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Research Article

Non-lightlike Helices Associated with Helical Curves, Relatively Normal-Slant Helices and Isophote Curves in Minkowski 3-space

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Abstract: In this paper, we introduce a new type of non-lightlike general helix that we name non-lightlike associated helix which is associated with a non-lightlike special surface curve. By using the Darboux frame of a surface curve, we generate the position vector of a non-lightlike associated helix in parametric form. We investigate special cases when the non-lightlike surface curve is a helical curve, a relatively normal-slant helix or an isophote curve. In every case, we obtain the position vector of the non-lightlike associated helix by solving differential equations and examples are given for the achieved results.

Keywords: Non-lightlike associated helix, non-lightlike isophote curve, non-lightlike relatively normal-slant helix.

1. Introduction

Geometrical structures of special type such as special surfaces or curves have always been a focus of interest for different disciplines. Without a doubt, the helix curve is the most fascinating of such special geometric structures. A general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix) and a necessary and sufficient condition that a curve to be a general helix is that the ratio of curvature κ to torsion τ be constant [3]. Helices arise in carbon nano-tubes, nano-springs, DNA double and collagen triple helix, α-helices, bacterial flagella in salmonella and escherichia coli, lipid bilayers, bacterial shape in spirochetes, aerial hyphae in actinomycetes, tendrils, horns, screws, springs, vines, helical staircases and sea shells [4, 14, 17]. Helical structures such as hyper-helices are used in fractal geometry [22]. In the realm of computer-aided design and computer graphics, helix shapes can be utilized for describing tool paths, simulating movement, and creating designs for roads, etc. [25].

Instead of tangent, by considering principal normal vector, a new type of special curve called slant helix has been defined by Izumiya and Takeuchi [10]. Later, further studies have been

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done. For instance, Ali investigated the position vector of spacelike slant helices, Ali and Turgut investigated the position vector of timelike slant helices in Minkowski 3-space [1, 2].

A surface curve is a curve that lies on a surface. While properties of any arbitrary curve are examined by Frenet frame, properties of surface curves can also be examined by Darboux frame $\{T, g, n\}$ (see Section 2 for details). On a surface, helical curves, relatively normal-slant helices and isophote curves have been defined considering the vectors of Darboux frame, by the property that the vector T, g and n makes a constant angle with a fixed straight line, respectively. Puig-Pey, Gálvez and Iglesias have studied helical surface curves and for the parametric and the implicit forms of a surface, they introduced a new method of generating helical tool paths [20]. In 2017, Macit and Düldül introduced relatively normal-slant helices and studied their axis in Euclidean 3-space [15]. El Haimi and Chahdi investigated the parametric equations of relatively normal-slant helices also in Euclidean 3-space [8]. Further studies have been done by Yadav and Pal, Yadav and Yadav in Minkowski 3-space [23, 24]. On the other hand, isophote curves have been studied in both Euclidean and Lorentzian spaces [5–7]. An isophote curve on a surface is also a result of Lambert's cosine law in optics. Lambert's cosine law indicates that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle between the surface normal and the light vector. According to this law, the intensity is irrespective of the actual viewpoint; hence the illumination is the same when viewed from any direction [12]. By considering Lambert's law Doğan and Yaylı introduced the geometric description of isophote curves in [7]. Isophote curves have many applications in different areas such as car body construction, local shading of a surface or geometry of surfaces of rotation and canal surfaces [11, 19, 21]. Öztürk, Nešović and Koç Öztürk have presented a method for numerical computing of general helices, relatively normal-slant helices, and isophote curves lying on a non-degenerate surface in Minkowski space $\mathbb{E}_1^3$ [18].

In [16], Önder defined new types of associated helices that are associated with special surface curves such as helical curves, relatively normal-slant helices and isophote curves in Euclidean 3-space. He introduced parametric forms of some special associated helices with respect to Darboux frame of special surface curves.

In this paper, we define new types of non-lightlike associated helices in Minkowski 3-space. We name these new helices as non-lightlike (spacelike or timelike) surface curve-connected (SCC) associated helices and we obtain parametrizations for such helices by considering helical curves, relatively normal-slant helices and isophote curves on a non-lightlike surface in Minkowski 3-space.

2. Preliminaries

Minkowski 3-space which is denoted by $\mathbb{E}_1^3$ is a real vector space endowed with the metric $\langle , \rangle = -dx^2 + dy^2 + dz^2$, where (x, y, z) is a rectangular coordinate system. This metric is also called Lorentzian metric. In $\mathbb{E}_1^3$, a vector u is called spacelike (resp. timelike or lightlike) if $\langle u, u\rangle > 0$ or $u = 0$ (resp. $\langle u, u\rangle < 0$ or $\langle u, u\rangle = 0$). Similarly, a curve is called spacelike (resp. timelike or lightlike) if its velocity vector is spacelike (resp. timelike or lightlike). In the case of surfaces, a surface is called spacelike (timelike or lightlike) if the induced metric on the surface is Riemannian (Lorentzian or degenerate), i.e., the normal vector on the surface is timelike (spacelike or lightlike, respectively) [13]. Throughout this paper, we only consider non-lightlike curves and surfaces. Therefore, whenever we talk about a surface or a curve, we assume that they are either spacelike or timelike.

The Lorentzian cross product for any vectors $u, v \in \mathbb{E}_1^3$ is defined by

$$u \times v = (u_2v_3 - u_3v_2, u_1v_3 - u_3v_1, u_2v_1 - u_1v_2),$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ [13]. The Frenet formulae $\{T, N, B\}$ for a unit speed non-lightlike curve α with arc-length parameter s is given by

$$T' = \kappa N, \;\; N' = \varepsilon_B \kappa T + \tau B, \;\; B' = \varepsilon_T \tau N, \tag{1}$$

where T, N, B are the tangent (velocity) vector, principal normal vector, binormal vector, respectively, $\varepsilon_T = \langle T, T\rangle$, $\varepsilon_B = \langle B, B\rangle$, $'$ denotes derivative with respect to s, κ is curvature and τ is torsion of the curve α. Here, ε_T and ε_B determines the Lorentzian character of the vectors T and B, respectively. If $\varepsilon_T = \varepsilon_B = 1$, then α is a spacelike curve with timelike principal normal vector. If $\varepsilon_T = 1$ and $\varepsilon_B = -1$, then α is a spacelike curve with spacelike principal normal vector. If $\varepsilon_T = -1$, then α is a timelike curve [13].

Let φ be a regular surface in $\mathbb{E}_1^3$ and $\alpha : I \subset \mathbb{R} \to \varphi$ be a non-lightlike smooth curve on φ. Then, the Darboux frame $\{T, g, n\}$ along the surface curve α is well defined and its formulae is given by

$$T' = \kappa_g g + \varepsilon_g k_n n, \;\; g' = \varepsilon_n \kappa_g T + \varepsilon_T \tau_g n, \;\; n' = k_n T + \tau_g g, \tag{2}$$

where T, $g = \varepsilon_g T \times n$, n are tangent vector of α, intrinsic normal, surface normal along α, respectively, k_n is normal curvature, κ_g is geodesic curvature, τ_g is geodesic torsion, $\varepsilon_T = \langle T, T\rangle$, $\varepsilon_g = \langle g, g\rangle$ and $\varepsilon_n = \langle n, n\rangle$. If $\varepsilon_T = \varepsilon_g = 1$, then both φ and α are spacelike. If $\varepsilon_T = 1$ and $\varepsilon_g = -1$, then φ is timelike and α is spacelike. Finally, if $\varepsilon_T = -1$ and $\varepsilon_n = 1$, then both φ and α are timelike [5, 6].

Considering Darboux vector fields defined in [9], we define following vector fields for non-lightlike surface curves on non-lightlike surfaces.

Definition 2.1 *Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, the vector fields D_n, D_r and D_o along α defined by*

$$D_n = -k_n g + \varepsilon_n \kappa_g n, \;\; D_r = -\tau_g T - \kappa_g n, \;\; D_o = \varepsilon_T \tau_g T + \varepsilon_g k_n g$$

are called normal Darboux vector field, rectifying Darboux vector field and osculating Darboux vector field, respectively.

Lemma 2.2 *[16] Let φ be a regular non-lightlike surface and α be a smooth non-lightlike curve on φ with Darboux frame $\{T, g, n\}$, normal curvature k_n, geodesic curvature κ_g and geodesic torsion τ_g. We have the followings:*

(i) α *is a geodesic curve* $\Leftrightarrow$ $\kappa_g = 0$.

(ii) α *is an asymptotic curve* $\Leftrightarrow$ $k_n = 0$.

(iii) α *is a line of curvature* $\Leftrightarrow$ $\tau_g = 0$.

Definition 2.3 *[24] Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, α is called a relatively normal-slant helix if the vector g makes a constant angle with a fixed unit direction.*

Definition 2.4 *[5, 6] Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, α is called an isophote curve if the vector n makes a constant angle with a fixed unit direction.*

Similar to the definition given by Önder in [16], we give the following definition for non-lightlike surface curves in Minkowski 3-space.

Definition 2.5 *Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux vector fields D_n, D_r and D_o. Then, α is called a D_i-Darboux slant helix if the Darboux vector field D_i makes a constant angle with a fixed unit direction, where $i \in \{n, r, o\}$.*

By using the above definitions, we introduce helices associated with special surface curves in the following section.

3. Helices Associated with Surface Curves in $\mathbb{E}_1^3$

Let φ be a regular non-lightlike surface and $\alpha : I \subset \mathbb{R} \to \varphi$ be a smooth, unit speed non-lightlike curve with arc-length parameter s, Frenet frame $\{T, N, B\}$ and Darboux frame $\{T, g, n\}$. We consider another non-lightlike curve $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^3$ which is given by the parametrization

$$\beta(s) = \alpha(s) + x(s)T(s) + y(s)g(s) + z(s)n(s), \tag{3}$$

where $x = x(s)$, $y = y(s)$ and $z = z(s)$ are smooth functions of s. The non-lightlike curve β is called non-lightlike associated curve of surface curve α" or SCC-associated curve", where SCC stands for surface curve connected. As well as the associated curve β might be on φ, it might be totally apart from φ. The position that β is on φ or not relies on the values which the functions x, y, z take. We investigate special cases for the functions x, y, z in the following subsections.

Moreover to the definition of the curve β, considering that β is a general helix it would be called SCC-associated helix. Now, let us differentiate the equation (3) with respect to s by using (1) and (2). As the result of this differentiation, we get

$$\beta'(s) = R_1(s)T(s) + R_2(s)g(s) + R_3(s)n(s), \tag{4}$$

where $R_1 = R_1(s)$, $R_2 = R_2(s)$ and $R_3 = R_3(s)$ are smooth functions of s which are defined by

$$R_1 = x' + \varepsilon_n \kappa_g y + k_n z + 1, \quad R_2 = \kappa_g x + y' + \tau_g z, \quad R_3 = \varepsilon_g k_n x + \varepsilon_T \tau_g y + z'. \tag{5}$$

In the following subsections, we investigate special cases when β is a helix and it is associated with a special surface curve.

3.1. Non-lightlike Helices Associated with Helical Curves on a Surface in $\mathbb{E}_1^3$

In this first subsection, we assume that the tangent vector β' of the non-lightlike associated curve β of any arbitrary non-lightlike surface curve α is linearly dependent with the tangent vector of α. For this special case, from (4), we get $R_1 \neq 0$, $R_2 = 0$, $R_3 = 0$ and thus $\beta'(s) = R_1(s)T(s)$. Let s_β be the arc-length parameter of the associated curve β. Then, from $\beta'(s) = R_1(s)T(s)$, we obtain $ds_\beta = \pm R_1 ds$ and the Frenet vectors of β are computed as

$$\begin{cases} T_\beta = \pm T, \quad N_\beta = \pm \dfrac{1}{\sqrt{|\varepsilon_g \kappa_g^2 + \varepsilon_n k_n^2|}} (\kappa_g g + \varepsilon_g k_n n), \\ B_\beta = \dfrac{\varepsilon_{B_\beta}}{\sqrt{|\varepsilon_g \kappa_g^2 + \varepsilon_n k_n^2|}} (\varepsilon_n \kappa_g n - k_n g) = \varepsilon_{B_\beta} \dfrac{D_n}{\|D_n\|}, \end{cases} \tag{6}$$

where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta \rangle$ and T_β, N_β, B_β are tangent vector, principal normal vector, binormal vector of β, respectively. By using Definition 2.1 and (6), we obtain the following Theorem 3.1:

Theorem 3.1 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(k_n, \kappa_g) \neq (0,0)$ which lies on a regular surface φ with the condition that β' and $\alpha' = T$ are linearly dependent. Then, followings are equivalent:*

(i) β is a helix.

(ii) α is a helical curve on φ.

(iii) α is a D_n-Darboux slant helix on φ.

Remark 3.2 *The non-lightlike helix curve β which is associated with a non-lightlike helical surface curve α can be referred to as: Non-lightlike helical curve-connected associated helix or non-lightlike HCC-associated helix.*

Let us now, investigate special cases when x, y or z vanishes, respectively. Such special cases allow us to determine the position vector of β in parametric form. From (5), we have the following system

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 \neq 0, \quad \kappa_g x + y' + \tau_g z = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' = 0. \tag{7}$$

Case 1: $x = 0$. Then, from (7) we have the system

$$\varepsilon_n \kappa_g y + k_n z + 1 \neq 0, \quad y' + \tau_g z = 0, \quad \varepsilon_T \tau_g y + z' = 0. \tag{8}$$

If $\tau_g \neq 0$, then the solution of system (8) depends on the sign of ε_T. Let $\varepsilon_T = 1$. By using a variable change $t = \int \tau_g(s)ds$, for constants $c_1, c_2 \in \mathbb{R}$ the solution of the system (8) is calculated as

$$y = -c_1 \sinh\left(\int \tau_g(s)ds\right) - c_2 \cosh\left(\int \tau_g(s)ds\right),$$

$$z = c_1 \cosh\left(\int \tau_g(s)ds\right) + c_2 \sinh\left(\int \tau_g(s)ds\right),$$

which we substitute in (3) and obtain the parametric form of the position vector of β as follows

$$\begin{aligned} \beta(s) = \alpha(s) &- \left[c_1 \sinh\left(\int \tau_g(s)ds\right) + c_2 \cosh\left(\int \tau_g(s)ds\right)\right] g(s) \\ &+ \left[c_1 \cosh\left(\int \tau_g(s)ds\right) + c_2 \sinh\left(\int \tau_g(s)ds\right)\right] n(s). \end{aligned} \tag{9}$$

In this case, α, β are spacelike curves and φ is a non-lightlike, i.e., spacelike or timelike, surface.

Let $\varepsilon_T = -1$. Then, for constants $c_3, c_4 \in \mathbb{R}$ the solution of system (8) is given by

$$y = c_3 \cos\left(\int \tau_g(s)ds\right), \quad z = c_4 \sin\left(\int \tau_g(s)ds\right),$$

which similarly leads to the parametric form of the position vector of β as follows

$$\beta(s) = \alpha(s) + c_3 \cos\left(\int \tau_g(s)ds\right) g(s) + c_4 \sin\left(\int \tau_g(s)ds\right) n(s). \tag{10}$$

In this case, α, β are timelike curves and φ is a timelike surface.

If $\tau_g = 0$, then, from second and third equations of system (8), we get $y = c_5$ and $z = c_6$, respectively, where $c_5, c_6 \in \mathbb{R}$ are constants. Therefore, position vector of β curve is given by $\beta(s) = \alpha(s) + c_5 g(s) + c_6 n(s)$.

We can give the following theorem and corollary as results of the above investigation.

Theorem 3.3 *The spacelike (resp. timelike) associated curve β given in (9) (resp. (10)) is a general helix if and only if α is a spacelike (resp. timelike) helical curve on a non-lightlike (resp. timelike) surface φ.*

Remark 3.4 *The spacelike (resp. timelike) associated curve (9) (resp. (10)) can be referred to as: Spacelike (resp. timelike) helical curve-connected associated helix of type 1 or spacelike (resp. timelike) HCC-associated helix of type 1.*

Corollary 3.5 *The helical curve α is a line of curvature if and only if non-lightlike HCC-associated helix has the parametrization $\beta(s) = \alpha(s) + c_5 g(s) + c_6 n(s)$, where $c_5, c_6 \in \mathbb{R}$ are constants.*

Case 2: $y = 0$. From (7), it follows

$$x' + k_n z + 1 \neq 0, \quad \kappa_g x + \tau_g z = 0, \quad \varepsilon_g k_n x + z' = 0, \tag{11}$$

with the condition $(\kappa_g, \tau_g) \neq (0,0)$. If $k_g \neq 0$, then we get $x = -\frac{\tau_g}{\kappa_g} z$ from second equation of system (11). We substitute this equality in the third equation of system (11) and get the differential equation

$$z' - \frac{\varepsilon_g k_n \tau_g}{\kappa_g} z = 0$$

whose solution is $z = c_7 \exp\left(\int \frac{\varepsilon_g k_n \tau_g}{\kappa_g} ds \right)$, where $c_7 \in \mathbb{R}$ is constant. Hence, the position vector of β is given by

$$\beta(s) = \alpha + c_7 \exp\left(\int \frac{\varepsilon_g k_n \tau_g}{\kappa_g} ds \right) \left(-\frac{\tau_g}{\kappa_g} T + n \right). \tag{12}$$

If $\kappa_g = 0$ and $k_n \neq 0$, then we obtain $x = z = 0$ and therefore $\beta(s) = \alpha(s)$.

By the investigation above, the followings can be given.

Theorem 3.6 *The non-lightlike associated curve β given by (12) is a general helix if and only if α is a non-lightlike helical curve on φ.*

Remark 3.7 *The associated curve (12) can be referred to as: spacelike (timelike) helical curve-connected associated helix of type 2 or spacelike (timelike) HCC-associated helix of type 2.*

Corollary 3.8 *(i) The non-lightlike helical curve α is an asymptotic curve with $\kappa_g \neq 0$ if and only if non-lightlike HCC-associated helix of type 2 has the parametrization $\beta(s) = \alpha(s) - \frac{c_5 \tau_g}{\kappa_g} T + c_7 n$, where $c_7 \in \mathbb{R}$ is constant.*

(ii) The non-lightlike helical curve α is a line of curvature if and only if non-lightlike HCC-associated helix of type 2 has the parametrization $\beta(s) = c_7 n$, where $c_7 \in \mathbb{R}$ is constant.

Case 3: $z = 0$. In this case, from (7), we have the following system

$$x' + \varepsilon_n \kappa_g y \neq 0, \quad \kappa_g x + y' = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y = 0, \tag{13}$$

with $(k_n, \tau_g) \neq (0,0)$. If $k_n \neq 0$, then from third equation of system (13), we have $x = -\dfrac{\varepsilon_T \tau_g}{\varepsilon_g k_n} y$. By substituting x in second equation of system (13), we get the following differential equation

$$y' - \frac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} y = 0,$$

whose solution is $y = c_8 \exp\left(\int \dfrac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} ds\right)$, where $c_8 \in \mathbb{R}$ is constant. Hence, the position vector of β is given by

$$\beta(s) = \alpha(s) + c_8 \exp\left(\int \frac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} ds\right)\left(-\frac{\varepsilon_T \tau_g}{\varepsilon_g k_n} T + g\right). \tag{14}$$

If $k_n = 0$, then it follows $x = y = 0$ and $\beta(s) = \alpha(s)$.

By the investigation above, we can give the followings.

Theorem 3.9 *The non-lightlike associated curve β given by (14) is a general helix if and only if α is a non-lightlike helical curve on φ.*

Remark 3.10 *The non-lightlike associated curve (14) can be referred to as: Non-lightlike helical curve-connected associated helix of type 3 or non-lightlike HCC-associated helix of type 3.*

Corollary 3.11 *(i) The non-lightlike helical curve α is a geodesic curve if and only if non-lightlike HCC-associated helix of type 3 has the parametrization $\beta(s) = \alpha(s) - \dfrac{c_8 \varepsilon_T \tau_g}{\varepsilon_g k_n} T + c_6 g$, where $c_8 \in \mathbb{R}$ is constant.*

(ii) The non-lightlike helical curve α is a line of curvature if and only if non-lightlike HCC-associated helix of type 3 has the parametrization $\beta(s) = \alpha(s) + c_8 g$, where $c_8 \in \mathbb{R}$ is constant.

3.2. Non-lightlike Helices Associated with Relatively Normal-slant Helices in $\mathbb{E}_1^3$

This subsection is to investigate non-lightlike associated helices of relatively normal-slant helices. In order to do the mentioned investigation, we assume that tangent vector β' of the associated curve β is linearly dependent with intrinsic normal vector field g of a surface curve α. Then, from

(4), it follows $\beta'(s) = R_2(s)g(s)$ and thus the Frenet vectors T_β, N_β, B_β of β are calculated as

$$\begin{cases} T_\beta = \pm g, \quad N_\beta = \pm \dfrac{1}{\sqrt{\left|\varepsilon_T \kappa_g^2 + \varepsilon_n \tau_g^2\right|}} \left(\varepsilon_n \kappa_g T + \varepsilon_T \tau_g n\right), \\ B_\beta = -\dfrac{\varepsilon_{B_\beta}}{\sqrt{\left|\varepsilon_T \kappa_g^2 + \varepsilon_n \tau_g^2\right|}} \left(\kappa_g n + \tau_g T\right) = \varepsilon_{B_\beta} \dfrac{D_r}{\|D_r\|}, \end{cases} \tag{15}$$

where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta \rangle$. We can give the following theorem by using (15) and Definition 2.1.

Theorem 3.12 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(\kappa_g, \tau_g) \neq (0,0)$ who lies on a regular surface φ with the condition that β' and intrinsic normal g are linearly dependent. Then, followings are equivalent:*

(i) β is a helix.

(ii) α is a relatively normal-slant helix on φ.

(iii) α is a D_r-Darboux slant helix on φ.

Remark 3.13 *The non-lightlike helix β which is associated with relatively normal-slant helix α can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix or non-lightlike RNS-HC-associated helix.*

Investigating when x, y, z functions have special values leads us to the following cases. From (5), we have

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad \kappa_g x + y' + \tau_g z \neq 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' = 0. \tag{16}$$

Case 1: $x = 0$. Then, the system (16) is reduced to

$$\varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad y' + \tau_g z \neq 0, \quad \varepsilon_T \tau_g y + z' = 0 \tag{17}$$

with $(k_n, \kappa_g) \neq (0,0)$. If $\kappa_g \neq 0$, then first and third equations of system (16) yields the following linear differential equation

$$z' - \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} z = \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g},$$

whose solution can be calculated as

$$z = \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right],$$

where $c_9 \in \mathbb{R}$ is constant. Then, position vector of associated curve beta is given by

$$\begin{aligned} \beta(s) = \alpha(s) &- \frac{1 + k_n \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right]}{\varepsilon_n \kappa_g} g \\ &+ \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right] n. \end{aligned} \tag{18}$$

If $\kappa_g = 0$ and $\tau_g \neq 0$, then from the first equation of system (16), we get $z = -\dfrac{1}{k_n}$. Since $z' = \dfrac{k_n'}{k_n^2}$, from the third equation of system (16), it follows $y = -\dfrac{k_n'}{\varepsilon_T k_n^2 \tau_g}$. Thus, associated curve beta is given with the position vector

$$\beta(s) = \alpha(s) - \frac{k_n'}{\varepsilon_T k_n^2 \tau_g} g - \frac{1}{k_n} n. \tag{19}$$

Theorem 3.14 *The non-lightlike associated curve β given in (18) (resp. (19)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.15 *The non-lightlike associated curve (18) (resp. (19)) can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix of type 1 or non-lightlike RNS-HC-associated helix of type 1.*

Corollary 3.16 *(i) The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization $\beta(s) = \alpha - \dfrac{1}{\varepsilon_n \kappa_g} g + \left(\displaystyle\int \dfrac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_7 \right) n$.*

(ii) The non-lightlike relatively normal-slant helix α is a geodesic curve on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization in (19).

(iii) The non-lightlike relatively normal-slant helix α is a line of curvature on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) - \dfrac{c_7 k_n + 1}{\varepsilon_n \kappa_g} g + c_7 n$.

Case 2: $y = 0$. The system (16) becomes

$$x' + k_n z = 0, \quad \kappa_g x + \tau_g z \neq 0, \quad \varepsilon_g k_n x + z' = 0. \tag{20}$$

If $k_n \neq 0$, then, from system (20), the following differential equation is derived

$$z'' - \frac{k_n'}{k_n} z' - \varepsilon_g k_n^2 z = \varepsilon_g k_n, \tag{21}$$

whose homogeneous part can be obtained with the aid of a variable change $t = \int k_n ds$ as follows

$$\frac{d^2 z}{dt^2} - \varepsilon_g z = 0. \tag{22}$$

The differential equation (22) has two different types of solutions with respect to the value of ε_g.

Let $\varepsilon_g = 1$. In this case, β is a spacelike curve. Then, the general solution of (21) is obtained as follows

$$\begin{aligned} z = {} & c_{10}\cosh\left(\int k_n ds\right) + c_{11}\sinh\left(\int k_n ds\right) \\ & - \cosh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right) ds + \sinh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right) ds, \end{aligned} \tag{23}$$

where $c_{10}, c_{11} \in \mathbb{R}$ are constants. This leads us to

$$\begin{aligned} x = {} & -c_{10}\sinh\left(\int k_n ds\right) - c_{11}\cosh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right) ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right) ds \end{aligned} \tag{24}$$

since $x = -\dfrac{z'}{k_n}$ from the third equation of system (20). In this case, β is a spacelike curve and α is a spacelike (resp. timelike) curve on a spacelike (resp. timelike) surface. Thus, by using (23) and (24), the position vector of spacelike associated curve β is given as follows

$$\begin{aligned} \beta(s) = {} & \alpha(s) + \Bigg[-c_{10}\sinh\left(\int k_n ds\right) - c_{11}\cosh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right) ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right) ds\Bigg] T \\ & + \Bigg[c_{10}\cosh\left(\int k_n ds\right) + c_{11}\sinh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right) ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right) ds\Bigg] n. \end{aligned} \tag{25}$$

Let $\varepsilon_g = -1$. In this case, T and n become spacelike vectors. Then, we get φ is a timelike surface, α is a spacelike curve and β is a timelike curve. Similar to the previous case, the general solution of (21) is obtained as follows

$$\begin{aligned} z = {} & c_{12}\cos\left(\int k_n ds\right) + c_{13}\sin\left(\int k_n ds\right) \\ & + \cos\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right) ds - \sin\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right) ds, \end{aligned}$$

where $c_{12}, c_{13} \in \mathbb{R}$ are constants and thus

$$\begin{aligned} x = {} & -c_{12}\sin\left(\int k_n ds\right) + c_{13}\cos\left(\int k_n ds\right) \\ & - \sin\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right) ds - \cos\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right) ds. \end{aligned}$$

Hence, the position vector of timelike associated curve β is stated as

$$\begin{aligned}\beta(s) = \alpha(s) + \Big[& -c_{12}\sin\left(\int k_n ds\right) + c_{13}\cos\left(\int k_n ds\right) \\ & - \sin\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right)ds - \cos\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right)ds\Big]T \\ & + \Big[c_{12}\cos\left(\int k_n ds\right) + c_{13}\sin\left(\int k_n ds\right) \\ & + \cos\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right)ds - \sin\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right)ds\Big]n.\end{aligned} \tag{26}$$

If $k_n = 0$, then from first and third equations of system (20), we get $x = -s + c_{19}$, $z = c_{20}$, respectively, and therefore the position vector of β is given by

$$\beta(s) = \alpha(s) + (-s + c_{14})T + c_{15}n, \tag{27}$$

where $c_{14}, c_{15} \in \mathbb{R}$ are constants. Now, we can give the followings:

Theorem 3.17 *The spacelike (resp. timelike and non-lightlike) associated curve β given by (25) (resp. (26) and (27)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.18 *The associated curves (25) and (26) can be referred to as: Spacelike and timelike relatively normal-slant helix-connected associated helix of type 2 or spacelike and timelike RNS-HC-associated helix of type 2, respectively.*

Corollary 3.19 *The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization in (27).*

Case 3: $z = 0$. In this case, from system (16), we obtain

$$x' + \varepsilon_n \kappa_g y + 1 = 0, \quad \kappa_g x + y' \neq 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y = 0. \tag{28}$$

with $(k_n, \tau_g) \neq (0,0)$. If $\tau_g \neq 0$, then from the third equation of system (28), we have $y = -\dfrac{\varepsilon_g k_n}{\varepsilon_T \tau_g}$. Substituting y in first equation of (28), it follows $x' - \dfrac{\varepsilon_g \varepsilon_n k_n \kappa_g}{\varepsilon_T \tau_g} x + 1 = 0$, where $\dfrac{\varepsilon_g \varepsilon_n}{\varepsilon_T} = -1$. Then, following differential equation is obtained

$$x' + \frac{k_n \kappa_g}{\tau_g} x = -1,$$

whose general solution is

$$x = \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right)ds + c_{16}\right],$$

where $c_{16} \in \mathbb{R}$ is constant. Hence, we obtain y as follows

$$y = -\frac{\varepsilon_g k_n}{\varepsilon_T \tau_g} \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{16}\right],$$

and the position vector of associated curve β is given by

$$\beta(s) = \alpha(s) + \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{16}\right]\left(T - \frac{\varepsilon_g k_n}{\varepsilon_T \tau_g} g\right). \tag{29}$$

If $\kappa_g \neq 0$ and $\tau_g = 0$, then from the system (28), we get $x = 0$ and $y = -\dfrac{1}{\varepsilon_n \kappa_g}$. Thus, the position vector of associated curve β is given by

$$\beta(s) = \alpha(s) - \frac{1}{\varepsilon_n \kappa_g} g. \tag{30}$$

Theorem 3.20 *The non-lightlike associated curve β given by (29) (resp. (30)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.21 *The non-lightlike associated curve (29) (resp. (30)) can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix of type 3 or non-lightlike RNS-HC-associated helix of type 3.*

Corollary 3.22 *(i) The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{16})T$, where $c_{16} \in \mathbb{R}$ is constant.*

(ii) The non-lightlike relatively normal-slant helix α is a geodesic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{16})T + \frac{(-s+c_{16})\varepsilon_g k_n}{\varepsilon_T \tau_g} g$, where $c_{16} \in \mathbb{R}$ is constant.

(iii) The non-lightlike relatively normal-slant helix α is a line of curvature on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization in (30).

3.3. Non-lightlike helices associated with isophote curves in $\mathbb{E}_1^3$

In this final subsection of Section 3, we investigate non-lightlike helices associated with isophote curves. Let the tangent vector β' of associated curve β be linearly dependent with the unit surface normal along an arbitrary non-lightlike curve α on an oriented surface φ. Then, from (4), we have $R_1 = R_2 = 0$ and $\beta'(s) = R_3(s)n(s)$. Arc-length parameter and Frenet vectors T_β, N_β, B_β of β

are calculated as $ds_\beta = \pm R_3 ds$ and

$$\begin{cases} T_\beta = \pm n, \quad N_\beta = \pm \dfrac{1}{\sqrt{\left|\varepsilon_T k_n^2 + \varepsilon_g \tau_g^2\right|}} \left(k_n T + \tau_g g\right), \\ B_\beta = \dfrac{\varepsilon_{B_\beta}}{\sqrt{\left|\varepsilon_T k_n^2 + \varepsilon_g \tau_g^2\right|}} \left(\varepsilon_g k_n g + \varepsilon_T \tau_g T\right) = \varepsilon_{B_\beta} \dfrac{D_o}{\|D_o\|}, \end{cases} \tag{31}$$

respectively, where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta. \rangle$. From (31) and Definition 2.1, we can give the following theorem.

Theorem 3.23 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(k_n, \tau_g) \neq (0,0)$ who lies on a regular surface φ with the condition that β' and unit surface normal n along α are linearly dependent. Then, followings are equivalent:*

(i) β is a helix.

(ii) α is an isophote curve on φ.

(iii) α is a D_o-Darboux slant helix on φ.

Remark 3.24 *The non-lightlike helix β associated with isophote curve α can be referred to as: Non-lightlike isophote curve-connected associated helix or non-lightlike ICC-associated helix.*

We now investigate special cases when x, y, z functions have special values. From (5), we get

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad \kappa_g x + y' + \tau_g z = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' \neq 0. \tag{32}$$

Case 1: $x = 0$. Then, from (32), we have

$$\varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad y' + \tau_g z = 0, \quad \varepsilon_T \tau_g y + z' \neq 0, \tag{33}$$

with $(k_n, \kappa_g) \neq (0,0)$. If $\tau_g \neq 0$, then from second equation of system (33), we have $z = -\frac{y'}{\tau_g}$ and by substituting this equality in the third equation of system (33), we obtain the following differential equation

$$y' - \frac{\varepsilon_n \kappa_g \tau_g}{k_n} y = \frac{\tau_g}{k_n},$$

whose general solution is

$$y = \exp\left(\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right) \frac{\tau_g}{k_n} ds + c_{17}\right), \tag{34}$$

where c_{17} is a real constant. Since $z = -\frac{y'}{\tau_g}$, it follows

$$z = -\frac{1}{k_n} - \frac{\varepsilon_n \kappa_g}{k_n} \exp\left(\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right) \frac{\tau_g}{k_n} ds + c_{17}\right). \tag{35}$$

Therefore, for the position vector of associated curve β, we obtain

$$\begin{aligned}\beta(s) &= \alpha(s) + \exp\left(\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\frac{\tau_g}{k_n}ds + c_{17}\right)g \\ &-\left[\frac{1}{k_n} + \frac{\varepsilon_n\kappa_g}{k_n}\exp\left(\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\frac{\tau_g}{k_n}ds + c_{17}\right)\right]n.\end{aligned} \tag{36}$$

If $k_n \neq 0$ and $\tau_g = 0$, then from the second equation of system (33), we get $y = c_{18}$ for a real constant c_{18}. Substituting this result in first equation of system (33) yields $z = -\frac{c_{18}\varepsilon_n\kappa_g + 1}{k_n}$. Therefore, the position vector of associated curve β is obtained as

$$\beta(s) = \alpha(s) + c_{18}g - \frac{c_{18}\varepsilon_n\kappa_g + 1}{k_n}n. \tag{37}$$

We state our findings with the following theorem and corollaries.

Theorem 3.25 *The non-lightlike associated curve β given by (36) (resp. (37)) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.26 *The non-lightlike associated curve (36) (resp. (37)) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 1 or non-lightlike ICC-associated helix of type 1.*

Corollary 3.27 *(i) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is an asymptotic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) - \frac{1}{\varepsilon_n\kappa_g}g - \frac{k_g'}{\varepsilon_n\kappa_g^2\tau_g}n$.*

(ii) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + \int \frac{\tau_g}{k_n}dsg - \frac{1}{k_n}n$.

(iii) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is a line of curvature if and only if non-lightlike ICC-associated helix has the parametrization (37).

Case 2: $y = 0$. From system (32), we have

$$x' + k_n z + 1 = 0, \quad \kappa_g x + \tau_g z = 0, \quad \varepsilon_g k_n x + z' \neq 0, \tag{38}$$

with $(\kappa_g, \tau_g) \neq (0,0)$. If $\tau_g \neq 0$, then, from the second equation of system (38), we get $z = -\frac{\kappa_g}{\tau_g}x$ which we substitute in the first equation of system (38) and obtain the following differential equation

$$x' - \frac{k_n\kappa_g}{\tau_g}x = -1,$$

whose general solution is

$$x = \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{19}\right], \tag{39}$$

where c_{19} is a real constant. Since $z = -\frac{\kappa_g}{\tau_g} x$, the position vector of the associated curve β is obtained as

$$\beta(s) = \alpha(s) + \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{19}\right]\left(T - \frac{\kappa_g}{\tau_g} n\right). \tag{40}$$

If $k_n \neq 0$ and $\tau_g = 0$, then, second and first equations of system (38) yield $x = 0$ and $z = -\frac{1}{k_n}$, respectively. Thus, the position vector of associated curve β is given by

$$\beta(s) = \alpha - \frac{1}{k_n} n. \tag{41}$$

Now, we give the following theorem and corollaries.

Theorem 3.28 *The non-lightlike associated curve β given by (40) (resp. (41)) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.29 *The non-lightlike associated curve (40) (resp. (41)) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 2 or non-lightlike ICC-associated helix of type 2.*

Corollary 3.30 *(i) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is an asymptotic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{19})T + \frac{\kappa_g(s - c_{19})}{\tau_g} n$.*

(ii) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{19})T$.

(iii) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is a line of curvature if and only if non-lightlike ICC-associated helix has the parametrization in (41).

Case 3: $z = 0$. In this case, from (32) we obtain

$$x' + \varepsilon_n \kappa_g y + 1 = 0, \quad \kappa_g x + y' = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y \neq 0. \tag{42}$$

If $\kappa_g = 0$, then, from system (42), we get $x = -s + c_{29}$ and $y = c_{30}$, where c_{20}, c_{21} are real constants. Then, the position vector of the associated curve β is given by

$$\beta(s) = \alpha(s) + (-s + c_{20})T + c_{21} g. \tag{43}$$

If $\kappa_g \neq 0$, then from second equation of system (42), we have $x = -\dfrac{y'}{\kappa_g}$. We take the derivative of x and substitute it in the first equation of system (42) and obtain the following differential equation

$$y'' - \frac{k_g'}{\kappa_g} y' - \varepsilon_n \kappa_g^2 y = \kappa_g,$$

whose homogeneous part can be achieved by a parameter change $t = \int \kappa_g ds$ as

$$\frac{d^2 y}{dt^2} - \varepsilon_n y = 0. \tag{44}$$

The solution of (44) depends on the value of ε_n which could be either 1 or -1. If $\varepsilon_n = 1$, then we get

$$\begin{aligned} y =& \ c_{22} \cosh\left(\int \kappa_g ds\right) + c_{23} \sinh\left(\int \kappa_g ds\right) \\ & - \cosh\left(\int \kappa_g ds\right) \int \sinh\left(\int \kappa_g ds\right) ds + \sinh\left(\int \kappa_g ds\right) \int \cosh\left(\int \kappa_g ds\right) ds, \\ x =& \ - c_{22} \sinh\left(\int \kappa_g ds\right) - c_{23} \cosh\left(\int \kappa_g ds\right) \\ & + \sinh\left(\int \kappa_g ds\right) \int \sinh\left(\int \kappa_g ds\right) ds - \cosh\left(\int \kappa_g ds\right) \int \cosh\left(\int \kappa_g ds\right) ds, \end{aligned} \tag{45}$$

where c_{22}, c_{23} are real constants.

If $\varepsilon_n = -1$, then we get

$$\begin{aligned} y =& \ c_{24} \cos\left(\int \kappa_g ds\right) + c_{25} \sin\left(\int \kappa_g ds\right) \\ & - \cos\left(\int \kappa_g ds\right) \int \sin\left(\int \kappa_g ds\right) ds + \sin\left(\int \kappa_g ds\right) \int \cos\left(\int \kappa_g ds\right) ds, \\ x =& \ c_{24} \sin\left(\int \kappa_g ds\right) - c_{25} \cos\left(\int \kappa_g ds\right) \\ & - \sin\left(\int \kappa_g ds\right) \int \sin\left(\int \kappa_g ds\right) ds - \cos\left(\int \kappa_g ds\right) \int \cos\left(\int \kappa_g ds\right) ds, \end{aligned} \tag{46}$$

where c_{24}, c_{25} are real constants. In either cases,

$$\beta(s) = \alpha(s) + xT + yg, \tag{47}$$

where x, y are as defined in (45) or (46).

Theorem 3.31 *The non-lightlike associated curve β given by (47) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.32 *The non-lightlike associated curve (47) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 3 or non-lightlike ICC-associated helix of type 3.*

Corollary 3.33 *The non-lightlike isophote curve α is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization (43).*

4. Examples

Example 4.1 *Let the spacelike surface φ be given by the parametrization $\varphi(u,v) = (\cosh u, \sinh u, v)$ and*

$$\alpha(u) = \left(\cosh\left(\frac{u}{\sqrt{2}}\right), \sinh\left(\frac{u}{\sqrt{2}}\right), \frac{u}{\sqrt{2}}\right)$$

be a spacelike helix on φ. Then, elements of Darboux frame of α are calculated as

$$T(s) = \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),$$

$$g(s) = \left(\sinh\left(\frac{s}{\sqrt{2}}\right), \cosh\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\right), \quad n(s) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right), \sinh\left(\frac{s}{\sqrt{2}}\right), 0\right),$$

$k_n = \frac{1}{2}$, $\kappa_g = 0$ *and* $\tau_g = \frac{1}{2}$. *Since $\kappa_g = 0$, α is a geodesic curve on φ. On the other hand, since g and n are Lorenztian circles or arc of a Lorenztian circle, then we have that α is also a relatively normal-slant helix and an isophote curve on φ. Figure 1 shows some β curves associated with α considering the obtained results in Section 3.*

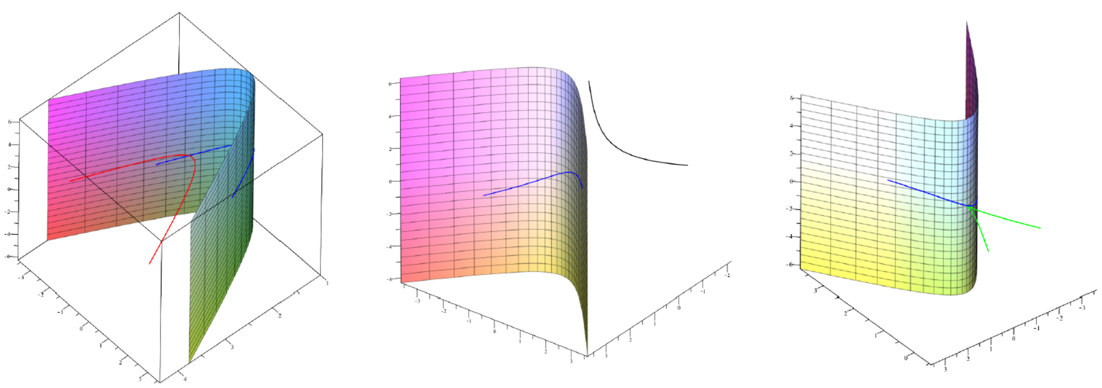

Figure 1: Spacelike surface curve α (blue), spacelike HCC-associated helix of type 1 (red), spacelike RNS-HC-associated helix of type 1 (black) and spacelike ICC-associated helix of type 2 (green), respectively

Example 4.2 *Let the timelike surface φ be given by the parametrization $\varphi(u,v) = \left(\sqrt{3}u, v\cos(u), v sin(u)\right)$,*

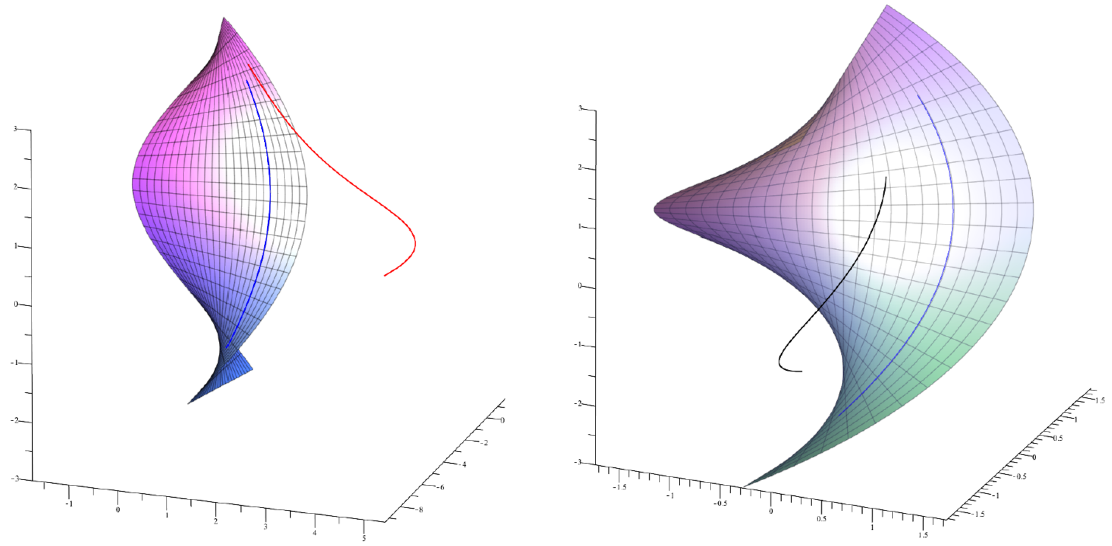

Figure 2: Timelike surface curve α (blue), timelike HCC-associated helix of type 3 (red), timelike RNS-HC-associated helix of type 3 (black), respectively

$v \in (-\sqrt{3}, \sqrt{3})$ *and*

$$\alpha(s) = \left(\sqrt{\frac{3}{2}}s, \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right)\right)$$

be a timelike helix on φ. The elements of Darboux frame of α are calculated as

$$n(s) = \left(-\frac{\sqrt{2}s}{2\sqrt{3-\frac{s^2}{2}}}, \frac{\sqrt{3}\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{3-\frac{s^2}{2}}}, \frac{\sqrt{3}\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{3-\frac{s^2}{2}}}\right), \quad k_n = \frac{1}{2}\cosh\left(\frac{\pi}{2}\right), \; k_n = \frac{1}{2}\sinh\left(\frac{\pi}{2}\right) \text{ and}$$

$\tau_g = \frac{\sqrt{3}}{2}$. Since g is a Lorenztian circle or an arc of a Lorenztian circle, then we have that α is also a relatively normal-slant helix. Figure 2 shows some β curves associated with α considering the obtained results in Section 3.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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