Yuzuncu Yil University Journal of the Institute of Natural & Applied Sciences, Volume 28, Issue 3 (December), 908-918, 2023

Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

LMI Approach for Asymptotical Stability of Riemann–Liouville Nonlinear Fractional Neutral Systems with Time-Varying Delays

Erdal KORKMAZ*, Abdulhamit ÖZDEMİR

Muş University, Faculty of Science and Literature, Department of Mathematics, 49250, Muş, Türkiye
Erdal KORKMAZ, ORCID No:0000-0002-6647-9312, Abdulhamit ÖZDEMİR, ORCID No:0000-0002-5310-6285
*Corresponding author e-mail: e.korkmaz@alparslan.edu.tr

Article Info

Received: 02.02.2023
Accepted: 19.06.2023
Online December 2023

DOI:10.53433/yyufbed.1246729

Keywords
Asymptotical stability,
Convex optimization problem,
Fractional neutral systems,
Riemann-Liouville derivative

Abstract: In this paper, we have delivered asymptotic stability results for solutions to non-autonomous nonlinear neutral systems. The acquired stability results are independent of the delays, and the delays are also both time-variable and unbounded. Additionally, the results were described as a convex optimization problem, and an example was used to examine the results' feasibility and efficacy.

Zaman-Değişken Gecikmeli Riemann–Liouville Lineer Olmayan Kesirli Nötr Sistemlerin Asimptotik Kararlılığına LMI Yaklaşımı

Makale Bilgileri

Geliş: 02.02.2023
Kabul: 19.06.2023
Online Aralık 2023

DOI:10.53433/yyufbed.1246729

Anahtar Kelimeler
Asimptotik kararlılık,
Kesir mertebeli nötr sistemler,
Konveks optimizasyon problemi,
Riemann-Liouville türevi

Öz: Bu çalışmada, otonom olmayan ve lineer olmayan kesirli nötr sistemlerin asimptotik kararlılığı üzerine sonuçlar elde edilmiştir. Elde edilen kararlılık sonuçları gecikmeden bağımsızdır ve gecikmeler aynı zamanda hem zaman-değişken olup hem de sınırlı değildir. Ayrıca bu çalışmadaki sonuçlar birer konveks optimizasyon problemi olarak ifade edilmiştir ve sonuçların uygulanabilirliği ve etkinliğini araştırmak için bir örnek kullanılmıştır.

1. Introduction

Fractional calculus which has a past of over 300 years, is said to have begun with a question asked by L'hospital to Leibniz in 1695 (Podlubny, 1999). Fractional calculus, which has been the subject of many scientific studies until today, continues to exist as a popular field of study in recent years, especially since it has found a concrete response in fields such as biology, physics, the various fields of engineering (Hale, 1977; Kilbas et al., 2006). In general, Caputo fractional derivative has been used in studies on fractional differential equations, few of these studies are fractional derivatives of Riemann Liouville (see references and their references). Lyapunov's second approach, which is employed by researchers, is one of the most effective ways for analyzing the behavior of fractional differential equation solutions (Duarte-Mermoud et al., 2015; Liu et al., 2016a and 2016b; Liu et al., 2017; Korkmaz & Özdemir, 2019; Altun & Tunç, 2020). Heymans & Podlunny (2006) demonstrated that Riemann-Liouville fractional derivatives with initial conditions can have physical meaning in a variety of viscoelasticity situations. Yang et al. (2017) worked on the Lyapunov stability analysis of nonlinear fractional systems with impulses. Deng et al. (2007) established a characteristic equation for multiple delayed fractional-order systems using the Laplace transform. Chen et al. (2016) discovered two novel delay-dependent sufficient conditions that ensure the stability of some fractional-order neural networks on a bounded-time interval by using the inequality approach. Chen et al. (2014) discovered a novel sufficient condition for guaranteeing local asymptotic stability of a variety of nonlinear fractional systems with fractional-order $\alpha: 1 < \alpha < 2$ by utilizing the Laplace transform, the generalized Gronwall inequality, and the Mittag-Leffler function. Li et al. (2015) obtained certain sufficient stability criteria by employing the relationship between the characteristic equations of integer order systems and fractional order systems. Qian et al. (2010) developed stability conditions for fractional-order systems with the Riemann-Liouville derivative, which includes linear systems, time-delayed systems, and perturbed systems.

What motivates us in this paper is that while most of the research done on Caputo fractional derivative has been used, the Riemann-Liouville fractional derivative has not been studied sufficiently, and on the other hand, delayed neutral nonlinear differential equation systems are still an open problem. The Caputo derivative is more useful to real-world issues since its initial conditions are physically well-understood. The major benefit of the Riemann-Liouville derivative, on the other hand, is the features of composition of the Riemann-Liouville derivative and the Riemann-Liouville integral. In fact, the Riemann-Liouville derivative is a continuous operator of order α.

2. Material and Methods

This section introduces several essential definitions of fractional calculus, as well as certain sufficient lemmas.

Notation. R^n indicates n-dimensional Euclidean space. $R^{n\times n}$ represents the set of all $n \times n$ real matrices. $\|x\|$ denotes the Euclidean norm of a real vector x. $\|A\|$ denotes the spectral norm of matrix A. $A > 0$ (or $A < 0$) indicates that the symmetric matrix A is positive definite (or negative definite).

Definition 1. The fractional-order integral and derivative of Riemann-Liouville are denoted by

$$_{t_0}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds\,, \quad (\alpha > 0) \tag{1}$$

$$_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{\alpha+1-n}}ds\,, \quad (n-1 \le \alpha < n), \tag{2}$$

respectively (Podlubny, 1999).

Lemma 1. Let $\alpha > \beta > 0$, then

$$_{t_0}D_t^{\beta}\left({}_{t_0}D_t^{-\alpha}x(t)\right) = {}_{t_0}D_t^{\beta-\alpha}x(t) \tag{3}$$

holds for "sufficiently good" functions $x(t)$. This relationship holds especially if $x(t)$ is integrable (Kilbas et al., 2006).

Lemma 2. If $x(t) \in R^n$ is a real vector of a differentiable function, then

$$\frac{1}{2}\,{}_{t_0}D_t^{\alpha}\big(x^T(t)Px(t)\big) \le x^T(t)P\,{}_{t_0}D_t^{\alpha}x(t), \quad \forall\alpha \in (0,1), \forall t \ge t_0, \tag{4}$$

inequality holds, where $P \in R^{n\times n}$ is a square, symmetric, positive semi-definite, and constant matrix (Liu et al., 2017).

3. Results

In this section, we obtain some stability criteria of nonlinear neutral fractional-order systems with time-dependent lags. For this, we employed the linear matrix inequality.

Consider the nonlinear neutral fractional-order system shown below:

$$ {}_{t_0}D_t^{\alpha}x(t) = Ax(t) + B_1 f_1\big(t, x(t)\big) + B_2 f_2\Big(t, x\big(t - \tau_1(t)\big)\Big) + C\,{}_{t_0}D_t^{\alpha}x\big(t - \tau_2(t)\big), \tag{5}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ is a real vector, $0 < \alpha < 1$ is a real number, for all $t > t_0$, $\tau_1(t), \tau_2(t) > 0$ are time-varying delays, $A, B_1, B_2, C \in R^{n\times n}$ are known constant matrices. $f_j\big(t, x(t)\big)$ $(j = 1,2)$ are vector-valued time-varying nonlinear functions with $f_j(t, 0) = 0$ and satisfies the following Lipschitz condition for all $(t, x), (t, \hat{x}) \in R \times R^n$

$$\|f_j(t, x) - f_j(t, \hat{x})\| \le a_j \|M_j(x - \hat{x})\|, \quad j = 1,2 \tag{6}$$

where a_j are positive scalars, and M_j are constant matrices of the proper dimension. Consequently, due to $f_j(t, 0) = 0$, we have

$$\|f_j(t, x)\| \le a_j \|M_j x\|, \quad j = 1,2. \tag{7}$$

Thus, by satisfying the condition (7) of equation (5), we guarantee the uniqueness of the zero solution of (5).

Theorem 1. If $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \le d_i < 1$, $(i = 1,2)$ and there exist P, R and S symmetric positive matrices that satisfy

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & \Pi_{34} \\ \Pi_{14}^T & \Pi_{24}^T & \Pi_{34}^T & \Pi_{44} \end{pmatrix} < 0, \tag{8}$$

LMI condition, then the zero solution of nonlinear fractional-order neutral system (5) is asymptotically stable, where

$$\begin{aligned}
\Pi_{11} &= A^T P + PA + A^T(R + mS)A + a_1^2 M_1^T M_1 + a_2^2 M_2^T M_2, \\
\Pi_{12} &= PB_1 + A^T(R + mS)B_1, \\
\Pi_{13} &= PB_2 + A^T(R + mS)B_2, \\
\Pi_{14} &= PC + A^T(R + mS)C, \\
\Pi_{22} &= B_1^T(R + mS)B_1 - I, \\
\Pi_{23} &= B_1^T(R + mS)B_2, \\
\Pi_{24} &= B_1^T(R + mS)C, \\
\Pi_{33} &= B_2^T(R + mS)B_2 - (1 - d_1)I, \\
\Pi_{34} &= B_2^T(R + mS)C, \\
\Pi_{44} &= C^T(R + mS)C - (1 - d_2)R,
\end{aligned}$$

I unit matrix and m is a positive constant.

Proof. Consider the Lyapunov-Krasovskii functional, which is described as:

$$\begin{aligned} V(t) = \; & {}_{t_0}D_t^{\alpha-1}\left(x^T(t)Px(t)\right) + \int_{t-\tau_2(t)}^{t} \left({}_{t_0}D_t^{\alpha}x(s)\right)^T R\left({}_{t_0}D_t^{\alpha}x(s)\right) ds \\ & + a_2^2 \int_{t-\tau_1(t)}^{t} x^T(s)M_2^T M_2 x(s) ds + \int_{t-m}^{t}\int_{\theta}^{t} \left({}_{t_0}D_s^{\alpha}x(s)\right)^T S\left({}_{t_0}D_s^{\alpha}x(s)\right) ds d\theta. \end{aligned}$$

Because the matrices P, R and S are all positive definite, the functional $V(t)$ is also positive definite. Therefore, from Lemma 1 and Lemma 2, the derivative of $V(t)$ is derived by utilizing the trajectories of system (5) as stated below:

$$\begin{aligned} \dot{V}(t) = \; & {}_{t_0}D_t^{\alpha}\left(x^T(t)Px(t)\right) + a_1^2 x^T(t)M_1^T M_1 x(t) - a_1^2 x^T(t)M_1^T M_1 x(t) \\ & + \left({}_{t_0}D_t^{\alpha}x(t)\right)^T R\left({}_{t_0}D_t^{\alpha}x(t)\right) - \left(1-\tau_2'(t)\right)\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T R\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right) \\ & + a_2^2 x^T(t)M_2^T M_2 x(t) - \left(1-\tau_1'(t)\right)a_2^2 x^T\left(t-\tau_1(t)\right)M_2^T M_2 x\left(t-\tau_1(t)\right) \\ & + m\left({}_{t_0}D_t^{\alpha}x(t)\right)^T S\left({}_{t_0}D_t^{\alpha}x(t)\right) - \int_{t-m}^{t} \left({}_{t_0}D_s^{\alpha}x(s)\right)^T S\left({}_{t_0}D_s^{\alpha}x(s)\right) ds \\ \leq \; & 2x^T(t)PD_t^{\alpha}x(t) + a_1^2 x^T(t)M_1^T M_1 x(t) + a_2^2 x^T(t)M_2^T M_2 x(t) \\ & + \left({}_{t_0}D_t^{\alpha}x(t)\right)^T (R+mS)\left({}_{t_0}D_t^{\alpha}x(t)\right) - (1-d_1)f_2^T\left(t, x\left(t-\tau_1(t)\right)\right) I f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & - f_1^T(t,x(t)) I f_1(t,x(t)) - (1-d_2)\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T R\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right) \\ = \; & 2x^T(t)P\left[Ax(t) + B_1 f_1\left(t,x(t)\right) + B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) + C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right] \\ & + a_1^2 x^T(t)M_1^T M_1 x(t) + a_2^2 x^T(t)M_2^T M_2 x(t) \\ & + \left[Ax(t) + B_1 f_1\left(t,x(t)\right) + B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) + C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right]^T \\ & \times (R+mS)\left[Ax(t) + B_1 f_1\left(t,x(t)\right) + B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) + C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right] \\ & - f_1^T(t,x(t)) I f_1(t,x(t)) - (1-d_1)f_2^T\left(t, x\left(t-\tau_1(t)\right)\right) I f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & - (1-d_2)\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T R\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right) \\ = \; & x^T(t)\left(A^T P + PA + a_1^2 M_1^T M_1 + a_2^2 M_2^T M_2 + A^T(R+mS)A\right)x(t) + 2x^T(t)PB_1 f_1\left(t,x(t)\right) \\ & + 2x^T(t)PB_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) + 2x^T(t)PC_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right) + x^T(t)A^T(R+mS)B_1 f_1\left(t,x(t)\right) \\ & + x^T(t)A^T(R+mS)B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) + x^T(t)A^T(R+mS)C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right) \\ & + \left(f_1\left(t,x(t)\right)\right)^T B_1^T(R+mS)Ax(t) + \left(f_1\left(t,x(t)\right)\right)^T B_1^T(R+mS)B_1 f_1\left(t,x(t)\right) \\ & + \left(f_1\left(t,x(t)\right)\right)^T B_1^T(R+mS)B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & + \left(f_1\left(t,x(t)\right)\right)^T B_1^T(R+mS)C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right) \\ & + \left(f_2\left(t, x\left(t-\tau_1(t)\right)\right)\right)^T B_2^T(R+mS)Ax(t) + \left(f_2\left(t, x\left(t-\tau_1(t)\right)\right)\right)^T B_2^T(R+mS)B_1 f_1\left(t,x(t)\right) \\ & + \left(f_2\left(t, x\left(t-\tau_1(t)\right)\right)\right)^T B_2^T(R+mS)B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & + \left(f_2\left(t, x\left(t-\tau_1(t)\right)\right)\right)^T B_2^T(R+mS)C_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right) \\ & + \left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T C^T(R+mS)Ax(t) + \left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T C^T(R+mS)B_1 f_1\left(t,x(t)\right) \\ & + \left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T C^T(R+mS)B_2 f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & + \left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T C^T(R+mS)C\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right) \\ & - f_1^T(t,x(t)) I f_1(t,x(t)) - (1-d_1)f_2^T\left(t, x\left(t-\tau_1(t)\right)\right) I f_2\left(t, x\left(t-\tau_1(t)\right)\right) \\ & - (1-d_2)\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right)^T R\left({}_{t_0}D_t^{\alpha}x\left(t-\tau_2(t)\right)\right). \end{aligned}$$

So, we can estimate

$$\dot{V}(t) \leq \eta^T \Pi \eta, \tag{9}$$

where

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & \Pi_{34} \\ \Pi_{14}^T & \Pi_{24}^T & \Pi_{34}^T & \Pi_{44} \end{pmatrix},$$

and

$$\begin{aligned} \Pi_{11} &= A^T P + PA + A^T(R + mS)A + a_1^2 M_1^T M_1 + a_2^2 M_2^T M_2, \\ \Pi_{12} &= PB_1 + A^T(R + mS)B_1, \\ \Pi_{13} &= PB_2 + A^T(R + mS)B_2, \\ \Pi_{14} &= PC + A^T(R + mS)C, \\ \Pi_{22} &= B_1^T(R + mS)B_1 - I, \\ \Pi_{23} &= B_1^T(R + mS)B_2, \\ \Pi_{24} &= B_1^T(R + mS)C, \\ \Pi_{33} &= B_2^T(R + mS)B_2 - (1 - d_1)I, \\ \Pi_{34} &= B_2^T(R + mS)C, \\ \Pi_{44} &= C^T(R + mS)C - (1 - d_2)R, \end{aligned}$$

$$\eta = \left(x^T(t), f_1^T\big(t, x(t)\big), f_2^T\left(t, -x^T\big(t - \tau_1(t)\big)\right), \left({}_{t_0}D_t^{\alpha} x\big(t - \tau_2(t)\big)\right)^T\right)^T.$$

It is clear that $\dot{V}(t)$ is negative definite, due to the axiom of the Theorem 1. Then the zero solution of nonlinear fractional-order neutral system (5) is asymptotically stable, which completes the proof.

Theorem 2. If $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \leq d_i < 1$, $(i = 1,2)$ and there exist P, Q and R symmetric positive matrices that satisfy

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} \\ \Omega_{14}^T & \Omega_{24}^T & \Omega_{34}^T & \Omega_{44} \end{pmatrix} < 0, \tag{10}$$

LMI condition, then the zero solution of nonlinear fractional-order neutral system (5) is asymptotically stable, where

$$\begin{aligned}
\Omega_{11} &= PA + A^T P + a_1^2 M_1^T M_1 + a_2^2 M_2^T M_2 + Q + mA^T RA,\\
\Omega_{12} &= -A^T PC,\\
\Omega_{13} &= PB_1 + mA^T RB_1,\\
\Omega_{14} &= PB_2 + mA^T RB_2,\\
\Omega_{22} &= -(1-d_2)Q,\\
\Omega_{23} &= -C^T PB_1,\\
\Omega_{24} &= -C^T PB_2,\\
\Omega_{33} &= mB_1^T RB_1 - I,\\
\Omega_{34} &= mB_1^T RB_2,\\
\Omega_{44} &= mB_2^T RB_2 - (1-d_1)I,
\end{aligned}$$

I unit matrix and m is a positive constant.

Proof. Consider the Lyapunov-Krasovskii functional, which is described as:

$$\begin{aligned}
V(t) = \;& {}_{t_0}D_t^{\alpha-1}\left(\left(x(t) - Cx(t-\tau_2(t))\right)^T P\left(x(t) - Cx(t-\tau_2(t))\right)\right)\\
&+a_2^2\int_{t-\tau_1(t)}^{t} x^T(s)M_2^T M_2 x(s)ds + \int_{t-\tau_2(t)}^{t} x^T(s)Qx(s)ds\\
&+\int_{t-m}^{t}\int_{\theta}^{t}\left({}_{t_0}D_s^{\alpha}\left(x(s) - Cx(s-\tau_2(s))\right)\right)^T R\left({}_{t_0}D_s^{\alpha}\left(x(s) - Cx(s-\tau_2(s))\right)\right)dsd\theta.
\end{aligned}$$

Because the matrices P, Q and R are all positive definite, the functional $V(t)$ is also positive definite. Therefore, from Lemma 1 and Lemma 2, the derivative of $V(t)$ is derived by utilizing the trajectories of system (5) as stated below:

$$\begin{aligned}
\dot{V}(t) = \;& {}_{t_0}D_t^{\alpha}\left(\left(x(t) - Cx(t-\tau_2(t))\right)^T P\left(x(t) - Cx(t-\tau_2(t))\right)\right) + a_1^2 x^T(t)M_1^T M_1 x(t)\\
&-a_1^2 x^T(t)M_1^T M_1 x(t) + a_2^2 x^T(t)M_2^T M_2 x(t) - \left(1-\tau_1'(t)\right)a_2^2 x^T\left(t-\tau_1(t)\right)M_2^T M_2 x\left(t-\tau_1(t)\right)\\
&+x^T(t)Qx(t) - \left(1-\tau_2'(t)\right)x^T\left(t-\tau_2(t)\right)Qx\left(t-\tau_2(t)\right)\\
&+m({}_{t_0}D_t^{\alpha}\left(x(t) - Cx(t-\tau_2(t))\right)^T R({}_{t_0}D_t^{\alpha}\left(x(t) - Cx(t-\tau_2(t))\right)\\
&-\int_{t-m}^{t}\left({}_{t_0}D_s^{\alpha}\left(x(s) - Cx(s-\tau_2(s))\right)\right)^T R\left({}_{t_0}D_s^{\alpha}\left(x(s) - Cx(s-\tau_2(s))\right)\right)ds\\
\leq\;& 2\left(x(t) - Cx(t-\tau_2(t))\right)^T P_{t_0}D_t^{\alpha}\left(x(t) - Cx(t-\tau_2(t))\right)\\
&+a_1^2 x^T(t)M_1^T M_1 x(t) + a_2^2 x^T(t)M_2^T M_2 x(t) + x^T(t)Qx(t)\\
&+m\left({}_{t_0}D_s^{\alpha}\left(x(s) - Cx(s-\tau_2(s))\right)\right)^T R\left({}_{t_0}D_t^{\alpha}\left(x(t) - Cx(t-\tau_2(t))\right)\right)\\
&-f_1^T(t,x(t))If_1(t,x(t)) - (1-d_1)f_2^T\left(t,x(t-\tau_1(t))\right)If_2\left(t,x(t-\tau_1(t))\right)\\
&-(1-d_2)x^T\left(t-\tau_2(t)\right)Qx\left(t-\tau_2(t)\right)\\
=\;& 2\left(x(t) - Cx(t-\tau_2(t))\right)^T P\left[Ax(t) + B_1 f_1(t,x(t)) + B_2 f_2\left(t,x(t-\tau_1(t))\right)\right]\\
&+a_1^2 x^T(t)M_1^T M_1 x(t) + m\left[Ax(t) + B_1 f_1(t,x(t)) + B_2 f_2\left(t,x(t-\tau_1(t))\right)\right]^T R\\
&\times\left[Ax(t) + B_1 f_1(t,x(t)) + B_2 f_2\left(t,x(t-\tau_1(t))\right)\right] + x^T(t)M_2^T M_2 x(t) + a_2^2 x^T(t)Qx(t)\\
&-f_1^T(t,x(t))If_1(t,x(t)) - (1-d_1)f_2^T\left(t,x(t-\tau_1(t))\right)If_2\left(t,x(t-\tau_1(t))\right)\\
&-(1-d_2)x^T\left(t-\tau_2(t)\right)Qx\left(t-\tau_2(t)\right)\\
=\;& x^T(t)(PA + A^T P + a_1^2 M_1^T M_1 + a_2^2 M_2^T M_2 + mA^T RA + Q)x(t)\\
&-2x^T\left(t-\tau_2(t)\right)C^T PAx(t) + 2x^T(t)PB_1 f_1(t,x(t)) - 2x^T\left(t-\tau_2(t)\right)C^T PB_1 f_1(t,x(t))\\
&+2x^T(t)PB_2 f_2\left(t,x(t-\tau_1(t))\right) - 2x^T\left(t-\tau_2(t)\right)C^T PB_2 f_2\left(t,x(t-\tau_1(t))\right)\\
&+mx^T(t)A^T RB_1 f_1(t,x(t)) + mx^T(t)A^T RB_2 f_2\left(t,x(t-\tau_1(t))\right)
\end{aligned}$$

$$
\begin{aligned}
&+mf_1^T(t,x(t))B_1^T RAx(t)+mf_1^T(t,x(t))B_1^T RB_1f_1(t,x(t))\\
&+mf_1^T(t,x(t))B_1^T RB_2f_2\left(t,x\left(t-\tau_1(t)\right)\right)+mf_2^T\left(t,x\left(t-\tau_1(t)\right)\right)B_2^T RAx(t)\\
&+mf_2^T\left(t,x\left(t-\tau_1(t)\right)\right)B_2^T RB_1f_1(t,x(t))+mf_2^T\left(t,x\left(t-\tau_1(t)\right)\right)B_2^T RB_2f_2\left(t,x\left(t-\tau_1(t)\right)\right)\\
&-f_1^T(t,x(t))If_1(t,x(t))-(1-d_1)f_2^T\left(t,x\left(t-\tau_1(t)\right)\right)If_2\left(t,x\left(t-\tau_1(t)\right)\right)\\
&-(1-d_2)x^T\left(t-\tau_2(t)\right)Qx\left(t-\tau_2(t)\right).
\end{aligned}
$$

So, we can estimate

$$\dot{V}(t)\leq \eta^T\Omega\eta, \tag{11}$$

where

$$
\Omega=\begin{pmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14}\\
\Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24}\\
\Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34}\\
\Omega_{14}^T & \Omega_{24}^T & \Omega_{34}^T & \Omega_{44}
\end{pmatrix}<0,
$$

$$
\begin{aligned}
\Omega_{11}&=PA+A^TP+a_1^2M_1^TM_1+a_2^2M_2^TM_2+Q+mA^TRA,\\
\Omega_{12}&=-A^TPC,\\
\Omega_{13}&=PB_1+mA^TRB_1,\\
\Omega_{14}&=PB_2+mA^TRB_2,\\
\Omega_{22}&=-(1-d_2)Q,\\
\Omega_{23}&=-C^TPB_1,\\
\Omega_{24}&=-C^TPB_2,\\
\Omega_{33}&=mB_1^TRB_1-I,\\
\Omega_{34}&=mB_1^TRB_2,\\
\Omega_{44}&=mB_2^TRB_2-(1-d_1)I,
\end{aligned}
$$

$$\eta=\left(x^T(t),x^T\left(t-\tau_2(t)\right),f_1^T(t,x(t)),f_2^T\left(t,x^T\left(t-\tau_1(t)\right)\right)\right)^T.$$

It is clear that $\dot{V}(t)$ is negative definite, due to the axiom of the Theorem 2. Then the zero solution of the nonlinear fractional-order neutral system (5) is asymptotically stable, which completes proof.

Now, asymptotic stability conditions for nonlinear fractional-order neutral systems (5) are presented as a convex optimization problem. The efficacy of the obtained results is demonstrated through a practical example.

Corollary 1. If $\|C\|<1$, for all $t>t_0$, $\tau_i'(t)\leq d_i<1$, $(i=1,2)$ and there exist P,R and S symmetric matrices and a_1^2 and a_2^2 scalars such that the following convex optimization problem in the variables P,R and S matrices and a_1^2 scalar:

$$
\begin{aligned}
&\min \quad -a_1^2\\
&\text{s. t.} \quad P>0,\qquad R>0,\qquad S>0,\qquad \Pi<0.
\end{aligned} \tag{12}
$$

is feasible, then the zero solution of nonlinear fractional-order neutral system (5) is asymptotically stable, where Π is a matrix defined by (8).

Corollary 2. If $\|C\|<1$, for all $t>t_0$, $\tau_i'(t)\leq d_i<1$, $(i=1,2)$ and there exist P,R and S symmetric matrices and a_1^2 and a_2^2 scalars such that the following convex optimization problem in the variables P,Q and R matrices and a_1^2 scalar:

$$\begin{array}{ll} \min & -a_1^2 \\ \text{s. t.} & P > 0, \quad Q > 0, \quad R > 0, \quad \Omega < 0 \end{array} \tag{13}$$

is feasible, then the zero solution of nonlinear fractional-order neutral system (5) is asymptotically stable, where Ω is a matrix defined by (10).

Let $f_2\left(t, x\big(t - \tau_1(t)\big)\right) = x\big(t - \tau_1(t)\big)$, then the nonlinear fractional-order neutral system (5) can write as stated below:

$${}_{t_0}D_t^{\alpha} x(t) = Ax(t) + B_1 f_1\big(t, x(t)\big) + B_2 x\big(t - \tau_1(t)\big) + C_0 D_t^{\alpha} x\big(t - \tau_2(t)\big), \tag{14}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ is the real vector, $0 < \alpha < 1$ is a real number, for all $t > t_0$, $\tau_1(t), \tau_2(t) > 0$ are time-varying delays, $A, B_1, B_2, C \in R^{n \times n}$ are known constant matrices. $f_1(t, x)$ is a vector-valued time-varying nonlinear function with $f_1(t, 0) = 0$ and satisfies the following Lipschitz condition for all $(t, x) \in R \times R^n$

$$\|f_1(t, x)\| \leq a_1 \|M_1 x\|,$$

where a_1 is a positive scalar, M_1 is a constant matrix of the proper dimension.

Corollary 3. If $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \leq d_i < 1$, $(i = 1,2)$ and there exist P, Q, R and S symmetric matrices and a_1^2 scalar such that the following convex optimization problem in the variables P, Q, R and S matrices and a_1^2 scalar:

$$\begin{array}{ll} \min & -a_1^2 \\ \text{s. t.} & P > 0, \quad Q > 0, \quad R > 0, \quad S > 0, \\ & \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{12}^T & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{13}^T & \Sigma_{23}^T & \Sigma_{33} & \Sigma_{34} \\ \Sigma_{14}^T & \Sigma_{24}^T & \Sigma_{34}^T & \Sigma_{44} \end{pmatrix} < 0, \end{array} \tag{15}$$

is feasible, then the zero solution of the nonlinear fractional-order neutral system (14) is asymptotically stable, where

$$\begin{aligned}
\Sigma_{11} &= A^T P + PA + A^T (R + mS) A + Q + a_1^2 M_1^T M_1, \\
\Sigma_{12} &= PB_1 + A^T (R + mS) B_1, \\
\Sigma_{13} &= PB_2 + A^T (R + mS) B_2, \\
\Sigma_{14} &= PC + A^T (R + mS) C, \\
\Sigma_{22} &= B_1^T (R + mS) B_1 - I, \\
\Sigma_{23} &= B_1^T (R + mS) B_2, \\
\Sigma_{24} &= B_1^T (R + mS) C, \\
\Sigma_{33} &= B_2^T (R + mS) B_2 - (1 - d_1) Q, \\
\Sigma_{34} &= B_2^T (R + mS) C, \\
\Sigma_{44} &= C^T (R + mS) C - (1 - d_2) R,
\end{aligned}$$

I unit matrix and m is a positive constant.

Corollary 4. If $\|C\| < 1$, for all $t > t_0$, $\tau_i'(t) \leq d_i < 1$, $(i = 1,2)$ and there exist P, Q_1, Q_2 and R symmetric matrices and a_1^2 scalar such that the following convex optimization problem in the variables P, Q_1, Q_2 and R matrices and a_1^2 scalar:

$$\min \quad -a_1^2 \tag{16}$$

$$\text{s. t.} \quad P > 0, \qquad Q_1 > 0, \qquad Q_2 > 0, \qquad R > 0,$$
$$\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ \Delta_{12}^T & \Delta_{22} & \Delta_{23} & \Delta_{24} \\ \Delta_{13}^T & \Delta_{23}^T & \Delta_{33} & \Delta_{34} \\ \Delta_{14}^T & \Delta_{24}^T & \Delta_{34}^T & \Delta_{44} \end{pmatrix} < 0,$$

is feasible, then the zero solution of the nonlinear fractional-order neutral system (14) is asymptotically stable, where

$$\begin{aligned}
\Delta_{11} &= PA + A^TP + a_1^2 M_1^T M_1 + Q_1 + Q_2 + mA^TRA, \\
\Delta_{12} &= -A^TPC, \\
\Delta_{13} &= PB_2 + mA^TRB_2, \\
\Delta_{14} &= PB_1 + mA^TRB_1, \\
\Delta_{22} &= -(1-d_2)Q_2, \\
\Delta_{23} &= -C^TPB_2, \\
\Delta_{24} &= -C^TPB_1, \\
\Delta_{33} &= mB_2^TRB_2 - (1-d_1)Q_1, \\
\Delta_{34} &= mB_2^TRB_1, \\
\Delta_{44} &= mB_1^TRB_1 - I,
\end{aligned}$$

I unit matrix and m is a positive constant.

Example 1. Consider the fractional neutral system given as:

$$ {}_{t_0}D_t^{\alpha}x(t) = Ax(t) + B_1 f_1(t, x(t)) + B_2 f_2\left(t, x(t - \tau_1(t))\right) + C_{t_0}D_t^{\alpha}x(t - \tau_2(t)), \tag{17}$$

where $\alpha \in (1,0)$, $\tau_1(t) = 0.5t + 0.3sin(t)$, $\tau_2(t) = 0.3t + 0.1cos(t)$,

$$f_1(t, x(t)) = a_1 \begin{pmatrix} sin(x_2(t)) \\ cos(x_1(t)) \end{pmatrix}, \quad f_2\left(t, x(t - \tau_1(t))\right) = 0.5 \begin{pmatrix} e^{-0.5t} sin\left(x_2(t - \tau_1(t))\right) \\ e^{-0.3t} cos\left(x_1(t - \tau_1(t))\right) \end{pmatrix}$$

$$A = \begin{bmatrix} -70 & 10 \\ 10 & -50 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

For $f_1(t, x(t))$, we have

$$\begin{aligned}
f_1(t, x(t))^2 &= a_1^2 \left(sin^2(x_2(t)) + cos^2(x_1(t))\right) \\
&\leq a_1^2 (x_1^2(t) + x_2^2(t)) \\
&= a_1^2 x^T(t) M_1^T M_1 x(t),
\end{aligned}$$

where $M_1 = I_2$. For $f_2\left(t, x(t - \tau_1(t))\right)$, we have

$$\begin{aligned}
\left\| f_2\left(t, x(t - \tau_1(t))\right) \right\|^2 &= 0.25 \left(e^{-t} sin^2\left(x_2(t - \tau_1(t))\right) + e^{-0.6t} cos^2\left(x_1(t - \tau_1(t))\right)\right) \\
&\leq 0.25(x_1^2(t) + x_2^2(t)) \\
&= 0.25 x^T(t - \tau_1(t)) M_2^T M_2 x(t - \tau_1(t)),
\end{aligned}$$

where $M_2 = I_2$.

Let us choose $d_1 = 0.8$, $d_2 = 0.4$, $m = 0.001$. Hence, we have a solution for the convex optimization problem (12) as $a_{1_{max}} = 1.7796$, where

$$P = \begin{bmatrix} 0.0746 & 0 \\ 0 & 0.0746 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0011 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0865 & 0 \\ 0 & 0.0865 \end{bmatrix},$$

and

$$\Pi = \begin{pmatrix} -1.0448 & 0.0557 & -0.0919 & 0.1198 & -0.0919 & 0.1198 & -0.0009 & 0.0012 \\ 0.0557 & -0.9334 & 0.1198 & 0.1476 & 0.1198 & 0.1476 & 0.0012 & 0.0015 \\ -0.0919 & 0.1198 & -0.8802 & 0 & 0.1198 & 0 & 0.0012 & 0 \\ 0.1198 & 0.1476 & 0 & -0.8802 & 0 & 0.1198 & 0 & 0.0012 \\ -0.0919 & 0.1198 & 0.1198 & 0 & -0.0802 & 0 & 0.0012 & 0 \\ 0.1198 & 0.1476 & 0 & 0.1198 & 0 & -0.0802 & 0 & 0.0012 \\ -0.0009 & 0.0012 & 0.0012 & 0 & 0.0012 & 0 & -0.0007 & 0 \\ 0.0012 & 0.0015 & 0 & 0.0012 & 0 & 0.0012 & 0 & -0.0007 \end{pmatrix}.$$

On the other hand, when all conditions are equal, we have that a solution for convex optimization problem (13) as $a_{1_{max}} = 1.7802$, where

$$P = \begin{bmatrix} 0.0763 & 0 \\ 0 & 0.0763 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0940 & 0 \\ 0 & 0.0940 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0825 & 0 \\ 0 & 0.0825 \end{bmatrix},$$

and

$$\Omega = \begin{pmatrix} -6.7873 & 1.4124 & 0.5338 & -0.0763 & 0.6968 & 0.0094 & 0.6968 & 0.0094 \\ 1.4124 & -3.9626 & -0.0763 & 0.3813 & 0.0094 & 0.7156 & 0.0094 & 0.7156 \\ 0.5338 & -0.0763 & -0.0495 & 0 & -0.0763 & 0 & -0.0763 & 0 \\ -0.0763 & 0.3813 & 0 & -0.0495 & 0 & -0.0763 & 0 & -0.0763 \\ 0.6968 & 0.0094 & -0.0763 & 0 & -0.9906 & 0 & 0.0094 & 0 \\ 0.0094 & 0.7156 & 0 & -0.0763 & 0 & -0.9906 & 0 & 0.0094 \\ 0.6968 & 0.0094 & -0.0763 & 0 & 0.0094 & 0 & -0.1906 & 0 \\ 0.0094 & 0.7156 & 0 & -0.0763 & 0 & 0.0094 & 0 & -0.1906 \end{pmatrix}.$$

When comparing Corollary 1 and 2, it can be observed that the tolerable bound $a_{1_{max}} = 1.7802$ in Corollary 2 is greater than $a_{1_{max}} = 1.7796$ in Corollary 1.

4. Discussion and Conclusion

In conclusion, two different theorems are provided to establish the asymptotic stability of solutions for some type of nonlinear fractional-order neutral systems having time delays. We've converted each of these theorems into a convex optimization problem to see which is more efficient. The effectiveness of the theorems is demonstrated using an example, and it is found that the efficiency of the two theorems differs only slightly. The solutions to the convex optimization problems were calculated using the LMI toolbox.

References

Altun, Y., & Tunç, C. (2020). On the asymptotic stability of a nonlinear fractional-order system with multiple variable delays. *Applications and Applied Mathematics*, 15(1), 458-468.

Chen, L. P., He, Y. G., Chai, Y., & Wu, R. C. (2014). New results on stability stabilization of a class of nonlinear fractional-order systems. *Nonlinear Dynamics*, 75, 633-641. doi:10.1007/s11071-013-1091-5

Chen, L. P., Liu, C., Wu, R. C., He, Y. G., & Chai, Y. (2016). Finite-time stability criteria for a class of fractional-order neural networks with delay. *Neural Computing and Applications*, 27, 549-556. doi:10.1007/s00521-015-1876-1

Deng, W. H., Li, C. P., & Lu, J. H. (2007). Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dynamics*, 48, 409-416. doi:10.1007/s11071-006-9094-0

Duarte-Mermoud, M. A., Aguila-Camacho, N., Gallegos, J. A., & Castro-Linares, R. (2015). Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, 22, 650-659. doi:10.1016/j.cnsns.2014.10.008

Hale, J. (1977). *Theory of Functional Differential Equations*. New York, USA: Springer-Verlag.

Heymans, N., & Podlubny, I. (2006). Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheologica Acta,* 45(5), 765-771. doi:10.1007/s00397-005-0043-5

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Application of Fractional Differential Equations.* New York, USA: Elsevier.

Korkmaz, E., & Özdemir, A. (2019). On stability of fractional differential equations with Lyapunov functions. *MAUN Fen Bilimleri Dergisi*, 7(1), 635-638. doi:10.18586/msufbd.559400

Li, H., Zhou, S., & Li, H. (2015). Asymptotic stability analysis of fractional-order neutral systems with time delay. *Advances in Continuous and Discrete Models*, 2015, 325-335.

Liu, S., Jiang, W., Li, X., & Zhou, X. F. (2016a). Lyapunov stability analysis of fractional nonlinear systems. *Applied Mathematics Letters*, 51, 13-19. doi:10.1016/j.aml.2015.06.018

Liu, S., Wu, X., Zhou, X. F., & Jiang, W. (2016b). Asymptotical stability of Riemann-Liouville fractional nonlinear systems. *Nolinear Dynamics*, 86, 65-71. doi:10.1007/s11071-016-2872-4

Liu, S., Wu, X., Zhang, Y. J., & Yang, R. (2017). Asymptotical stability of Riemann-Liouville fractional neutral systems. *Applied Mathematics Letters*, 69, 168-173. doi:10.1016/j.aml.2017.02.016

Podlubny, I. (1999). *Fractional Differential Equations*. New York, USA: Academic Press.

Qian, D., Li, C., Agarwal, R. P., & Wong, P. J. Y. (2010). Stability analysis of fractional differential system with Riemann-Liouville derivative. *Mathematical and Computer Modelling*, 52(5-6), 862-874. doi:10.1016/j.mcm.2010.05.016

Yang, X., Li, C., Huang, T., & Song, Q. (2017). Mittag-Leffler stability analysis of nonlinear fractional-order systems with impulses. *Applied Mathematics and Computation*, 293, 416-422. doi:10.1016/j.amc.2016.08.039