left\langle \mathcal{X}^{\vec{\tau}}\eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}}\right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}}\eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}}\right\rangle\right\} \to 0 \tag{4.31}$$

as $\tau_1, \ldots, \tau_n \to \infty$; in other words, for $\tau_1, \ldots, \tau_n$ sufficiently large, we may combine (4.29), (4.30), and (4.31) and produce a contradiction. □

Because viscosity solutions are both viscosity sub- and supersolutions, Theorem 4.1 implies that solutions to (4.26) and (4.27) are unique. Observing that viscosity solutions to (4.26) are viscosity supersolutions to (4.28) and that viscosity solutions to (4.27) are viscosity subsolutions to (4.28), we may therefore conclude that solutions to (4.28) are unique by an application of the lemma below.

Lemma 4.5 (cf. [2, Lemma 5.6])**.** *Let u^{ε} and u_{ε} be solutions to the Dirichlet Problems (4.26) and (4.27) respectively. Given $\delta > 0$, there exists $\varepsilon > 0$ such that*

$$u_{\varepsilon} \leq u^{\varepsilon} \leq u_{\varepsilon} + \delta.$$

References

[1] F. Beatrous, T. Bieske and J. Manfredi: *The Maximum Principle for Vector Fields*, Contemp. Math., **370** (2005), Amer. Math. Soc. Providence, RI, 1–9.

[2] T. Bieske: *On Infinite Harmonic Functions on the Heisenberg Group*, Comm. in PDE, **27** (3 & 4) (2002), 727–762 .

[3] T. Bieske: *Lipschitz Extensions on Generalized Grushin Spaces*, Michigan Math. J., **53** (1) (2005), 3–31.

[4] T. Bieske: *A Sub-Riemannian Maximum Principle and its Application to the* p*-Laplacian in Carnot Groups*, Ann. Acad. Sci. Fenn., **37** (2012), 119–134 .

[5] A. Bellaïche: *The Tangent Space in Sub-Riemannian Geometry*, In *Sub-Riemannian Geometry*; Bellaïche, André., Risler, Jean-Jacques., Eds.; Progress in Mathematics; Birkhäuser: Basel, Switzerland. **144**, 1–78 (1996).

[6] M. Crandall, H. Ishii, P.-L. Lions: *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bull. Amer. Math. Soc., **27** (1) (1992), 1–67.

[7] R. Jensen: *Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient*, Arch. Rational. Mech. Anal., **123** (1993), 51–74.

[8] P. Juutinen: *Minimization Problems for Lipschitz Functions via Viscosity Solutions*, Ann. Acad. Sci. Fenn. Math. Diss., **115** (1998).

THOMAS BIESKE
UNIVERSITY OF SOUTH FLORIDA
DEPARTMENT OF MATHEMATICS AND STATISTICS
4202 E. FOWLER AVE. CMC342
TAMPA, FL 33620, USA
ORCID: 0000-0003-2029-0562
E-mail address: tbieske@usf.edu

ZACHARY FORREST
UNIVERSITY OF SOUTH FLORIDA
DEPARTMENT OF MATHEMATICS AND STATISTICS
4202 E. FOWLER AVE. CMC342
TAMPA, FL 33620, USA
ORCID: 0000-0002-6636-0047
E-mail address: zachary9@usf.edu